Sharp tail estimates for perturbed
stochastic volatility models

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Abstract
Sharp tail asymptotics are established for generalized Heston and Stein-Stein stochastic volatility models, with a
perturbed drift for the volatility process. This builds on the work of Gulisashvili\&Stein [Gul10],[GS10][GS10I] and
Friz et al.[FGGS10], [DFJV11].

1 Introduction

Gulisashvili et al.[Gul10],[GS10],[GS10I] compute an asymptotic expansion for the tail behaviour of the stock price
density and a large-strike expansion for call option prices and implied volatility under the uncorrelated Heston, Hull-
White and Stein-Stein stochastic volatility models. For the case of non-zero correlation, large-strike asymptotics are
also obtained in Friz et al.[FGGS10] for the Heston model using saddlepoint methods, and for the Stein-Stein model in
Deuschel et al.[DFJV11] using Laplace’s method on Wiener space, combined with Girsanov’s theorem and a stochastic
Taylor expansion. The much publicized SABR model for $\beta = 1$ is a special case of the Hull-White model when the
drift of the volatility process is zero. We believe there is an error in the formula in [GS10] which is corrected in the
forthcoming article [HZ14] (and extended to the case when $\rho \leq 0$); for this reason we do not pursue the Hull-White or
SABR models in this article.

In this note, as a simple application of Girsanov’s theorem, we show how to adapt the expansions for the Heston
and Stein-Stein models when we perturb the drift of the volatility process. The density expansions still hold but require
an additional Girsanov pre-factor which involves an expectation respect to the log stock price conditioned to end at a
certain level, i.e. a bridge process. We also obtain sharp estimates for call options and implied volatility, but here the
price we pay for perturbing the drift is that we longer know the exact value of the constant (which is independent of
the log-moneyness) that appears in front of the call option expansions. However this constant is essentially a fourth
order term in the large-strike regime for the models considered; hence our estimates are still considerably sharper than
the crude logarithmic tail estimates obtained by computing moment explosions in e.g. [AP07], [Jour04], [LM07].

To our knowledge this is the first article to compute density and large-strike expansions for call prices and implied
volatility for (semi) non-parametric stochastic volatility models. This is useful in practice because a non-parametric
drift gives practitioners greater flexibility in calibrating the model to observed short/large maturity smiles (see e.g.
[FJL12] for the Heston model) and/or the variance swap term structure, and potentially gives more degrees of freedom
to minimize over when performing global calibrations numerically.

2 The main results

We work on a model $(\Omega, \mathcal{F}, \mathbb{P})$ throughout, with a filtration $\mathcal{F}_t$ supporting two independent Brownian motions which
satisfies the usual conditions.
2.1 The perturbed Heston model

Consider a perturbed Heston stochastic volatility model for a log forward price process $X_t = \log S_t$ defined by the following stochastic differential equations

\[
\begin{align*}
\frac{dX_t}{dt} &= -\frac{1}{2}Y_t dt + \sqrt{\gamma_t}dW_t^1, \\
\frac{dY_t}{dt} &= [\kappa(\theta - Y_t) + \sigma\sqrt{\gamma_t}g(Y_t)]dt + \sigma\sqrt{Y_t}dW_t^2
\end{align*}
\]

(1)

with $Y_0 = y_0, \kappa, \sigma > 0$ and $dW_t^1dW_t^2 = 0$, where $g = h'$ for some bounded function $h$ and we assume that $e^{-h}\mathcal{L}e^h = \mathcal{L}h + \frac{1}{2}\sigma^2y^2h'^2 \in C_b$, where $\mathcal{L} = \kappa(\theta - y)\partial_y + \frac{1}{2}\sigma^2y\partial_y^2$ is the infinitesimal generator for the unperturbed $Y$ process, i.e. with $g \equiv 0$. We further assume that $X_0 = \log S_0 = 1$, without loss of generality, because the law of the log return $X_t - X_0$ does not depend on $X_0$.

We first define the measure $Q$ via

\[
\frac{dQ}{dP}\big|_{\mathcal{F}_t} = e^{-\int_0^t g(Y_s)dW_s^2 - \frac{1}{2}\int_0^t g(Y_s)^2ds}
\]

(2)

and set $\tilde{W}_t^2 = W_t^2 + \int_0^t g(Y_s)ds$. Then by Girsanov’s theorem, $\tilde{W}_t^2$ is standard Brownian motion under $Q$ and $(X_t, Y_t)$ follows an unperturbed Heston model with zero correlation under $Q$.

Lemma 2.1 Let $p_t(x)$ denote the log stock price density for the model in (8), and $q_t(x)$ denote the corresponding log stock price density for the unperturbed model. Then we have

\[
p_t(x) = \psi(x)q_t(x)
\]

(3)

where

\[
\psi(x) = \mathbb{E}^Q(e^{h(Y_t)} - h(y_0) - \int_0^t e^{-h}\mathcal{L}e^h(Y_s)ds | X_t = x)
\]

and $0 < \inf_x \psi(x) \leq \sup_x \psi(x) < \infty$, and there exists positive constants $C_1, C_2$ independent of $K$ such that

\[
C_1 \mathbb{E}(S_t - K)^+ \leq \mathbb{E}^Q(S_t - K)^+ \leq C_2 \mathbb{E}(S_t - K)^+.
\]

(4)

Proof. See Appendix A. ■

Proposition 2.2 Let $p_t(x)$ denote the density of $X_t$ Then we have the following right tail behaviour for $p_t(x)$

\[
p_t(x) = \psi(x)A_1e^{-(A_3-1)x}e^{A_2\sqrt{\gamma}x}x^{-\frac{1}{4}+\frac{2\theta}{\sigma^2}} [1 + O(x^{-\frac{1}{4}})] \quad (x \to \infty)
\]

(5)

for some positive constants $A_1, A_2, A_3$, which are given in [GS10I].

Proof. Let $q_t(x)$ denote the density of $X_t$ for the unperturbed Heston model. Then re-writing Theorem 2.1 in [GS10I] in terms of $q_t(x)$ we obtain

\[
q_t(x) = A_1e^{-(A_3-1)x}e^{A_2\sqrt{\gamma}x}x^{-\frac{1}{4}+\frac{2\theta}{\sigma^2}} [1 + O(x^{-\frac{1}{4}})] \quad (x \to \infty).
\]

(6)

Combining this with (4), we obtain the result. ■

Proposition 2.3 We have the following large-strike behaviour for call options on $S_t$:

\[
\log \mathbb{E}(S_t - K)^+ = -(A_3 - 2)x + A_2\sqrt{x} + \left(-\frac{3}{4} + \frac{\kappa\theta}{\sigma^2}\right)\log x + O(1) \quad (K \to \infty)
\]

where $x = \log K$.

Proof. From Theorem 7.3, part b) in [Gul10], we have

\[
\mathbb{E}(S_t - K)^+ \sim \text{const.} \times x^{-\frac{1}{4}+a/c^2}e^{A_2\sqrt{\gamma}e^{-(A_3-2)x}} \quad (K \to \infty),
\]

(7)

where $a = \kappa\theta$, $c = \sigma$, and the constants are given in Theorem 2.2 in [GS10I]. Combining this with (4) and taking logs, we obtain the result. ■
2.2 The perturbed Stein-Stein model

Consider a perturbed Stein-Stein stochastic volatility model for a stock or forward price process $S_t$ defined by the following stochastic differential equations

\[
\begin{align*}
\{ & dX_t = -\frac{1}{2}Y_t^2 dt + Y_t dW_t^1, \\
& dY_t = \kappa(\theta - Y_t) + \sigma g(Y_t) dt + \sigma dW_t^2
\end{align*}
\]

with $X_0 = 1$, $Y_0 = y_0$, $\kappa, \sigma > 0$ and $dW_t^1 dW_t^2 = 0$, where $g = h'$ for some bounded function $h$ and we assume that $e^{-h} L e^{h} = L h + \frac{1}{2} \sigma^2 h'' \in C_b$, where $L$ where $L = \kappa(\theta - Y_t) \partial_y + \frac{1}{2} \sigma^2 \partial_y^2$ is the infinitesimal generator for the unperturbed $Y_t$ process, i.e., with $g \equiv 0$.

Similar to before, we let $Q$ denote a probability measure under which $(X_t, Y_t)$ is governed by an unperturbed Stein-Stein model i.e. with $g \equiv 0$.

**Proposition 2.4** Let $p_t(x)$ denote the density of $X_t$. Then we have the following right tail behaviour for $p_t(x)$

\[
p_t(x) = \psi(x) B_1 e^{-(B_3-1)x} e^{B_2 \sqrt{x} x^{-\frac{1}{2}}} [1 + O(x^{-\frac{1}{4}})] \quad (x \to \infty)
\]

where $\psi(x) = E^Q(e^{h(Y_t) - h(y_0)} - \int_0^t e^{-h} L e^{h} ds | X_t = x)$ as before, and some positive constants $B_1, B_2, B_3$, which are given in [GS10I].

**Proof.** Lemma 2.1 also holds for the Stein-Stein model by an almost identical argument. Now let $q_t(x)$ denote the density of $X_t$ for the unperturbed Stein-Stein model. Then re-writing Theorem 2.1 in [GS10I] in terms of $q_t(x)$ we obtain

\[
q_t(x) = B_1 e^{-(B_3-1)x} e^{B_2 \sqrt{x} x^{-\frac{1}{2}}} [1 + O(x^{-\frac{1}{4}})] \quad (x \to \infty)
\]

Combining this with (3), we obtain the result. ■

**Proposition 2.5** We have the following large-strike behaviour for call options on $S_t$:

\[
\log E(S_t - K)^+ = -(B_3 - 2)x + B_2 \sqrt{x} - \frac{1}{2} \log x + O(1) \quad (K \to \infty)
\]

where $x = \log K$.

**Proof.** By exactly the same argument, Lemma 2.1 also holds for the Stein-Stein model. Then from Theorem 7.3, part a) in [Gul10], we have

\[
E(S_t - K)^+ \sim \text{const.} \times x^{-\frac{1}{2}} e^{B_2 \sqrt{x} e^{-(B_3 - 2)x}} \quad (K \to \infty)
\]

where the constants are given in Theorem 2.1 in [GS10I]. Combining this with (4) and taking logs, we obtain the result. ■

2.3 Non-zero correlation

We now consider the perturbed Heston model with correlation $\rho \leq 0$, which we write in the usual way as

\[
\begin{align*}
\{ & dX_t = -\frac{1}{2}Y_t dt + \rho \sqrt{Y_t} dW_t^1 + \sqrt{1-\rho^2} Y_t dW_t^2, \\
& dY_t = \kappa(\theta - Y_t) + \sigma \sqrt{Y_t} g(Y_t) dt + \sigma \sqrt{1-\rho^2} dW_t^2
\end{align*}
\]

where $\rho = \sqrt{1-\rho^2}$, and $W_1, W_2$ are independent Brownian motions.

**Proposition 2.6** If $\sqrt{Y_t} g$ is bounded and $e^{-h} L e^{h} \in C_b$, then large-strike expansion in Theorem 2.3 also holds for $\rho \leq 0$, for some come choice of the constants $A_2, A_3$.

**Proof.** See Appendix B. ■

**Remark 2.1** An almost identical analysis holds for the Stein Stein model (if we instead imposes that $yg$ is bounded), because the expansion in (11) also holds for $\rho \leq 0$ for some choice of the constants $B_1, B_3$ (see section 5.2 in [DFJ11] for details).
2.4 One-sided estimates

For any of the models considered above, if we only know that $h(y)$ and $-e^{-h}Le^h$ are both bounded from above (or similarly both bounded from below), then trivially we can still obtain one-sided tail estimates for the density and call option prices, which allows us to relax the technical conditions on $g$.

3 Implied Volatility

Let $L(x) = -\log C(x)$, where $C(x)$ is the price of a European call expressed as a function of the log-money $x$ for an unspecified model. Then from Corollary 7.3 in Gao&Lee[GL11], if

$$L(x) = \alpha_1 x + \alpha_2 x^{\frac{1}{2}} + \alpha_3 \log x + \alpha_0 + O(x^{-r})$$

(12)

for some $r \in (0, \frac{1}{2})$, then we have the following large-strike behaviour for the dimensionless Black-Scholes implied volatility $V$ at log-money $x$:

$$V(x) = \beta_1 x^{\frac{1}{2}} + \beta_0 + \beta_2 x^{-1} + \beta_3 \log x + O(x^{-r+\frac{1}{2}}) \quad (x \to \infty),$$

for some constants which depend on $\alpha_1, \alpha_2, \alpha_3, \alpha_0$ which are given in [GL11]. We have seen that both the perturbed Heston and Stein-Stein models admit expansions of the form in (12), but we do not the value of $\alpha_0$. Moreover, $\beta_2$ is the first term in the expansion whose value depends on $\alpha_0$; thus for the Heston and Stein-Stein models we can say that

$$V(x) = \beta_1 x^{\frac{1}{2}} + \beta_0 + \beta_2 x^{-\frac{1}{2}} + O(x^{-\frac{1}{2}}) \quad (x \to \infty).$$

References


A Proof of Lemma 2.1

Let $\mathcal{L}$ denote the generator of the $Y$ process. Then from Girsanov’s theorem we can easily show that

$$\frac{d\mathbb{P}}{d\mathbb{Q}}_{|F_t} = e^{h(Y_t) - h(y_0) - \int_0^t e^{-h} \mathcal{L} e^h(Y_s) ds}$$

1(or see Lemma 3.6 in [Feng99]). Thus

$$\mathbb{E}(S_t - K)^+ = \mathbb{Q}^0(e^{h(Y_t) - h(y_0) - \int_0^t e^{-h} \mathcal{L} e^h(Y_s) ds}(S_t - K)^+).$$

indeed for any bounded continuous function $f$, we have

$$\mathbb{E}(f(X_t)) = \mathbb{Q}^0(e^{h(Y_t) - h(y_0) - \int_0^t e^{-h} \mathcal{L} e^h(Y_s) ds} f(X_t))$$

and (3) follows by the standard conditioning argument. Finally, we know that $h$ and $e^{-h} \mathcal{L} e^h$ are bounded by assumption.

B Proof of Proposition 2.6

Re-writing the model under the measure $\mathbb{Q}$ defined via (2), we see that

$$\begin{cases}
    dX_t = (-\frac{1}{2} Y_t - \sigma \sqrt{T_t} g(Y_t))dt + \rho \sqrt{T_t} d\tilde{W}_t^2 + \tilde{\rho} \sqrt{T_t} dW_t^1, \\
    dY_t = \kappa(\theta - Y_t)dt + \sigma \sqrt{T_t} dW_t^2
\end{cases}$$

where $W^1, \tilde{W}^2$ are independent standard Brownians under $\mathbb{Q}$. We know that $|\sqrt{T} g| \leq A$ for some $K > 0$. Then $|X_t - X_t^1| \leq At$, where $X_t^1$ is the solution to the same SDEs but without the $\sigma \sqrt{T} g(Y_t)$ drift term. Then

$$\mathbb{E}(S_t - e^x)^+ \leq \mathbb{E}(e^{X_t^1 + At} - e^x)^+ = e^{At} \mathbb{E}(e^{X_t^1} - e^{x - At})^+.$$

and the corresponding lower bound is similarly obtained. Moreover, (7) also holds for $\rho \leq 0$ for some come choice of the constants $A_2, A_3$ (see Eq 4.2 in [FGGS10]). From this simple observation we see that for $\rho \leq 0$, we have

$$\log \mathbb{E}(S_t - K)^+ = -(B_3 - 2)(x - At) + B_2 \sqrt{x - At} - \frac{1}{2} \log(x - At) + O(1)$$

$$= -(B_3 - 2)x + B_2 \sqrt{x} - \frac{1}{2} \log x + O(1) \quad (K \to \infty).$$

Thus Proposition 2.3 also holds for $\rho \leq 0$.

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[1] we thank Rohini Kumar for pointing this out.