The large-maturity smile for the SABR and CEV-Heston models

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Abstract: Large-time asymptotics are established for the SABR model with $\beta = 1, \rho \leq 0$ and $\beta < 1, \rho = 0$. We also compute large-time asymptotics for the CEV model in the large-time, fixed-strike regime and a new large-time, large-strike regime, and for the uncorrelated *CEV-Heston* model. Finally, we translate these results into a large-time estimates for implied volatility using the recent work of Gao&Lee[GL11] and Tehranchi[Teh09].

Keywords and phrases: SABR model, Brownian exponential functional, CEV model, Large deviations, Heston model.

1. Introduction

Large-time asymptotic estimates for call options are particularly useful when the maturity of the option is such that standard PDE or Monte Carlo methods break down, or as an independent check for the accuracy of these numerical schemes. We can also glean useful information about the qualitative behaviour of the implied volatility smile at large maturities, under different modelling assumptions. For example, exponential Lévy models and stochastic volatility models with an ergodic volatility process (e.g the CIR process for the Heston model or an Ornstein-Uhlenbeck process) have the same qualitative behaviour at large maturities- they both exhibit an asymptotic large-maturity smile, when we work in the so-called *large-time, large-strike* parametrization (see below for details and references). For Heston and exponential Lévy models, this smile can be computed in terms of a Legendre transform of a limiting cumulant generating function. For the usual *fixed*-strike regime, the implied volatility tends to a non-zero constant which is independent of the strike; this is in contrast to the CEV and SABR models discussed in this article and in [Forde10], where the implied volatility tends to zero in the large-time limit. These results are particularly suited to the realm of long-dated foreign exchange options and swaptions where liquidity is lower, with maturities of 15 years and beyond. Expiries can extend to 75 years for GBP denominated caps, floors and swaptions.

Using the Gärtner-Ellis theorem from large deviations theory, [FJ11] compute the asymptotic implied volatility smile for the Heston model when $\kappa > 0$, $\kappa > \rho\sigma$, in the large-time, large log-moneyness regime and [FJM10] compute the correction term using saddlepoint methods; the large-time smile mimicks the large-time smile for the Barndorff-Nielsen NIG model, and [GJ11] show that the asymptotic smile can actually be computed in closed-form via the SVI parameterization. [JM12] derive similar results for a displaced Heston model, and relax the aforementioned conditions on κ, ρ, σ. Using a similar approach, [JKRM12] have recently extended the results in [FJ11] to a general class of affine stochastic volatility models (with jumps), which includes the Heston model with state-independent jumps, the Bates model with state-dependent jumps and the Barndorff-Nielsen-Shephard model. Under mild assumptions, they show that the limiting smile necessarily corresponds to the smile generated by an exponential Lévy model.

In [Forde10], we compute a closed-form expression for the stock price density under the *modified* SABR model (see Islah[Isl09]) with zero correlation, for $\beta = 1$ and $\beta < 1$, using the known density for the Brownian exponential functional for $\mu = 0$ given in Matsumoto&Yor[MY05], and then

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reversing the order of integration using Fubini's theorem. We then derived a large-time asymptotic expansion for the Brownian exponential functional for $\mu = 0$, and used this to characterize the large-time behaviour of the stock price distribution for the modified SABR model.

In [FK13], we establish a large-time large deviation principle (LDP) for the re-scaled log stock price $\frac{1}{t}X_t = \frac{1}{t}\log S_t$ for a general correlated stochastic volatility model: $dS_t = S_t\sigma(Y_t)dW^1_t, dY_t = -\alpha Y_t dt + dW^2_t$, $dW^1_t dW^2_t = \rho dt$, under a mild sublinear growth condition on $\sigma(.)$. The rate function is given by $I(x) = \inf_{\mu\in\mathcal{P}(\mathbb{R})}[\frac{(x-M(\mu))^2}{2\nu(\mu)} + I_\alpha(\mu)]$ for some linear functionals $M(\mu), \nu(\mu)$ which depend on ρ, where $I_\alpha(.)$ is the rate function for the occupation measure μ_t of the OU process Y given in [DV76]. Using the LDP, we then translate these results into large-time asymptotics for call options and implied volatility, and we extend our analysis to incorporate stochastic interest rates, by deriving a similar LDP for a three-factor model with a CIR short rate process. [1]

In section 2 of this paper, we first define the Brownian exponential functional $A^{(-\mu)}_t$ with negative drift $-\mu$, and we recall the result in Dufresne[Duf90] that $A^{(-\mu)}_\infty = \lim_{t\to\infty} A^{(-\mu)}_t$ is distributed as one-half the reciprocal of a gamma random variable. In section 3, we combine this result with the "mixing formula" for correlated stochastic volatility models given in [Wil96],[RT97] to compute the asymptotic log stock price density and the large-time behaviour of European put options for the SABR model with $\beta = 1, \rho \le 0$. In this case the mixing distribution is just the distribution of $A^{(-\frac{1}{2})}_\infty$. The key observation here is that we can re-write the irksome stochastic integral in the Willard formula as a linear function of the terminal instantaneous volatility Y_t, and we then just use the fact that the volatility process Y is a driftless geometric Brownian motion, and thus tends to zero almost surely as $t \to \infty$. In subsection 3.3, we characterize the tail behaviour of the asymptotic log stock price density and translate this into large-strike asymptotics for call options and implied volatility, using results of [Gul09] and [GL11]

In section 4, we adapt the arguments of section 3 for the case when $\beta < 1$; in this case the asymptotic stock price density is obtained by integrating the closed-form CEV put option formula over the distribution of $A^{(-\hat{\mu})}_\infty$ where $\hat{\mu} = \mu/\alpha$.

In section 5, we first compute call option prices in the large-time, fixed-strike regime under the standard CEV model $dS_t = \delta S^\beta_t dW_t$ by letting $t \to \infty$ in the closed-form solution for call options given in Cox[Cox75] in terms of the complementary non-central chi square distribution function. We find that the so-called "covered call price"

$$S_0 - \mathbb{E}(S_t - K)^+ = \mathbb{E}(S_t \wedge K) \;=\; cKt^{-1/2|\bar{\beta}|}\,(1 + o(1)) \tag{1.1}$$

for some constant $c = c(\delta, \beta, S_0)$, where $\bar{\beta} = \beta - 1 < 0$. We then use a result by Tehranchi[Teh09] (which has been independently proved and extended by Gao&Lee[GL11]) to translate this into a large-time estimate for the dimensionless implied variance $V_t(K)$:

$$V_t(K) = \frac{4}{|\bar{\beta}|}\log t - 4\log\log t - 4\log(\frac{1}{2}\pi\bar{c}^2\gamma) - 4k + o(1) \qquad (t\to\infty)$$

where $\bar{c} = c/S_0$, i.e. the large-time implied volatility tends to zero as $t \to \infty$ and the leading order term is independent of K, and the implied variance skew is linear in the log-strike. In subsection 5.3, we derive a large deviation principle for the CEV model in a large-time, *large-strike* regime. This shows (as for the Heston and exponential Lévy models) that the smile does not disappear as $t \to \infty$, but rather it spreads out, and this new parametrization is needed to see the smile effect at large maturities.

In section 6, we derive a large-time large deviation principle for the time-average of the Cox-Ingersoll-Ross process in terms of a Fenchel-Legendre transform, using similar arguments to those used for the Heston model in [FJ11],[FJM10]. In section 7 we introduce the so-called *CEV-Heston*

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model as the CEV process evaluated at a CIR time-change in the large-time, large-strike regime, and establish a joint large deviation principle for the average integrated variance and the terminal stock price, by first establishing a weak LDP and then proving exponential tightness. Finally, we prove an LDP for the stock price itself as a simple application of the contraction principle, and we show that the rate function has a unique minimum at zero.

2. The large-time density of the Brownian exponential for $\mu < 0$

In this paper, we let $B = \{B_t, t \geq 0\}$ denote a one-dimensional Brownian motion started at zero defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $B_t^{(\mu)} = \{B_t + \mu t, t \geq 0\}$ denote the corresponding Brownian motion with constant drift $\mu \in \mathbb{R}$. We consider the exponential functional

$$A_t^{(-\mu)} = \int_0^t e^{2B_s^{(-\mu)}} ds\,,$$

which is the time-integral of geometric Brownian motion. $A_t^{(-\mu)}$ is closely related to the time-integral of the instantaneous variance for the SABR and Hull-White stochastic volatility models, and $A_t^{(-\mu)}$ is also used in pricing arithmetic Asian options under the Black-Scholes model. From section 4 in Matsumoto&Yor[MY05], we have the following double integral representation for the density of $1/(2A_t^{(-\mu)})$ when $\mu < 1$

$$\begin{aligned} f^{(-\mu)}(a,t) &= \frac{2}{\sqrt{2\pi^3 t}}\, e^{\pi^2/2t - \mu^2 t/2} e^{-a} a^{-(-\mu+1)/2} \\ &\cdot \int_0^\infty \eta^{-\mu} e^{-\eta^2} \int_0^\infty e^{-\xi^2/(2t)} e^{-2\sqrt{a}\,\eta\cosh(\xi)} \sinh(\xi)\sin(\pi\xi/t)\, d\xi\, d\eta\,. \end{aligned} \tag{2.1}$$

Gulisashvili&Stein[GS06],[GS10] have derived sharp tail estimates for $A_t^{(-\mu)}$ using saddlepoint methods.

We make the following assumption throughout:

Assumption 2.1. $\mu > 0$.

Under Assumption 2.1, $A_\infty^{(-\mu)} = \lim_{t\to\infty} A_t^{(-\mu)}$ is finite a.s., and we have the following theorem from Dufresne[Duf90] (see also [MY05]):

Theorem 2.2. *For $\mu > 0$, $A_\infty^{(-\mu)}$ is distributed as $Z = (2\gamma_\mu)^{-1}$, where γ_μ denotes a gamma random variable with parameter μ.*

Corollary 2.3. *$A_t^{(-\mu)} \to A_\infty^{(-\mu)}$ a.s., so $A_t^{(-\mu)} \xrightarrow{w} A_\infty^{(-\mu)}$ and the cdf of $A_\infty^{(-\mu)}$ has no atoms; thus (by the Lemma on page 181 in [Will91]), we have*

$$\lim_{t\to\infty} \mathbb{P}(A_t^{(-\mu)} > a) \;=\; \mathbb{P}(A_\infty^{(-\mu)} > a) \;=\; \mathbb{P}(Z > a)\,.$$

The density of γ_μ is the usual Gamma density

$$\mathbb{P}(\gamma_\mu \in dx) = \frac{1}{\Gamma(\mu)} x^{\mu-1} e^{-x} dx \qquad\qquad (x > 0)\,,$$

and from this we obtain the density of $A_\infty^{(-\mu)}$ as

$$f(a) \;=\; \frac{1}{da}\mathbb{P}(A_\infty^{(-\mu)} \in da) = \frac{1}{\Gamma(\mu)} (\frac{1}{2a})^{\mu-1} e^{-1/2a} \frac{1}{2a^2} \qquad\qquad (a > 0)\,. \tag{2.2}$$

From the time-scaling property of Brownian motion we see that

$$\int_0^t e^{2\alpha(B_s - \mu s)} ds \overset{\text{(law)}}{=} \int_0^t e^{2(B_{\alpha^2 s} - \alpha\mu s)} ds \;=\; \frac{1}{\alpha^2}\int_0^{\alpha^2 t} e^{2(B_u - \mu u/\alpha)} du = \frac{1}{\alpha^2} A_{\alpha^2 t}^{(-\hat{\mu})}\,, \tag{2.3}$$

where

$$\hat{\mu} \;=\; \mu/\alpha\,. \tag{2.4}$$

3. The SABR model for $\beta = 1, \rho \leq 0$

From here on, we work on a model $(\Omega, \mathcal{F}, P)$ throughout, with a filtration $(\mathcal{F}_t)_{t\geq 0}$ supporting two Brownian motions, and satisfying the usual conditions.

We now consider the well known SABR model with $\beta = 1$ and correlation $\rho \leq 0$, $|\rho| < 1$ defined by the following stochastic differential equations

$$\begin{cases} dS_t = S_t Y_t dW_t\,, \\ dY_t = \alpha Y_t dB_t\,, \end{cases} \tag{3.1}$$

where $dW_t dB_t = \rho dt$ and $\alpha > 0$. It will be convenient to re-write the model in terms of the log stock price $X_t = \log S_t$ and two independent Brownian motions B and W as follows:

$$\begin{cases} dX_t = -\frac{1}{2}Y_t^2 dt + Y_t(\bar{\rho} dW_t + \rho dB_t)\,, \\ dY_t = \alpha Y_t dB_t\,, \end{cases}$$

where $dW_t dB_t = 0$ and $\bar{\rho} = \sqrt{1-\rho^2}$. The correlation ρ has to be non-positive to ensure that (S_t) is a martingale (see e.g. Jourdain[Jour04]).

Noting that $Y_t = y_0 e^{\alpha(B_t - \frac{1}{2}\alpha t)}$ and using (2.3), we see that

$$T_t = \int_0^t Y_s^2 ds = y_0^2 \int_0^t e^{2\alpha(B_s - \frac{1}{2}\alpha s)} ds \overset{(\text{law})}{=} \sigma^2 A^{(-\frac{1}{2})}_{\alpha^2 t} \tag{3.2}$$

where

$$\sigma = y_0/\alpha\,. \tag{3.3}$$

3.1. Asymptotic behaviour of the log stock price density

The following theorem characterizes the behaviour of the asymptotic log return density in the large-time limit for the SABR model with $\beta = 1, \rho \leq 0$ (see also the left plot in Figure 1).

Proposition 3.1. *$S_\infty = \lim_{t\to\infty} S_t$ exists a.s., and $X_\infty - x_0 = \log \frac{S_\infty}{S_0}$ has density*

$$p_\infty(x) := \sigma e^{-(x+\rho\sigma)/(2\bar{\rho}^2)} \frac{K_1\left(\frac{\sqrt{(x^2+2x\rho\sigma+(\rho^2+\bar{\rho}^2)\sigma^2)}}{2\bar{\rho}^2}\right)}{2\pi\bar{\rho}\sqrt{x^2+2x\rho\sigma+(\rho^2+\bar{\rho}^2)\sigma^2}}\,, \tag{3.4}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind.

Proof. $S_t = e^{X_t}$ is a non-negative supermartingale, so $S_\infty = \lim_{t\to\infty} S_t$ exists a.s. (see e.g. Problem 3.16 in [KS91]). Moreover, we can re-write the log return $X_t - x_0$ as

$$X_t - x_0 = -\frac{1}{2}T_t + \rho\int_0^t Y_s dB_s + \bar{\rho}\int_0^t Y_s dW_s\,.$$

For the SABR model here, we can re-write the stochastic integral term as $\int_0^t Y_s dB_s = \frac{1}{\alpha}(Y_t - y_0)$. Using this and conditioning on $(Y_s; 0 \leq s \leq t)$ we have

$$\begin{aligned} \mathbb{E}(e^{i\theta(X_t - X_0)}) &= \mathbb{E}(e^{i\theta(-\frac{1}{2}T_t + \rho\frac{Y_t - y_0}{\alpha} + \bar{\rho}\int_0^t Y_s dW_s)}) \\ &= \mathbb{E}\big[e^{i\theta(-\frac{1}{2}T_t + \rho\frac{Y_t - y_0}{\alpha})}\mathbb{E}(e^{i\theta\bar{\rho}\int_0^t Y_s dW_s} \,|\, \sigma(Y_s; 0 \leq s \leq t))\big] \\ &= \mathbb{E}(e^{i\theta(-\frac{1}{2}T_t + \rho\frac{Y_t - y_0}{\alpha}) - \frac{1}{2}\bar{\rho}^2\theta^2 T_t})\,. \end{aligned}$$

But Y is a driftless geometric Brownian motion, so $Y_t \to 0$ a.s. as $t \to \infty$. Thus (because $X_\infty = \lim_{t\to\infty} \log \frac{S_t}{S_0} = \log \frac{S_\infty}{S_0}$ exists a.s.) we have (by the dominated convergence theorem)

$$\lim_{t\to\infty} \mathbb{E}(e^{i\theta(X_t - x_0)}) = \mathbb{E}(e^{i\theta(X_\infty - x_0)}) = \mathbb{E}(e^{i\theta(-\frac{1}{2}T_\infty - \rho\sigma) - \frac{1}{2}\bar{\rho}^2\theta^2 T_\infty})$$

where $\sigma = \frac{y_0}{\alpha}$ as before. Thus $X_\infty - x_0 \stackrel{\text{(law)}}{=} -\frac{1}{2}T_\infty - \rho\sigma + \bar{\rho}\sqrt{T_\infty}\,Z$ where $Z \sim N(0,1)$ is a Normal random variable independent of T_∞.

From (3.2) we know that $T_\infty \stackrel{\text{(law)}}{=} \sigma^2 A_\infty^{(-\frac{1}{2})}$. Let $p_a(x) = \frac{e^{-\frac{(x+\frac{1}{2}\sigma^2 a+\rho\sigma)^2}{2\bar{\rho}^2\sigma^2 a}}}{\sqrt{2\pi\bar{\rho}^2\sigma^2 a}}$ denote the density of $-\frac{1}{2}\sigma^2 t - \rho\sigma + \bar{\rho}\sqrt{\sigma^2 t}Z$ at $t = a$. Then

$$p_\infty(x) \quad = \quad \int_0^\infty p_a(x) f(a) da\,,$$

where $f(a)$ is defined in (2.2). Evaluating the last integral explicitly in e.g. Mathematica, we arrive at the closed-form expression in (3.4). □

Remark 3.2. *$X_t \to X_\infty$ a.s. so $X_t \stackrel{w}{\to} X_\infty$ and the distribution of X_∞ has no atoms, so we also have*

$$\lim_{t\to\infty} \mathbb{P}(X_t > x) \quad = \quad \mathbb{P}(X_\infty > x) \tag{3.5}$$

for all $x \in \mathbb{R}$.

3.2. Asymptotic behaviour of put option prices

The following theorem characterizes the behaviour of put option prices in the large-maturity limit for the SABR model with $\beta = 1, \rho \leq 0$ (see also the right plot in Figure 1).

Theorem 3.3. *Then we have the following large-time behaviour for put options under the SABR model with $\beta = 1$, $\rho \leq 0$:*

$$\begin{aligned} P_\infty(k) := \lim_{t\to\infty} \frac{1}{S_0}\mathbb{E}(K - S_t)^+ &= \frac{1}{S_0}\mathbb{E}(K - S_\infty)^+ \\ &= \mathbb{E}(P^{\mathrm{BS}}(e^{-\frac{1}{2}\rho^2 T_\infty - \frac{\rho}{\alpha}y_0}, e^k, 1, \bar{\rho}^2 T_\infty)) < \frac{K}{S_0}\,, \end{aligned} \tag{3.6}$$

where $K = S_0 e^k$ and $P^{\mathrm{BS}}(S, K, \sigma, \tau) = K\Phi^c(\frac{\log\frac{S}{K} - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}) - S\Phi^c(\frac{\log\frac{S}{K} + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}})$ is the usual Black-Scholes put option formula with zero interest rates, $\Phi(x) = \int_{-\infty}^x \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}dz$ and $\Phi^c(x) = 1 - \Phi(x)$.

Proof. From the mixing formula in [Wil96],[RT97] we have

$$\mathbb{E}(K - S_t)^+ = \mathbb{E}(P^{\mathrm{BS}}(S_0 e^{-\frac{1}{2}\rho^2 T_t + \rho\int_0^t Y_s dB_s}, K, 1, \bar{\rho}^2 T_t)) \tag{3.7}$$

so we can re-write (3.7) as

$$\mathbb{E}(K - S_t)^+ = \mathbb{E}(P^{\mathrm{BS}}(S_0 e^{-\frac{1}{2}\rho^2 T_t + \frac{\rho}{\alpha}(Y_t - y_0)}, K, 1, \bar{\rho}^2 T_t))\,. \tag{3.8}$$

Now, letting $t \to \infty$ and using the bounded convergence theorem, we obtain the first and second equalities in (3.6).

From the known density of $X_\infty - x_0$ given in Theorem 3.1 , we know that $\mathbb{P}(S_\infty = 0) = 0$, thus $\mathbb{E}(K - S_\infty)^+ < K$. □

Remark 3.4. *The trick of re-writing the stochastic integral here is only useful if the drift of the Y process is zero, or else we are left with an expression containing both $\int_0^t Y_s ds$ and $\int_0^t Y_s^2 ds$.*

3.3. Tail behaviour of the asymptotic density and the dimensionless implied volatility

Corollary 3.5. *Using the asymptotic relation $K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}$ as $x \to \infty$, we have the following tail behaviour for $p_\infty(x)$*

$$p_\infty(x) \quad \sim \quad \begin{cases} \frac{\sigma|x|^{-\frac{3}{2}}}{2\pi} & (x \to -\infty)\,, \\ \frac{\sigma x^{-\frac{3}{2}}}{2\pi}\, e^{-\rho\sigma/\bar{\rho}^2}\, e^{-x/\bar{\rho}^2} & (x \to \infty)\,, \end{cases} \tag{3.9}$$

where p_∞ is defined in (3.1)*, so we see that $p_\infty(x)$ has power decay in the left tail and exponential decay in the right tail.*

Remark 3.6. $p_\infty(x) = \frac{\sigma x^{-\frac{3}{2}}}{2\pi}\, e^{-\rho\sigma/\bar{\rho}^2}\, e^{-x/\bar{\rho}^2}[1+o(1)]$ *as $x \to \infty$. From this and a repeated application of Lemma 5.11 in [Gul09], we can easily show that*

$$\frac{1}{S_0}C_\infty(k) \quad = \quad \bar{\rho}^4 e^k p_\infty(k)\,[1+o(1)] \qquad\qquad (k \to \infty)\,, \tag{3.10}$$

where $C_\infty(k) = \lim_{t\to\infty}\mathbb{E}(S_t - S_0e^k)^+$ is the price of a call option with log-moneyness k as $t \to \infty$. Using Corollary 6.3 in Gao&Lee[GL11], we can then translate this into the following large-strike behaviour for the dimensionless Black-Scholes implied volatility $\sqrt{V}$:

$$|G_-(k, L - \log\frac{\sqrt{4\pi L}}{1-(k+L)^{-\frac{1}{2}}}) - \sqrt{V}| \quad = \quad O(\frac{\log L}{L^{\frac{3}{2}}}) \qquad\qquad (k \to \infty) \tag{3.11}$$

where $L = \log\frac{1}{C(k)/S_0}$ and $G_-(k,u) = \sqrt{2}\,[\sqrt{u+k} - \sqrt{k}]$.

3.4. Limiting behaviour of the implied volatility

Recall that $\mathbb{P}(S_\infty = 0) = 0$, and thus $\mathbb{E}(K - S_\infty)^+ < K$, and this is also true for any model for which $\mathbb{P}(S_\infty = 0) < 1$. This means that the asymptotic implied variance (i.e. the solution to $P_\infty(k) = P^{\mathrm{BS}}(1, e^k, 1, V_\infty(k)))$ is a finite constant.

To analyze the limiting behaviour of the implied variance, let $V_t(k) = \hat{\sigma}_t(k)^2 t$ denote the Black-Scholes implied variance at finite maturity t, where $\hat{\sigma}_t(k)$ is the implied volatility at log-moneyness k. Then using Theorem 3.3 and using the differentiability of the Black-Scholes put option formula as a function of the volatility, we easily show that

$$V_t(k) \nearrow V_\infty(k) \tag{3.12}$$

as $t \to \infty$, which implies that $\hat{\sigma}_t(k) \sim \sqrt{V_\infty(k)}\,t^{-\frac{1}{2}}$ as $t \to \infty$, i.e. for a fixed strike, the implied volatility for the SABR model tends to zero in the large-time limit.

3.5. Large-time asymptotics for a more general class of stochastic volatility models

We have the following partial generalization of Proposition 3.1:

Proposition 3.7. *Consider the the following more general stochastic volatility model*

$$\begin{cases} dS_t \;=\; S_t\sigma(Y_t)dW_t\,, \\ dY_t \;=\; \alpha\sigma(Y_t)dB_t\,, \end{cases} \tag{3.13}$$

with $dW_t dB_t = \rho dt$ and assume that $\frac{1}{K}y \le \sigma(y) \le Ky$ for some $K > 0$. Then $S_\infty = \lim_{t\to\infty} S_t$ exists a.s., and $X_\infty - x_0 = \log\frac{S_\infty}{S_0}$ has characteristic function

$$\mathbb{E}(e^{i\theta(X_\infty - x_0)}) \quad = \quad \mathbb{E}(e^{i\theta(-\frac{1}{2}\hat{T}_\infty - \rho\frac{y_0}{\alpha}) - \frac{1}{2}\bar{\rho}^2\theta^2\hat{T}_\infty})\,,$$

where $\hat{T}_\infty \;=\; \int_0^t \sigma(Y_s)^2 ds < \infty$ a.s., i.e. $X_\infty - x_0$ has a Gaussian mixture density.

Proof. Follows from the same argument used in the proof of Proposition 3.1. □

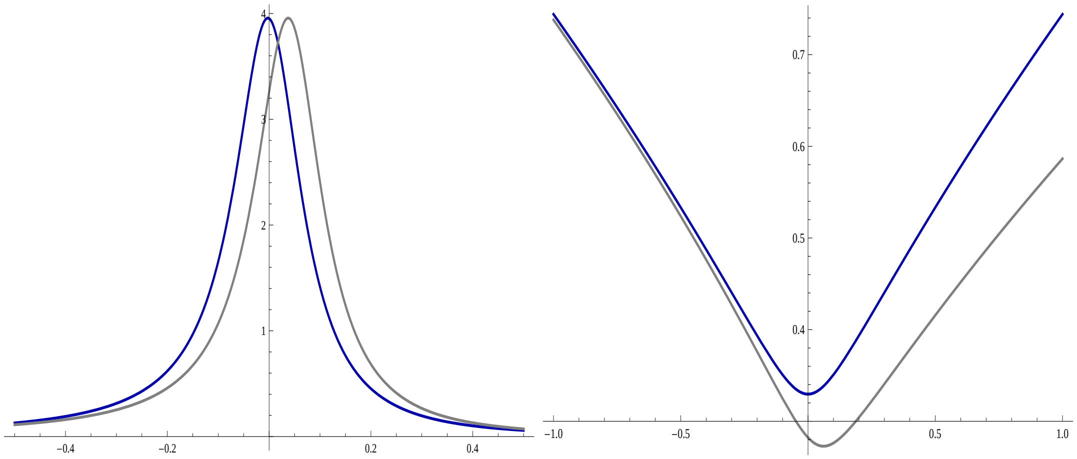

FIG 1. *Here we have plotted the asymptotic log return density $p_\infty(x)$ given in (3.4) and the asymptotic dimensionless implied volatility $\sqrt{V_\infty(k)}$ for the SABR model with $\sigma = y_0/\alpha = .1, \beta = 1$ and $\rho = 0$ (blue) and $\rho = -.4$ (grey).*

4. The general SABR model for $\beta < 1$, $\rho = 0$

The *constant elasticity of variance*(CEV) diffusion process of Cox[Cox75] is defined by the SDE

$$dS_t = \delta S_t^\beta \, dW_t \tag{4.1}$$

with $\beta \in (0,1)$, $\delta > 0$, $S_0 > 0$. The origin is an exit boundary for $\beta \in (\frac{1}{2}, 1)$, and a regular boundary for $\beta \leq \frac{1}{2}$, which we specify as absorbing to ensure that (S_t) is a martingale. Infinity is a natural, non-attracting boundary. The transition density for the CEV process is given by

$$p(t, S_0, S) = \frac{S^{-2\bar{\beta}-\frac{3}{2}} S_0^{\frac{1}{2}}}{\delta^2 |\bar{\beta}| t} \exp\left(-\frac{S_0^{-2\bar{\beta}} + S^{-2\bar{\beta}}}{2\delta^2 \bar{\beta}^2 t}\right) I_\nu\left(\frac{S_0^{-\bar{\beta}} S^{-\bar{\beta}}}{\delta^2 \bar{\beta}^2 t}\right) \qquad (S > 0), \tag{4.2}$$

where $\bar{\beta} = \beta - 1 < 0$, $\nu = \frac{1}{2(1-\beta)}$, and $I_\nu(.)$ is the modified Bessel function of the first kind (see Davydov&Linetsky[DavLin01]).

Now consider a mild generalization of the standard SABR model for $\beta < 1, \rho = 0$, defined by the stochastic differential equations

$$\begin{cases} dS_t = S_t^\beta Y_t dW_t \,, \\ dY_t = bY_t dt + \alpha Y_t dB_t \end{cases} \tag{4.3}$$

with $dW_t dB_t = 0$, $Y_0 = y_0 > 0$. The model is general in the sense that b can take any value less than or equal to $\frac{1}{2}\alpha^2$; $b = \frac{1}{2}\alpha^2$ is the critical drift value where the model becomes the *modified SABR model*, which is qualitatively very different and discussed at length in [Forde10]. The case $\beta < 1$ is most relevant to interest rates markets.

For the model in (4.3), using (3.2) we have that

$$S_t \overset{(\text{law})}{=} X_{\int_0^t Y_s^2 ds} \overset{(\text{law})}{=} X_{\sigma^2 A^{(-\frac{1}{2})}_{\alpha^2 t}} \,, \tag{4.4}$$

where X_t is a CEV process $dX_t = \delta X_t^\beta dW_t$ with $\delta = 1$ and independent of Y, with $X_0 = S_0$ and σ defined as in (3.3).

The following theorem characterizes the behaviour of put option prices in the large-maturity limit for the SABR model with $\beta \leq 1, \rho = 0$ (see also Figure 3).

Proposition 4.1. *Let $T_\infty = \sigma^2 A_\infty^{(-\hat{\mu})}$ as before, where $\hat{\mu}$ is defined in (2.4). Then we have the following large-time behaviour for put options under the SABR model with $\beta < 1$, $\rho = 0$:*

$$\begin{aligned} P_\infty(K) \;=\; \lim_{t\to\infty} \mathbb{E}(K - S_t)^+ \;&=\; \mathbb{E}(K - S_\infty)^+ \\ &=\; \mathbb{E}(P^{\text{CEV}}(S_0, K, T_\infty; \delta, \beta)) \;<\; K \,, \end{aligned} \tag{4.5}$$

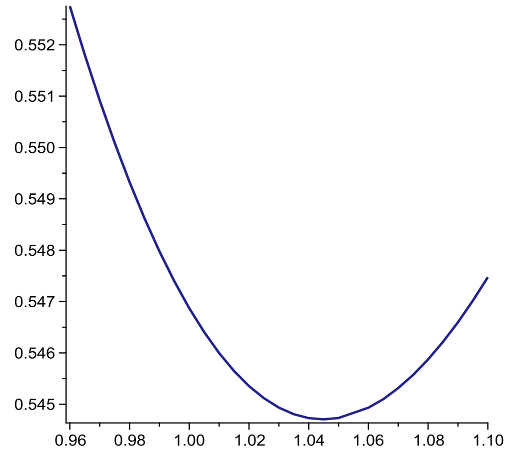

FIG 2. *Here we have plotted the asymptotic dimensionless implied volatility $\sqrt{V_\infty(k)}$ for the standard SABR model (i.e. $b=0$) with $S_0=1, y_0=1, \alpha=1, \delta=.2, \beta=.5$.*

with $\delta=1$, where $P^{\mathrm{CEV}}(S_0,K,\tau;\delta,\beta)$ is the price of a put option of strike K and maturity τ under the standard CEV model.

Proof. The proof just follows from the dominated convergence theorem and conditioning on the independent T_∞. □

Remark 4.2. *The original SABR model in [HKLW02] was written in the form $dF_t=\delta F_t^\beta Y_t dW_t$ for a forward price process $F_t=S_t e^{(r-q)(T-t)}$; our results can be trivially adjusted to deal with non-zero (constant) interest rate r and dividend rate q by including a discount factor e^{-rt} and replacing S_t by F_t (note that implied volatility is unaffected by r and q for the SABR model in this form). For a modified SABR model of the form $dS_t=S_t(r-q)dt+\delta Y_t S_t^\beta dW_t$, the time-change argument used in Proposition 4.1 only works if $r=q$, otherwise S becomes a time-inhomogenous diffusion once we condition on the Y process.*

Remark 4.3. *$P^{\mathrm{CEV}}(S_0,K,\tau;\delta,\beta)$ admits a closed-form formula, which is obtained just by combining the corresponding call option formula in* (5.2) *and the put-call parity. In this case, there is a non-zero probability of absorption at $S=0$ (see subsection 4.1 for an explicit computation of this probability).*

Remark 4.4. *Similar to* (3.12)*, we can easily show that*

$$V_t(K) \nearrow V_\infty(K) < \infty \qquad\qquad (t\to\infty)\,,$$

where $V_t(K)$ is the dimensionless implied variance at strike K, and $V_\infty(K)$ is the dimensionless implied variance associated with the asymptotic put price $P_\infty(K)$, which is the unique solution to $P_\infty(K)=P^{\mathrm{BS}}(S_0,K,1,V_\infty(K))$.

4.1. Probability of eventual absorption at zero for $\beta<1$

For the standard CEV model in (4.1), from e.g. page 312 in Lewis[Lew00], we have the following well known expression for the probability of absorption at $S=0$ by time t

$$\mathbb{P}(S_t=0) \quad = \quad G\Big(\frac{\gamma}{2},\frac{\frac{1}{2}S_0^{-2\bar\beta}}{\delta^2\bar\beta^2 t}\Big) \tag{4.6}$$

where $\gamma=1/|\bar\beta|$ and $G(\nu,x)=\frac{1}{\Gamma(\nu)}\int_x^\infty t^{\nu-1}e^{-t}dt$ is the *complementary incomplete Gamma function.*

Remark 4.5. *Substituting the asymptotic relation* $1-G(\nu,x)=x^{\gamma}\frac{1}{\Gamma(\nu+1)}(1+O(x))$ *(see page 260 in [AS72]) as* $x\to 0$ *into* (4.6)*, we obtain*

$$\mathbb{P}(S_t>0) \quad = \quad 1\;-\;G(\frac{\gamma}{2},\frac{\frac{1}{2}S_0^{-2\bar{\beta}}}{\delta^2\bar{\beta}^2 t})\;\sim\;(\frac{\frac{1}{2}S_0^{-2\bar{\beta}}}{\delta^2\bar{\beta}^2})^{\frac{\gamma}{2}}\frac{1}{\Gamma(\frac{\gamma}{2}+1)}t^{-\frac{\gamma}{2}} \qquad (t\to\infty)\,, \tag{4.7}$$

so we see that the probability of not hitting zero has power decay as $t\to\infty$*.*

The probability of eventual absorption for the uncorrelated general SABR model is then obtained by integrating $\mathbb{P}(S_t=0)$ with $\delta=y_0/\alpha$ over the density of $A_\infty^{(-\hat{\mu})}$ as

$$\mathbb{P}(S_\infty=0) \quad = \quad \int_0^\infty G(\frac{\gamma}{2},\frac{\frac{1}{2}S_0^{-2\bar{\beta}}}{\delta^2\bar{\beta}^2 a})\,\mathbb{P}(A_\infty^{(-\hat{\mu})}\in da)\,. \tag{4.8}$$

It may be possible to simplify this expression further, but for the sake of brevity we defer the details for future work.

5. The CEV model

In this section we characterize the large-time asymptotic behaviour of the CEV model defined in (4.1). The CEV model has been actively used in interest rates markets before the adoption of the SABR model.

5.1. Call option asymptotics for the large-time, fixed-strike regime

Proposition 5.1. *Let* $\gamma=1/|\bar{\beta}|$ *as before. Then we have the following large-time behaviour for call options under the standard CEV model in* (4.1)

$$S_0-\mathbb{E}(S_t-K)^+=\mathbb{E}(S_t\wedge K)\;=\;cKt^{-\frac{\gamma}{2}}\,(1+o(1)) \qquad (t\to\infty)\,, \tag{5.1}$$

where

$$c=c(\beta,\delta,S_0)=\frac{1}{\Gamma(1+\frac{\gamma}{2})}[\frac{1}{2}(\frac{S_0^{-2\bar{\beta}}}{\delta^2\bar{\beta}^2})]^{\frac{\gamma}{2}}\,.$$

Remark 5.2. *From* (5.1) *and* (4.7) *we see that* $\mathbb{P}(S_t>0)\sim\frac{1}{K}\mathbb{E}(S_t\wedge K)$ *as* $t\to\infty$*.*

Proof. From page 7 in [DavLin01] we have

$$\mathbb{E}(S_t-K)^+=S_0Q(y_0;n,\zeta)-K(1-Q(\zeta;n-2,y_0))\,, \tag{5.2}$$

where $n=2+1/|\bar{\beta}|$, $\zeta=S_0^{-2\bar{\beta}}/\delta^2\bar{\beta}^2 t$, $y_0=K^{-2\bar{\beta}}/\delta^2\bar{\beta}^2 t$ and $Q(x;u,v)$ is the complementary non-central chi-square distribution function with u degrees of freedom and non-centrality parameter v.

From Appendix A (see also Eq 26.4.6 on page 941 in [AS72] for the case $v=0$) we have $1-Q(\epsilon;u,\epsilon v)\;\sim\;\frac{1}{\Gamma(\frac{1}{2}u+1)}(\frac{1}{2}\epsilon)^{u/2}$ as $\epsilon\searrow 0$, Applying this to (5.2) as $t\to\infty$ we obtain

$$\begin{aligned}
\mathbb{E}(S_t-K)^+ &= S_0Q(y_0;n,\zeta)-K(1-Q(\zeta;n-2,y_0))\\
&= S_0\big[1-(\frac{1}{2}y_0)^{n/2}\frac{1}{\Gamma(\frac{1}{2}n+1)}\big](1+o(1))-K(\frac{1}{2}\zeta)^{(n-2)/2}\frac{1}{\Gamma(\frac{1}{2}n)}(1+o(1))\\
&= S_0-\frac{S_0}{\Gamma(\frac{1}{2}n+1)}(\frac{1}{2}\frac{K^{-2\bar{\beta}}}{\delta^2\bar{\beta}^2 t})^{\frac{1}{2}n}(1+o(1))-\frac{K}{\Gamma(\frac{1}{2}n)}(\frac{1}{2}(\frac{S_0^{-2\bar{\beta}}}{\delta^2\bar{\beta}^2 t}))^{\frac{1}{2}n-1}(1+o(1)) \qquad (t\to\infty)\,.
\end{aligned}$$

The rightmost term dominates the middle term as $t\to\infty$, and (5.1) follows. □

5.2. Implied volatility

Corollary 5.3. *We have the following large-time behaviour for the dimensionless implied variance $V_t(K)$ at strike K*

$$\begin{aligned} V_t(K) &= 8L - 4\log L + 4k - 4\log\pi + o(1) \\ &= \frac{4}{|\bar{\beta}|}\log t - 4\log\log t - 4\log(\frac{1}{2}\pi\bar{c}^2\gamma) - 4k + o(1) \end{aligned} \tag{5.3}$$

as $t\to\infty$, where $k=\log\frac{K}{S_0}$, $L=\log\frac{1}{C_+}$, $C_+=1-\frac{1}{S_0}\mathbb{E}(S_t-K)^+$, $\bar{c}=\frac{c}{S_0}$ and c,γ are defined in Proposition 5.1.

Proof. See Appendix B. □

Remark 5.4. *From* (5.3)*, we see that the leading order term is independent of K and depends only on β. The higher order $-4k$ term gives the implied variance skew, which is linear in log-strike as $t\to\infty$; to see the convexity effect, we have to work in the large-time, large-strike regime discussed in the next section. Note that the implied volatility $\hat{\sigma}_t(K)=\sqrt{V_t(K)/t}=O(\frac{\sqrt{\log t}}{t})$ and thus tends to zero as $t\to\infty$.*

Remark 5.5. *The* ***relative*** *error of the approximation in* (5.3) *is $o(\frac{1}{\log t})$, so convergence is slower than $O(t^{-p})$ for any $p>0$. Corollary 7.9 of the recent preprint by Gao&Lee[GL11] states that the error in* (5.3) *is actually $O(\frac{\log L}{L})$, and they also give a higher order expansion for $V_t(K)$. However, given the* $\log t$ *error terms that appear in all the aforementioned implied volatility approximations for the CEV model, in practice it is far more efficient to work directly with the call option asymptotics in* (5.1) *(see Figure 3).*

5.3. The Large-time, large-strike regime

We now consider a *large-time, large-strike* regime where the strike scales as $Kt^{1/|\bar{\beta}|}$ as $t\to\infty$, which is mathematically more interesting and is also the correct parametrization to use if we want to see the full smile effect at large maturities.

Proposition 5.6. *Let $\gamma=1/|\bar{\beta}|$ as before. Then for the CEV model in* (4.1)*, (S_t/t^γ) satisfies the large deviation principle on $[0,\infty)$ as $t\to\infty$ with continuous rate function*

$$I_{\mathrm{CEV}}(K)=\frac{K^{2|\bar{\beta}|}}{2\delta^2\bar{\beta}^2} \qquad (K\geq 0)\,.$$

Proof. From (4.2), we have the following large-time behaviour for S_t

$$I_{\mathrm{CEV}}(K) \;=\; -\lim_{\epsilon\to 0}\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}(|S_t-Kt^\gamma|<\epsilon) \;=\; \frac{K^{2|\bar{\beta}|}}{2\delta^2\bar{\beta}^2} \qquad (K\geq 0)\,, \tag{5.4}$$

which establishes the *weak* large deviation principle (see e.g. [DZ98] for a definition). We can establish exponential tightness by using the fact that $R_t=\frac{1}{\delta|\beta|}S_t^{-\bar{\beta}}$ is a Bessel process of order ν killed at the origin (see section 4 in Linetsky[Lin04]), and then bounding the tail cdf by the tail cdf for the usual reflecting Bessel process which can be obtained from the transition density given in chapter XI of Revuz&Yor[RY99], to show that for the all $\alpha>0$, there exists an K_α such that $\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(S_t/t^\gamma>K_\alpha)<-\alpha$. □

Corollary 5.7. *We have the following large-time behaviour for the distribution function of S_t*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}(S_t>Kt^\gamma)=I_{\mathrm{CEV}}(K)\,.$$

Proof. The proof just follows from the continuity and monotonicity of $I_{\mathrm{CEV}}(K)$. □

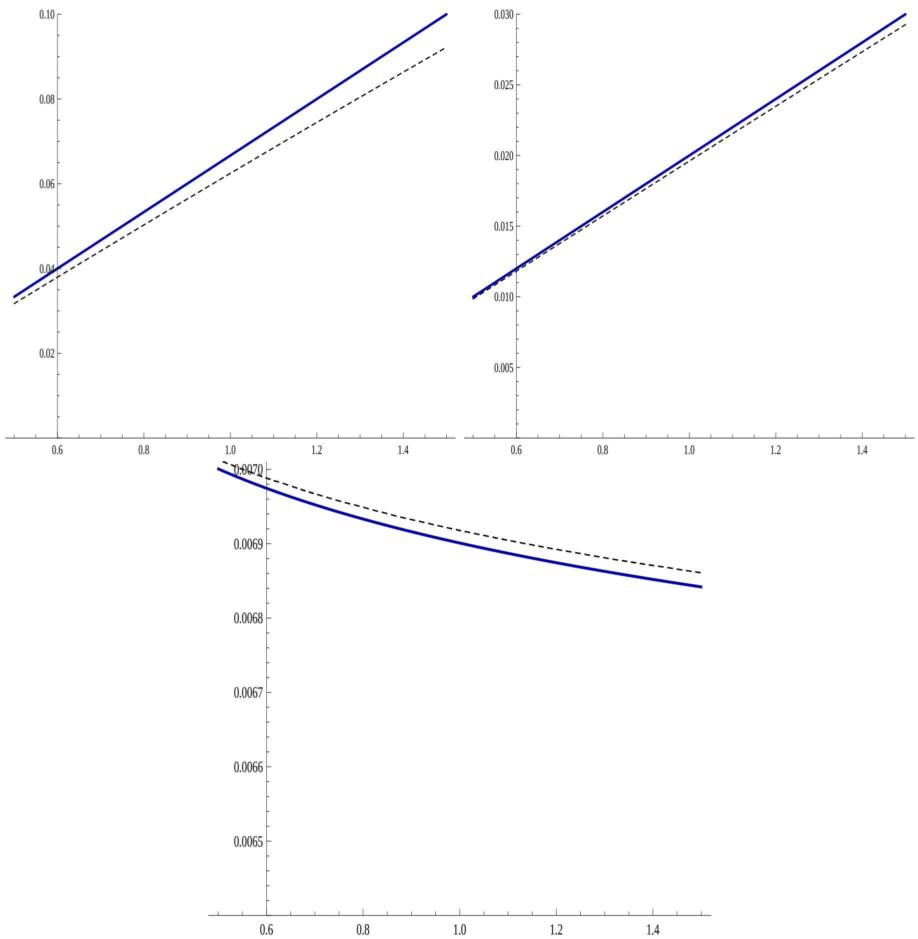

Fig 3. *Here we have plotted the true value of* $S_0 - \mathbb{E}(S_t - K)^+ = \mathbb{E}(S_t \wedge K)$ *as a function of* K *for the CEV model (in grey) against the large-time approximation given in* (5.1) *(blue) for* $t = 30$ *years (left) and* $t = 100$ *years (right), and* $S_0 = 1, \delta = 1, \beta = .5$. *In the third plot, we have plotted the true implied volatility (grey) verse the large-time approximation in* (5.1) *(blue) for* $S_0 = 1, \delta = 1, \beta = .5$ *for* $t = 2000000$. *The approximation does not perform as well as the large-time call approximation for the reason given in Remark 5.5, which is why we have chosen such a large* t*-value here.*

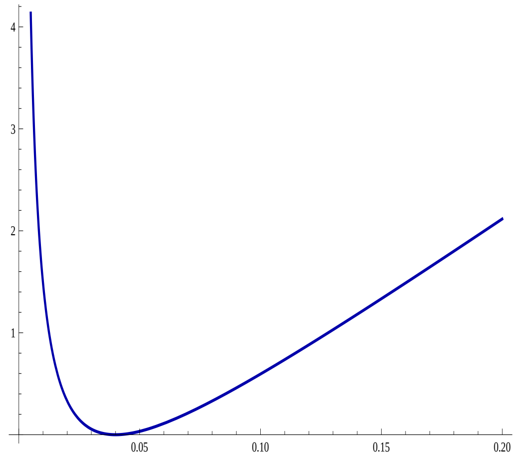

FIG 4. *Here we have plotted the rate function $I_{\mathrm{CIR}}(a)$ using a numerical root finding scheme in Mathematica, for $\kappa = 1.15, \theta = .04, \sigma = 0.2$. I_{CIR} attains its unique minimum of zero at $a = \theta$.*

Remark 5.8. *$Kt^{\gamma} \to \infty$ as $t \to \infty$, so this is a large-time, large-strike regime for the CEV model, similar to the large-time, large-strike regime for the Heston and exponential Lévy models discussed in [FLF11],[FJ11].*

Remark 5.9. *$I_{\mathrm{CEV}}(K)$ is concave for $\beta \in (\frac{1}{2}, 1)$, convex for $\beta \in (0, \frac{1}{2})$ and linear for $\beta = \frac{1}{2}$.*

6. The Cox-Ingersoll-Ross process

Let Y denote a Cox-Ingersoll-Ross (CIR) square root process defined by the stochastic differential equation

$$dY_t = \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t}\, dW_t^2 \tag{6.1}$$

where W_t^2 is standard Brownian motion, $Y_0 = y_0 > 0$, $\kappa, \theta, y_0, \sigma > 0$ and $2\kappa\theta > \sigma^2$, which ensures that $Y = 0$ is an unattainable boundary. The CIR process is used to model the instantaneous volatility process for the well known Heston stochastic volatility model.

The following proposition establishes a large-time large deviation principle and a refined saddle-point density estimate for $A_t = \frac{1}{t}\int_0^t Y_s ds$.

Proposition 6.1. *Consider the CIR process Y in (6.1). Then A_t satisfies a large-time large deviation principle with good rate function given by the Fenchel-Legendre transform of Λ*

$$I_{\mathrm{CIR}}(a) \quad = \quad \sup\{pa - \Lambda(p)\} \quad = \quad \frac{\kappa^2(a-\theta)^2}{2a\sigma^2} \tag{6.2}$$

defined as

$$\Lambda(p) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{p\int_0^t Y_s ds}) \quad = \quad \begin{cases} \frac{\kappa\theta}{\sigma^2}[\kappa - \sqrt{\kappa^2 - 2\sigma^2 p}], & \text{for } p \in (-\infty, p_+] \\ \infty, & \text{for } p \notin (-\infty, p_+], \end{cases}$$

where $p_+ = \frac{\kappa^2}{2\sigma^2}$. Clearly I_{CIR} attains its minimum value of zero at $a = \theta$.

Proof. Just follows from the Gärtner-Ellis theorem from large deviations theory, using a very similar argument to Theorem 2.1 in Forde&Jacquier[FJ11]. ∎

7. The CEV-Heston model

Combining the CEV model with a CIR time-change, we can define the uncorrelated *CEV-Heston model*, governed by the following stochastic differential equations

$$\begin{cases} dS_t = S_t^\beta \sqrt{Y_t} dW_t^1 \,, \\ dY_t = \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t} dW_t^2 \end{cases}$$

with $dW_t^1 dW_t^2 = 0$, $Y_0 = y_0 > 0$. Equivalently, by conditioning on $\int_0^t Y_s ds$ and using the independence of the Brownian motions, we can write $S_t = X_{\int_0^t Y_s ds}$, where X is now just the standard CEV process $dX_t = \delta X_t^\beta \, dW_t$ with $\delta = 1$. The standard Heston model corresponding to $\beta = 1$ is widely used in FX markets. The β paramater can give a market implied estimate of the default probability for a stock (the probability of hitting zero and staying there) which can then be compared with implied probabilities in CDS markets. This has an interesting practical application for trading equities versus credit, particularly in long dated contracts, and is known as capital structure arbitrage.

7.1. The large-time, large-strike regime

We now consider the *large-time, large-strike* regime for the CEV-Heston model, which was previously discussed for the standard CEV model in subsection 5.3.

Proposition 7.1. *$(S_t/t^\gamma, A_t)$ satisfies a joint large deviation principle (LDP) on $[0,\infty) \times (0,\infty)$ as $t \to \infty$, with good rate function*

$$I(K, a) = aI_{\mathrm{CEV}}(\frac{K}{a^\gamma}) + I_{\mathrm{CIR}}(a)$$

with $\delta = 1$.

Proof. See Appendix C. □

From Proposition 7.1 we obtain the following:

Proposition 7.2. *(S_t/t^γ) satisfies the LDP on $[0,\infty)$ as $t \to \infty$ with a good rate function given by*

$$I_{\mathrm{CEVH}}(K) \;=\; \inf_{a\in(0,\infty)} [aI_{\mathrm{CEV}}(\frac{K}{a^\gamma}) \;+\; I_{\mathrm{CIR}}(a)] \;\le\; \theta I_{\mathrm{CEV}}(\frac{K}{\theta^\gamma}) \qquad (K \ge 0)\,,$$

and the infimum of I is attained uniquely at $K = 0$, where $I(K) = 0$.

Proof. The LDP just follows from the contraction principle. Setting $a = \theta$ and using that $I_{\mathrm{CIR}}(\theta) = 0$, we see that $I_{\mathrm{CEVH}}(0) = 0$. Moreover, for any $K > 0$, we cannot find an $a \in (0,\infty)$ which simultaneously makes $aI_{\mathrm{CEV}}(\frac{K}{a^\gamma})$ and $I_{\mathrm{CIR}}(a)$ vanish, so $K = 0$ is the unique minimizer. The upper bound for $I_{\mathrm{CEVH}}(K)$ just follows from setting $a = \theta$. □

Remark 7.3. *It should be possible to translate Propositions 5.6 and 7.2 into asymptotics for call options and implied volatility in the large time, large-strike regime; for the sake of brevity and the fact that this regime is less relevant in practice, we defer the details for future work.*

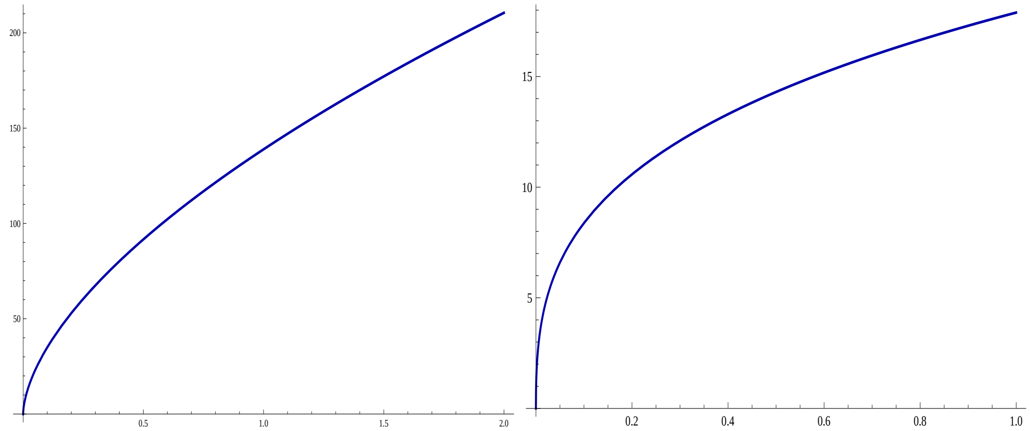

FIG 5. *Here we have plotted the rate function $I_{\mathrm{CEV}}(K)$ for the CEV model in the large-time, large-strike regime for $\delta = .2, \beta = .7, S_0 = 1$ on the left, and the rate function $I_{\mathrm{CEVH}}(K)$ for the CEV-Heston model in the large-time, large-strike regime, for $\delta = 1, \beta = .7, S_0 = 1$ and $\kappa = 1.15, \theta = .04, \sigma = 0.2$ on the right. For both models, I attains its unique minimum of zero at $a = \theta$.*

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Appendix A: Left tail asymptotics for the non-central chi-square distribution

The probability density function of a noncentral chi-square distribution is

$$f(x;u,v)=\frac{1}{2}e^{-(x+v)/2}(\frac{x}{v})^{u/4-\frac{1}{2}}I_{u/2-1}(\sqrt{vx})\,,$$

for $x>0$, and the corresponding distribution function is $1-Q(x;u,v)=\int_0^x f(x;u,v)dx$. From this, using the asymptotic result that $I_\nu(z)\sim\frac{1}{\Gamma(\nu+1)}(\frac{1}{2}z)^\nu$ as $z\to 0$, we have the asymptotic behaviour

$$\begin{aligned}
1-Q(\epsilon;u,\epsilon v) &= \int_0^\epsilon \frac{1}{2}e^{-(x+\epsilon v)/2}(\frac{x}{\epsilon v})^{\frac{1}{4}u-\frac{1}{2}}I_{\frac{1}{2}u-1}(\sqrt{\epsilon v x})dx \\
&\sim \int_0^\epsilon \frac{1}{2}e^{-\frac{1}{2}x}(\frac{x}{\epsilon v})^{\frac{1}{4}u-\frac{1}{2}}\frac{1}{\Gamma(\frac{1}{2}u)}(\frac{1}{2}\sqrt{\epsilon v x})^{\frac{1}{2}u-1}(1+o(1))dx \\
&\sim \frac{(\frac{1}{2})^{\frac{1}{2}u}}{\Gamma(\frac{1}{2}u)}\int_0^\epsilon e^{-\frac{1}{2}x}x^{\frac{1}{2}u-1}dx \\
&\sim \frac{(\frac{1}{2})^{\frac{1}{2}u}}{\Gamma(\frac{1}{2}u)}\int_0^\epsilon (1+o(1))x^{\frac{1}{2}u-1}dx \\
&\sim \frac{1}{\Gamma(\frac{1}{2}u)}\,\frac{1}{\frac{1}{2}u}\,(\frac{1}{2}\epsilon)^{\frac{1}{2}u}=\frac{1}{\Gamma(\frac{1}{2}u+1)}\,(\frac{1}{2}\epsilon)^{\frac{1}{2}u} \qquad \text{(A-1)}
\end{aligned}$$

as $\epsilon\to 0$.

Appendix B: Proof of Corollary 5.3

Let $V=V_t(K)$ denote the implied variance at strike K. Then from Theorem 3 in Tehranchi[Teh09] or Corollary 7.9 in Gao&Lee[GL11], we have

$$|8L-4\log L+4k-4\log\pi-V_t(k)|\;=\;o(1)\,,$$

where $L = \log \frac{1}{C_+}$, where $C_+ = 1 - \frac{1}{S_0}\mathbb{E}(S_t - K)^+ = \mathbb{E}(S_t \wedge K)$.

For the CEV model, $C_+ = \bar{c}Kt^{-\frac{\gamma}{2}}(1+o(1))$ as $t \to \infty$, so $L = -\log[\bar{c}Kt^{-\frac{\gamma}{2}}(1+o(1))] = \frac{\gamma}{2}\log t - \log \bar{c}K + o(1)$, so we have

$$\begin{aligned} V &= 8L - 4\log L + 4\log k - 4\log \pi + o(1) \\ &= 8[\frac{\gamma}{2}\log t - \log \bar{c}K + o(1)] - 4\log[\frac{\gamma}{2}\log t - \log cK + o(1)] + 4k - 4\log\pi + o(1)\,, \\ &= 4\gamma \log t - 8\log \bar{c}K - 4\log[\frac{\gamma}{2}\log t\,(1+o(1))] + 4k - 4\log\pi + o(1)\,, \\ &= 4\gamma\log t - 4\log\log t - 8\log\bar{c}K - 4\log\frac{\gamma}{2} + 4\log\frac{K}{S_0} - 4\log\pi + o(1) \\ &= 4\gamma\log t - 4\log\log t - 8\log\bar{c} - 4\log\frac{\gamma}{2} - 4k - 4\log\pi + o(1)\,. \end{aligned} \tag{A-1}$$

Appendix C: Proof of Proposition 7.1

Let $Z_t = S_t/t^\gamma$. We first note that $(Z_t, A_t) \overset{(\text{law})}{=} (X_{tA_t}/t^\gamma, A_t)$, where X is a standard CEV process $dX_t = \delta\, X_t^\beta\, dW_t^1$. We first note that

$$\begin{aligned} \mathbb{P}(|Z_t - K| < \frac{\delta}{\sqrt{2}}, |A_t - a| < \frac{\delta}{\sqrt{2}}) &\leq \mathbb{P}(\sqrt{|Z_t - K|^2 + |A_t - a|^2} < \delta) \\ &\leq \mathbb{P}(|Z_t - K| < \delta, |A_t - a| < \delta)\,. \end{aligned}$$

By Proposition 5.6, we know that X_t/t^γ satisfies the LDP as $t \to \infty$ with rate function $I_{\text{CEV}}(K)$. From this we see that

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t}\log \mathbb{P}(X_{ta}/t^\gamma > K) &= \lim_{t\to\infty} \frac{a}{u}\log\mathbb{P}(X_u/u^\gamma > \frac{K}{a^\gamma}) \\ &= aI_{\text{CEV}}(\frac{K}{a^\gamma})\,, \end{aligned}$$

and X_{ta}/t^γ satisfies the LDP as $t \to \infty$ with rate function $aI_{\text{CEV}}(\frac{K}{a^\gamma})$.

Thus for any $\epsilon > 0$, conditioning on A_t and using the LDP for A_t and the LDP for X_{ta}/t^γ, there exists a $t = t^*(\epsilon, \delta)$ such that for all $t > t^*$ we have

$$\begin{aligned} \mathbb{P}(|Z_t - K| < \delta, |A_t - a| < \delta) &= \mathbb{P}(|Z_t - K| < \delta \mid |A_t - a| < \delta)\ \mathbb{P}(|A_t - a| < \delta) \\ &\leq \exp\{-t[-\epsilon + \inf_{y\in B_\delta(K)} (a-\delta) I_{\text{CEV}}(\frac{y}{(a+\delta)^\gamma})]\} \\ &\quad \cdot \exp\{-t[-\epsilon + \inf_{a_1 \in \bar{B}_a(\delta)} I_{\text{CIR}}(a_1)]\}\,. \end{aligned} \tag{A-1}$$

From this we obtain

$$\limsup_{t\to\infty} \frac{1}{t}\log\mathbb{P}(|Z_t - K| < \delta, |A_t - a| < \delta) \;\leq\; -\inf_{y\in B_\delta(K)}(a-\delta)I_{\text{CEV}}(\frac{y}{(a+\delta)^\gamma}) - \inf_{a_1\in\bar{B}_a(\delta)} I_{\text{CIR}}(a_1)\,,$$

and by the continuity of $I_{\text{CIR}}(a)$ we obtain

$$\lim_{\delta\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(|Z_t - K| < \delta, |A_t - a| < \delta) \;\leq\; -[aI_{\text{CEV}}(\frac{K}{a^\gamma}) + I_{\text{CIR}}(a)]\,.$$

But $aI_{\text{CEV}}(\frac{K}{a})$ and $I_{\text{CIR}}(a)$ are both good rate functions in K and a respectively, and X_{ta}/t^γ (for a fixed) and A_t both satisfy the full LDP, so for all $\alpha > 0$ there exists $R(\alpha), a_{\min}(\alpha), a_{\max}(\alpha)$ such that

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}((Z_t, A_t) \notin ([0,R]\times[a_{\min}, a_{\max}])^c) \\ \leq\;& \limsup_{t\to\infty}\frac{1}{t}\log\left[\mathbb{P}(Z_t \notin [0,R]) + \mathbb{P}(A_t \in [a_{\min}, a_{\max}])^c\right] \\ \leq\;& \limsup_{t\to\infty}\frac{1}{t}\log 2\big[\,\mathbb{P}(Z_t \notin [0,R]) \vee \mathbb{P}(A_t \in [a_{\min}, a_{\max}])^c\big] \\ \leq\;& -I(R) \wedge (I_{\text{CIR}}(a_{\min}) \wedge I_{\text{CIR}}(a_{\max})) \;\leq\; -\alpha \end{aligned}$$

so (Z_t, A_t) is exponentially tight; hence (Z_t, A_t) satisfies the full LDP and the rate function is good.