The large-time smile and skew for exponential Lévy models

José E. Figueroa-López[*]   Martin Forde[†]   Antoine Jacquier[‡]

Abstract

We derive a full asymptotic expansion for call option prices and a third order approximation for implied volatility in the large-time, large log-moneyness regime for a general exponential Lévy model, by extending the saddlepoint argument used in Forde,Jacquier&Mijatović [*Proc. R. Soc. A*, 466(2124), 3593-3620, 2010] for the Heston model. As for the Heston model, there are two special log-moneyness values where the call option asymptotics are qualitatively different, and we use an Edgeworth expansion to deal with these cases. We also characterize the behaviour of the implied volatility *skew* at large-maturities; in particular we show that the derivative of the dimensionless implied variance with respect to log-moneyness exists and is less than or equal to 4 in the large-maturity limit, which is consistent with the bound on the right and left-side derivative given in Rogers&Tehranchi [*Finance and Stochastics*, 14(2), 235-248, 2010].

1   Introduction

Using the Gärtner-Ellis theorem from large deviations theory, Forde&Jacquier[FJ09] characterized the leading-order behaviour of call prices under the Heston model, in a new regime where the maturity and the log-moneyness are large. Using this result, they derived the implied volatility in the large-time, large-strike limit, and find that the large-time smile mimics the large-time smile for the Barndorff-Nielsen's NIG model. The implied volatility smile does not flatten as the maturity increases, but rather it spreads out, and the new regime is needed to capture this effect. Gatheral&Jacquier[GJ10] proved that the so-called SVI parameterisation is the true limit of the Heston implied volatility smile as the maturity tends to infinity. In this regime, there are *two* special values for the log-moneyness where the call option asymptotics exhibit qualitatively different behaviour from other strike values, and as a special case, we can prove the well-known result by Lewis[Lewis00] for the implied volatility in the usual large-time, *fixed*-strike regime, at leading order. Using similar tools from large deviations theory, Jacquier,Keller-Ressel&Mijatović[JKM11] recently extended the study of the large-strike, large-maturity implied volatility to the general class of affine stochastic volatility models (with jumps). Under mild assumptions, they proved that the limiting smile necessarily corresponds to the smile generated by an exponential Lévy process. [FJM10] use Laplace's method for contour integrals to compute the correction term for the implied volatility in this new regime for the Heston model. The correction term for implied volatility is important because it takes account of the initial level of the instantaneous volatility process as well and allows us to approximate call option prices in the large-time limit in contrast to the crude large deviations bounds in [FJ09].

In the small-maturity limit, the implied volatility of an out-of-the-money call option under an exponential Lévy model tends to infinity (see Roper[Rop10] and Tankov[Tnkv10]); Furthermore, Figueroa-López&Forde[FF11] developed a small-time second-order estimate for the out-of-the-money call option prices and derived an estimate for the dimensionless implied variance of order $O(|\log t|^{-2})$. The latter has recently been sharpened in Gao&Lee[GL11] using the leading order term in the call option expansion in Figueroa-López&Forde[FF11]. The long-term asymptotic behavior of the smile for exponential Lévy models and more general martingale models have been studied in Rogers&Tehranchi[RT10], where it is proved that for fixed log-moneyness k and large maturity, the implied volatility converges to a constant value that does not depend on k. This phenomenon is typically referred as the "smile-flattening" effect, which arises from the large deviation principle for i.i.d. random variables (see e.g. Cramér's theorem in [DeZ98]). For a general exponential Lévy model with mild conditions on the cumulant generating function, [GL11] derived an expansion of the form

---

$$\hat{\sigma}_t(x)^2 \; = \; \sigma_\infty^2 + \frac{a_1(x)}{t} \; + \; \frac{a_2(x)}{t^2} \; + \; o(\frac{(\log t)^2}{t^3}) \qquad\qquad (t \to \infty), \tag{1}$$

for the implied volatility $\hat{\sigma}_t(x)$ at log-moneyness x and maturity t, where $a_1(x)$ and $a_2(x)$ are respectively affine and quadratic in x. This sharpens the result of Tehranchi[Teh09b] who only computed the $a_1(x)$ term.

For the fixed-strike regime, [RT10] also characterized the behaviour of the large-time implied volatility *skew*, i.e. the derivative of the implied volatility with respect to the log-moneyness; in particular they showed that the absolute value of the right and left derivatives of the dimensionless implied variance with respect to the log-moneyness is less than or equal to 4 as the maturity tends to infinity. If the implied volatility is differentiable with respect to the log-moneyness, this result can be easily obtained from the simple no-arbitrage bounds on the slope of the implied volatility at *all* maturities given in section 3.1.1 in [Lee05]. We also refer the reader to Tankov[Tnkv10] for a review of these and other results on asymptotics for implied volatilities in exponential Lévy models.

In the classical paper of Lugannani&Rice[LR80], the authors derive an asymptotic expansion for the probability distribution of the sum of a large number of i.i.d. random variables. Their series take into account the mutual effect of the pole of the integrand at zero and the "principal saddlepoint" on the imaginary axis, and contains an erf (error function) term in the expansion which provides greater accuracy over the classic saddlepoint method for values close to the mean of the distribution. The corresponding saddlepoint expansion for the density, due to Daniels[Dan54], depends only on the saddlepoint. In this article, we derive a full asymptotic expansion for call options and a third order estimate for implied volatility (similar to (1)) for the large-time, large log-moneyness regime under a general exponential Lévy model, by adapting the methods in [FJM10]. Our approach applies the Laplace's method in a similar vein to Lugannani&Rice. As will become evident from our numerical results, the correction terms for the implied volatilities can dramatically improve the rough leading order approximation at large maturities.

Our paper is structured as follows. In Section 2 we state the main asymptotic result (Theorem 2.1) - a large-time expansion for call options in the large-time, large log-moneyness regime. In Section 3, we translate this into a large-time expansion for implied volatility (Corollary 3.2). We also provide numerical results in this section which confirm the accuracy of our asymptotic expansions. The asymptotic behavior of the implied volatility skew is considered in Section 4, while the proofs of the main results are deferred to the appendices.

2 Large-time asymptotics for call option prices

2.1 Preliminary definitions

In this note, we consider an exponential Lévy model $S_t = e^{X_t}$ for the price process of a risky asset. Here $X := (X_t)_{t \geq 0}$ is a Lévy process defined on a complete probability space $(\Omega, \mathbb{P}, \mathcal{F})$ with a Lévy triple (b, σ^2, ν) satisfying $\text{supp}(\nu) \neq \emptyset$ (hence, X has a non-null jump component) and such that $S_t = e^{X_t}$ is a $\mathbb{P}$-martingale relative to its own filtration. The latter property holds true if and only if $\int_{|y|>1} e^y \nu(dy) < \infty$ and the following martingale condition is satisfied

$$b + \frac{1}{2}\sigma^2 + \int_{-\infty}^{\infty} (e^y - 1 - y 1_{|y| \leq 1})\nu(dy) = 0. \tag{2}$$

Here $\mathbb{P}$ represents a risk-neutral pricing measure and, for simplicity, the risk-free interest rate and dividend yield are set to zero. Exponential Lévy models are one of the simplest and most natural generalizations of the classical Black-Scholes model. Among the better known models are the *Variance Gamma model* of [CMC98], the so-called *CGMY model*[1] of [CGMY02], and the *generalized hyperbolic motion* of [BN98],[EK95] (see also [Ebe01]). We refer the reader to Chapter 4 in [CT04] for more details.

Throughout the paper, ψ denotes the characteristic exponent of X defined by $\psi(p) := \log \mathbb{E}e^{ipX_1}$ and given by

$$\psi(p) = ibp - \frac{\sigma^2}{2}p^2 + \int \left(e^{ipy} - 1 - ipy 1_{\{|y| \leq 1\}}\right) \nu(dy),$$

for any $p \in \mathbb{R}$. We will assume that ψ is analytic with a strip of analyticity of the form $\{z \in \mathbb{C} : \text{Im}(z) \in (p_-, p_+)\}$ for some $p_- < 0 < 1 < p_+$ (see Section 7 in [Luk70] for definitions and sufficient conditions). We recall that for a Lévy

---

[1]The CGMY model was also considered by [Kop95] (see also [Nov94]]) under the name of the "truncated Lévy flight", while its application for financial modeling was also proposed in [CBP97] and [Mat00].

process X, $\mathbb{E}e^{pX_t} < \infty$ if and only if $\int_{\{|y|>1\}} e^{py}\nu(dy) < \infty$ and, thus,

$$p_- = \inf\{p < 0 : \int_{-\infty}^{-1} e^{py}\nu(dy) < \infty\}, \quad \text{and} \quad p_+ = \sup\{p > 1 : \int_1^\infty e^{py}\nu(dy) < \infty\}.$$

We also let

$$V(p) := \psi(-ip) = bp + \frac{\sigma^2}{2}p^2 + \int \left(e^{py} - 1 - py 1_{\{|y|\leq 1\}}\right)\nu(dy), \tag{3}$$

for $p \in (p_-, p_+)$. Note that V is strictly convex and V' is strictly increasing since

$$V'(p) = b + \sigma^2 p + \int y(e^{py} - 1_{\{|y|\leq 1\}})\nu(dy) < \infty, \quad V''(p) = \sigma^2 + \int y^2 e^{py}\nu(dy) \in (0,\infty),$$

for any $p \in (p_-, p_+)$. Furthermore, from the martingale property of (S_t), we know that $V(0) = V(1) = 0$ so we have

$$V'(p_-) < V'(0) < 0, \quad \text{and} \quad V'(p_+) > V'(1) > 0,$$

where we had set $V'(p_-) := \lim_{p \searrow p_-} V'(p)$ and $V'(p_+) := \lim_{p \nearrow p_+} V'(p)$. In particular, $V'(p_\pm) = \pm\infty$ when $\lim_{p \to \pm\infty} V'(p) = \pm\infty$. We shall also make use of the following notation:

$$x_- = V'(0) \quad \text{and} \quad x_+ = V'(1). \tag{4}$$

Throughout the paper, $p^*(x)$ denotes the unique number in (p_-, p_+) such that

$$V'(p^*(x)) = x,$$

for $x \in (V'(p_-), V'(p_+))$. Note that $p^*(x)$ is unique because $V''(p) > 0$ for $p \in (p_-, p_+)$. Also, $V^*(x)$ will denote the Legendre transform of $V(p)$ defined by

$$V^*(x) := \sup_{p \in (p_-, p_+)} (px - V(p)). \tag{5}$$

Finally, the formulae below follow directly from the definitions of V and p^*:

$$\text{(i) } V^*(x) = p^*(x)x - V(p^*(x)), \quad \text{(ii) } V'(p^*(x)) = x, \quad \text{(iii) } V^{*'}(x) = p^*(x). \tag{6}$$

2.2 The main asymptotic result

We now consider the asymptotic behavior of call option prices for large time-to-maturity values t and large positive (or negative) log-moneyness values x. Below, $\sim$ means an asymptotic expansion in the sense of page 16 in Olver[Olv74]. Concretely, we will write $f(t) \sim \sum_{n=0}^\infty a_n t^{-n}$ as $t \to \infty$ if for each $N \geq 0$,

$$f(t) - \left\{\sum_{n=0}^{N-1} a_n t^{-n}\right\} = O\left(t^{-N}\right), \qquad (t \to \infty). \tag{7}$$

Theorem 2.1 *For $x \in (V'(p_-), V'(p_+))$, we have the following asymptotic expansion for call options on S_t in the large-time, large log-moneyness regime:*

$$\frac{1}{S_0}\mathbb{E}\left(S_t - S_0 e^{xt}\right)^+ \sim (1 - e^{xt})1_{\{x<x_-\}} + 1_{\{x_-<x<x_+\}} + \frac{1}{2}1_{\{x=x_+\}} + \left(1 - \frac{1}{2}e^{x-t}\right)1_{\{x=x_-\}}$$

$$+ e^{-t(V^*(x)-x)}\frac{1}{\sqrt{2\pi t}}\sum_{n=0}^\infty A_n(x)\frac{1}{t^n}, \qquad (t \to \infty), \tag{8}$$

for some coefficients $A_n(x)$. For $x \notin \{x_-, x_+\}$, the formulae for the first two coefficients are

$$A_0(x) = \frac{1}{(p^{*2}(x) - p^*(x))\sqrt{V^{(2)}(p^*(x))}},$$

$$A_1(x) = \frac{2}{\sqrt{2\pi}}\Gamma(3/2)\left\{2q^{(2)} - \frac{2F^{(3)}q'}{F^{(2)}} + \left[\frac{5}{6}\left(\frac{F^{(3)}}{F^{(2)}}\right)^2 - \frac{F^{(4)}}{2F^{(2)}}\right]q\right\}\frac{1}{(2F^{(2)})^{3/2}}, \tag{9}$$

3

where $F(k) := -ikx - \psi(-k)$, $q(k) := \frac{1}{ik-k^2}$, and all derivatives of F and q are evaluated at $k := ip^*(x)$. For $x \in \{x_-, x_+\}$, we have

$$A_0(x_+) = -\frac{1}{\sqrt{V^{(2)}(1)}} - \theta_3^* \quad , \quad A_1(x_+) = \frac{35}{2}\theta_3^{*3} - 15\theta_3^*\theta_4^* + 3\theta_5^* - B_1(x_+),$$

$$A_0(x_-) = -\frac{1}{\sqrt{V^{(2)}(0)}} + \theta_3 \quad , \quad A_1(x_-) = B_1^*(x_-) - \frac{35}{2}\theta_3^3 + 15\theta_3\theta_4 - 3\theta_5. \tag{10}$$

where $\theta_n = \frac{1}{n!}\frac{V^{(n)}(0)}{[V^{(2)}(0)]^{\frac{1}{2}n}}$, $\theta_n^* = \frac{1}{n!}\frac{V^{(n)}(1)}{[V^{(2)}(1)]^{\frac{1}{2}n}}$, and B_1, B_1^* are defined as in Lemmas B.1 and B.2 of Appendix B.

Remark 2.1 It is easy to express (9) in terms of the real-valued function V given in (3). Indeed, we have that

$$A_1(x) = \frac{2}{\sqrt{2\pi}}\Gamma(3/2)\left\{-2\xi^{(2)} + \frac{2V^{(3)}\xi'}{V^{(2)}} + \left[-\frac{5}{6}\left(\frac{V^{(3)}}{V^{(2)}}\right)^2 + \frac{V^{(4)}}{2V^{(2)}}\right]\xi\right\}\frac{1}{(2V^{(2)})^{3/2}}, \tag{11}$$

where $\xi(p) := \frac{1}{p^2-p}$ and all derivatives of V and ξ are evaluated at $p^*(x)$.

Remark 2.2 A similar expansion to (8) has been independently obtained in the recent manuscript by Gao&Lee[GL11] (see Lemma 8.4 therein), but they do not consider the special cases $x \notin \{x_-, x_+\}$. [GL11] also impose a monotonicity condition on V along the approximate horizontal contour of steepest descent in Eqs. 8.5-8.7; it turns out that this condition is superfluous for our choice of Lévy process, because the condition is automatically satisfied, as shown in Lemma A.1 in the proof of Theorem 2.1.

Remark 2.3 In the case when $V'(p_-)$ or $V'(p_+)$ is finite and $x \notin (V'(p_-), V'(p_+))$ for $x \in \mathbb{R}$, we cannot find a real solution to the saddlepoint equation $V'(p^*) = x$. In this case, we can still use Cramér's theorem from large deviations theory to show that $(X_t/t)_{t\geq 0}$ satisfies a large deviation principle as $t \to \infty$ with rate function $V^*(x) = \sup_{p \in (p_-, p_+)}\{px - V(p)\}$ (we defer the details for future work). Also, note that $|A_0(x)| \to \infty$ as $x \to x_+$ and $x \to x_-$ because $p^*(x_+) = 1$, $p^*(x_-) = 0$ and $V''(p) \in (0, \infty)$ on (p_-, p_+), so in both cases the $\frac{1}{p^{*2}-p^*}$ term in $A_0(x)$ explodes.

Example 2.1 *We now proceed to evaluate some of the above quantities for the CGMY model in [CGMY02]. This process is a pure-jump Lévy process with Lévy measure*

$$\nu(dx) = \left(\frac{Ce^{-G|x|}}{|x|^{1+Y}}1_{\{x<0\}} + \frac{Ce^{-Mx}}{x^{1+Y}}1_{\{x>0\}}\right)dx, \tag{12}$$

for $C, G, M > 0$ and $0 < Y < 2$. Here we exclude the special case $Y = 1$ for simplicity. The characteristic exponent is given by

$$\psi(u) = C\Gamma(-Y)\left\{(M-iu)^Y + (G+iu)^Y - M^Y - G^Y\right\} + ibu. \tag{13}$$

for $Y \neq 1$ (see Section 4.5 in Cont&Tankov[CT04]). The martingale condition (2) implies that $M > 1$ and

$$b := -C\Gamma(-Y)\left\{(M-1)^Y + (G+1)^Y - M^Y - G^Y\right\},$$

so that $V(0) = V(1) = 0$. We first note that $p_+ = M$, $p_- = -G$. For $Y \in (0,1)$, we have $V'(p_\pm) \to \pm\infty$ as $p \to p_\pm$, while for $Y \in (1,2)$, we have

$$V'(p_-) = b - C\Gamma(-Y)Y(M+G)^{Y-1}, \quad V'(p_+) = b + C\Gamma(-Y)Y(M+G)^{Y-1}.$$

In both cases $p^*(x)$ has to be found numerically from the equation $(G+p^*(x))^{Y-1} - (M-p^*(x))^{Y-1} = (x-b)/C\Gamma(-Y)Y$, for $x \in (V'(p_-), V'(p_+))$.

3 Large-time asymptotics for implied volatility

3.1 The large-time, large log-moneyness regime

We first prove the following lemma which we use in the corollary that follows. The lemma characterizes the large-time behaviour of the Black-Scholes call option formula with a time-dependent volatility function. Below, $C^{\text{BS}}(S_0, K, \sigma, \tau)$ denotes the usual Black-Scholes call option formula with initial stock price S_0, strike K, volatility σ, maturity τ, and zero interest rates.

Lemma 3.1 *Let*

$$\tilde{\sigma}_t^2 := \sigma^2 + \frac{a_1}{t} + \frac{a_2}{t^2}\,, \tag{14}$$

for some constants $a_1, a_2 \in \mathbb{R}$ and t large enough so that $\tilde{\sigma}_t^2 > 0$. Then we have the following large-time behaviour for the Black-Scholes call option formula with time-dependent volatility $\tilde{\sigma}_t$:

$$
\begin{aligned}
\frac{1}{S_0} C^{\mathrm{BS}}(S_0, S_0 e^{xt}, \tilde{\sigma}_t, t) &= (1 - e^{xt}) 1_{\{x < -\frac{\sigma^2}{2}\}} + 1_{\{-\frac{\sigma^2}{2} < x < \frac{\sigma^2}{2}\}} + \frac{1}{2} \cdot 1_{\{x = \frac{\sigma^2}{2}\}} + (1 - \frac{1}{2} e^{-\frac{1}{2}\sigma^2 t}) 1_{\{x = -\frac{\sigma^2}{2}\}} \\
&+ e^{-t(V_{\mathrm{BS}}^*(x,\sigma) - x)} \frac{1}{\sqrt{2\pi t}} M_1 \left[A_0^{\mathrm{BS}}(x, \sigma) + A_1^{\mathrm{BS}}(x, \sigma, a_1, a_2) \frac{1}{t} + O(\frac{1}{t^2}) \right],
\end{aligned} \tag{15}
$$

as $t \to \infty$, where $V_{\mathrm{BS}}^*(x, \sigma) = \frac{(x + \frac{1}{2}\sigma^2)^2}{2\sigma^2}$ and, for $x \neq \pm\frac{1}{2}\sigma^2$, we have

$$A_0^{\mathrm{BS}}(x, \sigma) = \frac{\sigma^3}{x^2 - \frac{1}{4}\sigma^4}\,, \qquad A_1^{\mathrm{BS}}(x, \sigma, a_1, a_2) = -\frac{1}{2\sigma^3} \frac{\gamma_1 + \gamma_2 a_2}{(4x^2 - \sigma^4)^3}\,, \tag{16}$$

and $M_1 = M_1(x, \sigma, a_1) = \exp(\frac{a_1}{2\sigma A_0^{\mathrm{BS}}(x,\sigma)})$, $\gamma_2 = -\sigma^2(4x^2 - \sigma^4)^3$, $\gamma_1 = 4a_1(4x^2 - \sigma^4)(4a_1 x^4 - x^2 \sigma^4(a_1 + 12) - \sigma^8) + 32\sigma^{12} + 384\sigma^8 x^2$. For $x = \pm\frac{1}{2}\sigma^2$, we have $M_1 = 1$ and

$$A_0^{\mathrm{BS}}(x, \sigma) = \frac{1}{\sigma}\left(\frac{1}{2}a_1 - 1\right), \qquad A_1^{\mathrm{BS}}(x, \sigma, a_1, a_2) = -\frac{1}{48\sigma^3}(6a_1^2 - 48 - 24a_2\sigma^2 + a_1^3)\,. \tag{17}$$

Proof. We just substitute $\tilde{\sigma}_t$ into the Black-Scholes formula, and use the asymptotic result

$$\Phi^c(z) \sim \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \left[\frac{1}{z} - \frac{1}{z^3} + O(\frac{1}{z^5}) \right], \qquad (z \to +\infty)\,,$$

where $\Phi(z) = \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$ (see e.g. Olver[Olv74]). ∎

Now let $\hat{\sigma}_t(x)$ denote the implied volatility at log-moneyness x and time-to-maturity t, defined in the usual way as the unique solution to $C^{\mathrm{BS}}(S_0, S_0 e^x, \hat{\sigma}_t(x), t) = \mathbb{E}(S_t - S_0 e^x)^+$. The following corollary gives a third order approximation for the implied volatility in the large-time, large log-moneynesss regime:

Corollary 3.2 *For $x \notin \{x_-, x_+\}$ we have the following expansion for the implied volatility in the large-time, large log-moneyness regime:*

$$\hat{\sigma}_t(xt)^2 = \sigma(x)^2 + \frac{a_1(x)}{t} + \frac{a_2(x)}{t^2} + o(\frac{1}{t^2})\,, \tag{18}$$

where

$$
\sigma^2(x) = \begin{cases}
2\left[2V^*(x) - x - 2\sqrt{V^*(x)^2 - V^*(x)x}\right] & (x > x_+ \text{ or } x < x_-)\,, \\
2\left[2V^*(x) - x + 2\sqrt{V^*(x)^2 - V^*(x)x}\right] & (x \in (x_-, x_+))\,,
\end{cases} \tag{19}
$$

$$a_1(x) = 2\sigma(x) A_0^{\mathrm{BS}}(x, \sigma(x)) \log \frac{A_0(x)}{A_0^{\mathrm{BS}}(x, \sigma(x))}\,,$$

$$a_2(x) = \frac{1}{\gamma_2}\left[-2A_1(x)\sigma(x)^3 \frac{(4x^2 - \sigma(x)^4)^3}{M_1(x, \sigma(x), a_1(x))} - \gamma_1 \right]\,,$$

and A_0 is defined as in Theorem 2.1, while γ_1, γ_2 are defined as in Lemma 3.1 setting $\sigma = \sigma(x), a_1 = a_1(x)$. For $x \in \{x_-, x_+\}$ we have

$$a_1(x) = 2\left[1 - \frac{\sigma(x)}{\sqrt{V^{(2)}(p^*)}}\left(1 + \mathrm{sgn}(x)\frac{V^{(3)}(p^*)}{6V^{(2)}(p^*)}\right)\right]\,, \tag{20}$$

$$a_2(x) = -\frac{1}{24\sigma^2(x)}\left[-48\sigma^3(x)A_1(x) - 6a_1(x)^2 + 48 - a_1(x)^3\right]\,. \tag{21}$$

Proof. We give the proof for the case $x > x_-$, the other cases follow similarly. We first *assume* that the implied volatility admits an expansion of the form (18), and we then prove this rigorously by establishing upper and lower

bounds for the implied volatility. If we formally equate call prices under the exponential Lévy model and the Black-Scholes model with this maturity-dependent implied volatility expansion we obtain

$$e^{-t(V^*(x)-x))}\frac{1}{\sqrt{2\pi t}}[A_0(x)+A_1(x)\frac{1}{t}+O(\frac{1}{t^2})] = e^{-t(V_{\mathrm{BS}}^*(x,\sigma)-x))}\frac{1}{\sqrt{2\pi t}}M_1(x,\sigma,a_1)[A_0^{\mathrm{BS}}(x,\sigma)+A_1^{\mathrm{BS}}(x,\sigma,a_1,a_2)\frac{1}{t}+O(\frac{1}{t^2})].$$

Taking logs of both sides and cancelling terms and dividing by t, we have

$$-(V^*(x)-x)+\log A_0(x)\frac{1}{t}+\frac{A_1(x)}{A_0(x)}\frac{1}{t^2} = -(V_{\mathrm{BS}}^*(x,\sigma)-x)+\log[M_1(x,\sigma,a_1)A_0^{\mathrm{BS}}(x,\sigma)]\frac{1}{t}+\frac{A_1^{\mathrm{BS}}(x,\sigma,a_1,a_2)}{A_0^{\mathrm{BS}}(x,\sigma)}\frac{1}{t^2}+O(\frac{1}{t^3}).$$

We then just match coefficients (recall the definitions of $A_0^{\mathrm{BS}}(x,\sigma)$ and $A_1^{\mathrm{BS}}(x,\sigma,a_1,a_2)$ which depend on σ,a_1,a_2) and then solve in turn for σ,a_1,a_2. To establish tight bounds for $\hat{\sigma}_t(xt)$, consider $\delta>0$ and let

$$\tilde{\sigma}_{t,\delta}^2(x) = \sigma(x)^2 + \frac{a_1(x)}{t} + \frac{a_2(x)+\delta}{t^2}. \tag{22}$$

Combining Theorem 2.1 and Lemma 3.1, cancelling the coefficients that we have equated and noting the affine dependence of $A_1^{\mathrm{BS}}(x,\sigma,a_1,a_2)$ on a_2, we have

$$\log\frac{\mathbb{E}(S_t-S_0e^{xt})^+}{C^{\mathrm{BS}}(S_0,S_0e^{xt},\tilde{\sigma}_{t,\delta}(x),t)} \leq \frac{1}{A_0^{\mathrm{BS}}(x,\sigma(x))}[A_1^{\mathrm{BS}}(x,\sigma(x),a_1(x),a_2(x)+\delta) - A_1^{\mathrm{BS}}(x,\sigma(x),a_1(x),a_2(x))]\frac{1}{t}$$

$$= \frac{\delta}{2\sigma(x)A_0^{\mathrm{BS}}(x,\sigma(x))}\frac{1}{t} \tag{23}$$

for $t=t(\delta)$ sufficiently large. Proceeding similarly for the upper bound, and using the strict monotonicity of C^{BS} in the volatility argument, we have

$$\tilde{\sigma}_{t,-\delta}^2(x) \leq \hat{\sigma}_t(xt) \leq \tilde{\sigma}_{t,\delta}^2(x),$$

for t sufficiently large. This completes the proof since δ is arbitrary. ∎

3.2 The large-time, fixed log-moneyness regime

In this subsection, we briefly discuss how to obtain a second order estimate for the implied volatility in the large-time, *fixed* log-moneyness regime. A higher order estimate for implied volatility in this regime has been quoted in the recent preprint by [GL11], but their arguments are more involved, so we give a short self contained proof of the second order approximation here, which is all we need in the next section to characterize the volatility skew. To this end, we define

$$p_0 = p^*(0); \tag{24}$$

i.e. p_0 is the unique solution to $V'(p_0)=0$. Since $V''(p)>0$, for $p\in(p_-,p_+)$, and $V(0)=V(1)=0$, we have that $p_0\in(0,1)$. The following result in this subsection. A similar result has been obtained in [Teh09b] in a more general framework. We however provide here an explicit representation for the coefficients of the expansion.

Proposition 3.3 *For $x\in\mathbb{R}$, we have the following expansion for the implied volatility in the large-time,* fixed *log-moneyness regime*

$$\hat{\sigma}_t^2(x) = \sigma_\infty^2 + a(x)\frac{1}{t} + o(\frac{1}{t}). \tag{25}$$

where $\sigma_\infty^2 := \sigma^2(0) = 8V^*(0)$, $a(x) = -8[\log(\frac{1}{4}A_0\sigma_\infty)+(\frac{1}{2}-p_0)x]$ and $A_0=|A_0(0)|$.

Proof. By a similar argument to the proof of Theorem 2.1 we can prove the following large-time estimate for call options of fixed-strike

$$\mathbb{E}(S_t\wedge K) = S_0 - \mathbb{E}(S_t-K)^+ = \frac{A_0}{\sqrt{2\pi t}}e^{(1-p_0)x}e^{-V^*(0)t}[1+O(\frac{1}{t})] \qquad (t\to\infty). \tag{26}$$

Then, by Theorem 3.1 in [Teh09b], we have

$$\hat{\sigma}_t(x)^2 = -8\log\mathbb{E}(S_t\wedge K) - 4\log[-\log\mathbb{E}(S_t\wedge K)] + 4x - 4\log\pi + o(1)$$

$$= 8V^*(0)t - 8\log A_0 + 8(p_0-1)x + 4\log(2\pi t) - 4\log[\frac{1}{8}\sigma_\infty^2 t\,(1+O(\frac{1}{t}))] + 4x - 4\log\pi + o(1)$$

$$= 8V^*(0)t + 4\log 2 + 4\log 8 - 8\log A_0 - 8\log\sigma_\infty + (\frac{1}{2}-p_0)x + o(1). \tag{27}$$

∎

3.3 Numerics

The second and third order approximations for the implied volatility at large-maturities given in (18) appear to be very accurate even for relatively low time-to-maturity values. To illustrate this point, Figure 3.3 shows the first three approximations for the CGMY model with a time-to-maturity $T = 1.1$ years. The choice of parameters was motivated by the seminal paper of [CGMY02], where the CGMY model was calibrated using 2009 option data on Microsoft stock. The figure also illustrates the values of the second and third order approximations, computed using (20) and (21), at the two special x-values: $x_+ = 0.0518911$ and $x_- = -0.053822$.
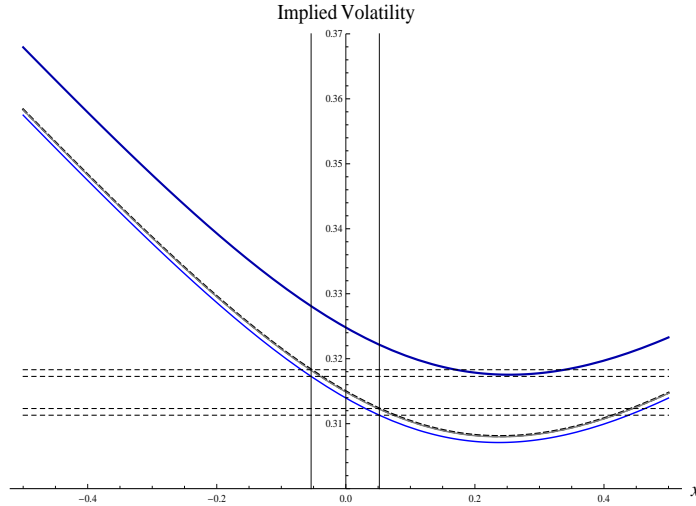

Figure 1: Here we have plotted the asymptotic implied volatility smile $\sigma(x)$ (dark blue), the second order approximation $[\sigma^2(x) + a_1(x)/t]^{\frac{1}{2}}$ (lighter blue), the third order approximation $[\sigma^2(x) + \frac{a_1(x)}{t} + \frac{a_2(x)}{t^2}]^{\frac{1}{2}}$ (black dashed) for the large-strike, large log-moneyness regime all taken from (18), and the smile obtained numerically using the inverse Fourier transform in (A.1) (grey) along the approximate horizontal contour of steepest descent $k_r + ip^*(x)$ for $k_r \in \mathbb{R}$, for the large-time, large log-moneyness regime under the CGMY model with the same parameters as in [CGMY02]: $C = 1.1, Y = 0.4456, M = 8.6, G = 5.09$ and $T = 1.1$ years. The two vertical lines correspond to the two special cases $x_+ = 0.0518911$ and $x_- = -0.053822$, and the horizontal dashed lines are the values of the second and third order approximations at the two special x-values, computed using (20) and (21).

4 Large-time asymptotics for the implied volatility skew

We now consider the asymptotic behavior of the skew $\frac{\partial \hat{\sigma}_t(x)}{\partial x}$ of the implied volatility $\hat{\sigma}_t(x)$:

Proposition 4.1 *We have the following large-time behaviour for the implied volatility skew $\frac{\partial \hat{\sigma}_t(x)}{\partial x}$ for all $x \in \mathbb{R}$:*

$$\lim_{t \to \infty} \frac{\partial}{\partial x}[\hat{\sigma}_t(x)^2 t] = a'(0) = 8(p_0 - \frac{1}{2}) \tag{28}$$

where $a(x)$ and p_0 are defined in Proposition (3.3) and (24), respectively.

Remark 4.1 $p_0 \in (0, 1)$, so $|\frac{\partial}{\partial x}[\hat{\sigma}_t(x)^2 t]| \leq 4$ as $t \to \infty$, which is consistent with the general bounds given in Theorem 5.1 in Rogers&Tehranchi[RT10]. Note that $\frac{\partial}{\partial x}\hat{\sigma}_t(x)^2 \sim a'(0)\frac{1}{t}$ as $t \to \infty$, so we recover the well-known fact that the skew flattens as the maturity tends to infinity. Note that (28) is also what we obtain from formal differentiation of (25) with respect to x, even though here we are interchanging taking limits in x and t.

Proof. The existence of the derivative $\frac{\partial}{\partial x}[\hat{\sigma}_t^2(x)]$ follows from Appendix C. Now let $K = S_0 e^x$ and $\sigma_\infty = \sigma(0)$ with σ as in (19). By a similar argument we have the following large-time behaviour for digital call options

$$\mathbb{P}(S_t > K) = \mathbb{P}(X_t > x) \sim \frac{e^{-p_0 x} e^{-V^*(0)t}}{p_0 \sqrt{2\pi V''(p_0)t}} \qquad (t \to \infty)$$

7

For the Black-Scholes model with volatility σ, $p_0 = \frac{1}{2}$ so we have

$$\mathbb{P}_\sigma^{\text{BS}}(S_t > K) \sim \frac{e^{-\frac{1}{2}x}e^{-\frac{1}{8}\sigma^2 t}}{\frac{1}{2}\sigma\sqrt{2\pi t}}, \qquad (t \to \infty),$$

where $\mathbb{P}_\sigma^{\text{BS}}(S_t > K) = \mathbb{P}(\sigma W_t - \sigma^2 t/2 > x)$; i.e., the probability of the event $\{S_t > K\}$ when the stock price process (S_t) follows the Black-Scholes model with zero interest rate, volatility σ, and log-moneyness x. Replacing σ by the asymptotic expansion in (25), we obtain

$$\mathbb{P}_{\sqrt{\sigma_\infty^2 + a(x)\frac{1}{t} + o(\frac{1}{t})}}^{\text{BS}}(S_t > K) \sim \frac{e^{-\frac{1}{2}x}e^{-\frac{1}{8}[\sigma_\infty^2 t + a(x)]}}{\frac{1}{2}\sigma_\infty\sqrt{2\pi t}}, \qquad (t \to \infty). \qquad (29)$$

Similarly, evaluating the Black-Scholes Vega at $\sigma = \sqrt{\sigma_\infty^2 + a(x)\frac{1}{t} + o(\frac{1}{t})}$ we have

$$\frac{\partial C^{BS}}{\partial \sigma} \sim S_0 \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}\sqrt{t} = S_0 e^{\frac{1}{2}x - \frac{1}{8}a(x)}e^{-\frac{1}{8}\sigma_\infty^2 t}\frac{1}{\sqrt{2\pi}}\sqrt{t}$$

where $d_1 = \frac{-x + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}$. Using that $\sigma_\infty = 8V^*(0)$ and (C.2), the exponentially small terms cancel and we have

$$\frac{\partial \hat{\sigma}_t(x)}{\partial x} = S_0 e^x \frac{\mathbb{P}^{\text{BS}}(S_t > K) - \mathbb{P}(S_t > K)}{\frac{\partial C^{BS}}{\partial \sigma}} \sim \left[\frac{e^{-\frac{1}{2}x}e^{-\frac{1}{8}a(x)}}{\frac{1}{2}\sigma_\infty} - \frac{e^{-p_0 x}}{p_0\sqrt{V''(p_0)}}\right]e^{\frac{1}{2}x + \frac{1}{8}a(x)}\frac{1}{t} \qquad (t \to \infty)$$

$$= \frac{1}{\sigma_\infty}\left[2 - \frac{4}{A_0 p_0 \sqrt{V''(p_0)}}\right]\frac{1}{t}. \qquad (30)$$

Noting that $V^*(0) = -V(p_0)$ and plugging in the definition of $A_0 = |A_0(0)|$ from (9), we have

$$\frac{1}{\sigma_\infty}\left[2 - \frac{4}{A_0 p_0\sqrt{V''(p_0)}}\right] = \frac{2}{\sigma_\infty}(2p_0 - 1), \qquad (31)$$

and the result follows by noting that $\frac{\partial}{\partial x}[\hat{\sigma}_t(x)^2 t] = 2\hat{\sigma}_t(x)\frac{\partial}{\partial x}\hat{\sigma}_t(x)t$ and $\hat{\sigma}_t(x) \to \sigma_\infty$ as $t \to \infty$ by Proposition 3.3. ∎

References

[AS72] Abramowitz, M. and I.Stegun (1972), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York: Dover Publications.

[BN98] Barndorff-Nielsen, O. (1998), "Processes of normal inverse Gaussian type", *Finance and Stochastics* **2**, 41–68.

[CM09] Carr, P., and D. Madan (2009), "Saddlepoint methods for option pricing", *The Journal of Computational Finance*, 13(1), 49-61.

[CMC98] Carr, P., Madan, D. & Chang, E. (1998), "The variance Gamma process and option pricing", *European Finance Review* **2**, 79–105.

[CGMY02] Carr, P., H. Geman, D. Madan, and M. Yor (2002), "The fine structure of asset returns: An empirical investigation", *Journal of Business*, 75, 303-325.

[CBP97] Cont, R., J. Bouchaud, and M. Potters (1997), "Scaling in financial data: Stable laws and beyond" *in Scale Invariance and Beyond, Dubrulle, B., Graner, F., and Sornette, D., eds.*.

[CT04] Cont, R. and P. Tankov (2004), "Financial modelling with Jump Processes", Chapman & Hall.

[Dan54] Daniels, H.E., (1954) "Saddlepoint approximations in statistics", *Ann. Math. Statist.*, 25, 631-650.

[DeZ98] Dembo, A. and O.Zeitouni (1998), "Large deviations techniques and applications", Jones and Bartlet publishers, Boston.

[EK95] Eberlein, E. & Keller, U. (1995), "Hyperbolic Distribution in Finance", *Bernoulli* **1**, 281–299.

[Ebe01] Eberlein, E. (2001), "Application of Generalized Hyperbolic Lévy Motions to Finance", In *Lévy Processes. Theory and Applications, by O.E. Barndorff-Nielsen, T. Mikosch, and S.I. Resnick, Eds.*, Birkhauser, 319-336.

[FF11] Figueroa-López, J.E., and M. Forde, M. (2011), "The small-maturity smile for exponential Lévy models", forthcoming in *SIAM Journal on Financial Mathematics*.

[FJ09] Forde, M., and A. Jacquier (2009), "The large maturity smile for the Heston model", forthcoming in *Finance and Stochastics*.

[FJM10] Forde, M., A.Jacquier and A.Mijatović (2010), "Asymptotic formulae for implied volatility in the Heston model", *Proceedings of the Royal Society A*, Volume 466, no. 2124, pp. 3593-3620.

[GJ10] Gatheral, J. and Jacquier, A. (2011), "Convergence of Heston to SVI", *Quantitative Finance 11 (8): 1129-1132*.

[GL11] K.Gao and Lee, R. (2011), "Asymptotics of Implied Volatility to Arbitrary Order", preprint, http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1768383.

[JKM11] Jacquier.A., M.Keller-Ressel and A. Mijatović (2011), "Large deviations and stochastic volatility with jumps: asymptotic implied volatility for affine models". Available at ArXiv 1108.3998.

[Kop95] Koponen, I. (1995), "Analytic approach to the problem of convergence of truncated Lévy flights toward the Gaussian stochastic process", *Phys. Rev. E*, 52:1197–1199.

[Lee04] Lee, R. (2004), "Option Pricing by Transform Methods: Extensions, Unification, and Error Control", *Journal of Computational Finance*, 7(3):51-86.

[Lee05] Lee, R. (2005), "Implied Volatility: Statics, Dynamics, and Probabilistic Interpretation", in *Recent Advances in Applied Probability*, Springer.

[Lewis00] Lewis, A. (2000), "Option valuation under stochastic volatility", Finance Press.

[LR80] Lugannani, R. and S.O.Rice (1980), "Saddlepoint approximations for the distribution of the sum of independent random variables", *Advances in Applied Probability* 12, 475-490.

[Luk70] Lukacs, E. (1970), "Characteristic Functions", *Charles Griffin & Company Limited*, London.

[Mat00] Matacz, A. (2000), "Financial modeling and option theory with the truncated Lévy process", *Int. J. Theor. Appl. Finance*, 3:143–160.

[Nov94)] Novikov, E.A. (1994), "Infinitely divisible distributions in turbulence", *Phys. Rev. E*, 50:3303–R3305.

[Olv74] Olver, F.W. (1974), "Asymptotics and Special Functions", Academic Press.

[RT10] L.C.G.Rogers and M.Tehranchi (2010), "Can the implied volatility surface move by parallel shifts?", *Finance and Stochastics*, 14(2) pp.235-248.

[Rop10] Roper, M. (2009), "Implied volatility: small time to expiry asymptotics in exponential Lévy models", Thesis, University of New South Wales.

[Sat99] Sato, K. (1999), "Lévy Processes and Infinitely Divisible Distributions", Cambridge studies in advance mathematics.

[Tnkv10] Tankov, P. (2010), "Pricing and hedging in exponential Lévy models: review of recent results". Paris-Princeton Lecture Notes in Mathematical Finance, Springer.

[Teh09a] Tehranchi, M. (2009), "Implied volatility: long maturity behavior", *Encyclopedia of Quantitative Finance*, Wiley, 2010.

[Teh09b] Tehranchi, M. (2009), "Asymptotics of implied volatility far from maturity", *Journal of Applied Probability*, 46 (3): 629–650.

A   Proof of Theorem 2.1 for the case $x \notin \{x_-, x_+\}$

From [Lee04] (see also, [FJM10]), we have the following Fourier representation for the price of a call option of log-moneyness xt:

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0 e^{xt})^+ = R_\alpha + \frac{e^{xt}}{\pi}\int_{0+i(\alpha+1)}^{+\infty+i(\alpha+1)} \mathrm{Re}[e^{ikxt}\frac{e^{t\psi(-k)}}{ik-k^2}]dk, \tag{A.1}$$

for $\alpha + 1 \in (p_-, p_+)$, where $R_\alpha = 1_{\{-1<\alpha<0\}} + (1-e^{xt})1_{\{\alpha<-1\}} + \frac{1}{2}1_{\{\alpha=0\}} + (1-\frac{e^{xt}}{2})1_{\{\alpha=-1\}}$. We now break the proof into four parts for clarity:

1. Computing the saddlepoint

Recall that, for each $x \in (V'(p_-), V'(p_+))$, $p^*(x) \in (p_-, p_+)$ is defined so that $V'(p^*(x)) = x$. We now define

$$F(k) = -ikx - \psi(-k). \tag{A.2}$$

Then, $F(k)$ has a saddlepoint at $k^*(x) = ip^*(x)$ since $F'(k) = -ix + \psi'(-k^*(x)) = -i(x - V'(p^*(x))) = 0$ (see, e.g, Definition 5.2 in [FJM10] for the definition of saddle point).

2. Re-writing the indicator functions

Setting $\alpha + 1 = p^*(x)$, we can re-write (A.1) as

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0 e^{xt})^+ = R_{p^*(x)-1} + \frac{e^{xt}}{\pi}\int_{0+ip^*(x)}^{+\infty+ip^*(x)} \mathrm{Re}[\frac{e^{-F(k)t}}{ik-k^2}]dk, \tag{A.3}$$

We now show that we have the following simplified expression for $R_{p^*(x)-1}$ (which we shall need later):

$$R_{p^*(x)-1} = (1-e^{xt})1_{\{x<x_-\}} + 1_{\{x_-<x<x_+\}} + \frac{1}{2}1_{\{x=x_+\}} + (1-\frac{1}{2}e^{x-t})1_{\{x=x_-\}}. \tag{A.4}$$

From the definitions of x_-, x_+ in (4) and the property (6)-(iii), we have $V^{*'}(x_+) = 1$ and $V^{*'}(x_-) = 0$. But $V'(x)$ is strictly increasing, so $p^*(x)$ is strictly increasing and, hence, $p^*(x_+, \infty) = (1, \infty)$, $p^*(x_-, x_+) = (0, 1)$, $p^*(-\infty, x_-) = (-\infty, 0)$ and, also, $p^*(x_+) = 1$ and $p^*(x_-) = 0$. Now since $\alpha + 1 = p^*(x)$, we have $\{-1 < \alpha < 0\} = \{x_- < x < x_+\}, \{\alpha < -1\} = \{x < x_-\}, \{\alpha = 0\} = \{x = x_+\}, \{\alpha = -1\} = \{x = x_-\}, \{\alpha > 0\} = \{x > x_+\}$, and the expression (A.4) follows.

3. Strict minimal point for $\mathrm{Re}(F(y + ip^*(x)))$

Lemma A.1 *We have the following property for F along the horizontal contour $y + ip^*(x)$ for $y \in \mathbb{R}\backslash\{0\}$:*

$$\mathrm{Re}(F(y + ip^*(x))) > \mathrm{Re}(F(ip^*(x))). \tag{A.5}$$

Proof. Recall that

$$F(k) = -ikx - \psi(-k) = -ikx - V(-ik) = -ikx + ibk + \frac{\sigma^2}{2}k^2 - \int(e^{-ikz} - 1 + ikz1_{|z|\leq1})\nu(dz),$$

From this we see that

$$\mathrm{Re}(F(y + ip^*(x))) = xp^*(x) - bp^*(x) - \frac{\sigma^2}{2}p^*(x) + \frac{\sigma^2}{2}y^2 - \int\{e^{zp^*(x)}\cos(zy) - 1 - zp^*(x)1_{|z|\leq1}\}\nu(dz). \tag{A.6}$$

Given that the first four terms on the right hand side of (A.6) are all strictly monotone increasing in $y \in [0, \infty)$, we only have to worry about the last term in (A.6). This term can be split into the following two finite integrals

$$\int\{1-\cos(zy)\}e^{zp^*(x)}\nu(dz) - \int\{e^{zp^*(x)} - 1 - zp^*(x)1_{|z|\leq1}\}\nu(dz). \tag{A.7}$$

The second term above does not depend on y and since $\{1 - \cos(zy)\}e^{zp^*(x)} \geq \{1 - \cos(zy)\}e^{zp^*(x)}|_{y=0} = 0$, the first term in (A.7) is always non-negative. Furthermore, it is strictly positive unless $\{1 - \cos(zy)\}e^{zp^*(x)} = 0$ for ν-a.e. z. This contradicts our assumption that $\mathrm{supp}(\nu) \neq \emptyset$. ∎

4. Applying Laplace's method along the horizontal contour

Evaluating the exponent $F(k) = -ikx - \psi(-k)$ at the saddlepoint $k^*(x) = ip^*(x)$, we have the important property that

$$F(k^*(x)) = p^*(x)x - V(p^*(x)) = V^*(x).$$

Also, note that $F(k)$ is analytic in the strip $\text{Im}(k) \in (p_-, p_+)$ and, from (A.1), we know that $\text{Re}(F(y + ip^*(x)) - F(ip^*(x))) > 0$ for $y \neq 0$ and $\text{Re}(F(y + ip^*(x)) - F(ip^*(x)))$ is bounded away from zero as $y \to \pm\infty$. Finally, using Laplace's method for contour integrals along the horizontal contour going through the saddlepoint $k^*(x) = ip^*(x)$, as in Theorem 7.1 of chapter 4 in Olver[Olv74], we obtain the result.

B   Proof of Theorem 2.1 for the case $x \in \{x_-, x_+\}$

Following the density transformation construction of Sato[Sat99] (see Definition 33.4 and Example 33.4 therein) and using the martingale condition (2), we define a probability measure $\mathbb{P}^*$ such that $\mathbb{P}^*(B) = \mathbb{E}\left(e^{X_t} 1_B\right)$, for any $t > 0$ and $B \in \mathcal{F}_t$. $\mathbb{P}^*$ is sometimes called the Share measure (see, e.g., Carr&Madan[CM09]). One can readily check that (X_t) is a Lévy process under $\mathbb{P}^*$ with characteristic triplet (b^*, σ^2, ν^*), where

$$\nu^*(dx) = e^x \nu(dx) \quad \text{and} \quad b^* = b + \int_{|x| \leq 1} x \left(e^x - 1\right) \nu(dx) + \sigma^2. \tag{B.1}$$

(see [FF11] for further details). Then,

$$V_S(p) := \psi_S(-ip) := \log \mathbb{E}^{\mathbb{P}^*}\left(e^{pX_1}\right) = \log \mathbb{E}(e^{X_1} e^{pX_1}) = V(p+1), \tag{B.2}$$

for $p \in (p_- - 1, p_+ - 1)$, and

$$V_S(0) = V(1) < \infty. \tag{B.3}$$

To compute the Fenchel-Legendre transform $V_S^*(x)$ of $V_S(p)$, we have to solve for the unique $p_S^*(x)$ such that $V_S'(p_S^*(x)) = V'(p_S^*(x) + 1) = x$. From this we see that $p_S^*(x) = p^*(x) - 1$, and

$$V_S^*(x) = p_S^*(x)x - V_S(p_S^*(x)) = p^*x - V(p) - x = V^*(x) - x. \tag{B.4}$$

We now give expansions for the tail probabilities of (X_t) under the pricing measure $\mathbb{P}$ and the Share measure $\mathbb{P}^*$. For the sake of brevity, we only present the proof for the Share measure case below. The proof for the case of $\mathbb{P}$ is similar.

Lemma B.1 *Let F be as in (A.2). Then, for $x \neq x_-$, we have the following asymptotic behaviour for X_t:*

$$\mathbb{P}(X_t > xt) \sim 1_{\{x < x_-\}} + e^{-tV^*(x)} \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^{\infty} B_n(x) \frac{1}{t^n} \qquad (t \to \infty) \tag{B.5}$$

for some coefficients $B_n(x)$, where $\sim$ here means asymptotic expansion as in (7). The formulae for the first two coefficients are

$$B_0(x) = \frac{1}{p^* \sqrt{V^{(2)}(p^*)}}, \qquad B_1(x) = \frac{2}{\sqrt{2\pi}} \Gamma(3/2) \left\{ 2\tilde{q}^{(2)} - \frac{2F^{(3)}\tilde{q}'}{F^{(2)}} + \left(\frac{5(F^{(3)})^2}{6(F^{(2)})^2} - \frac{F^{(4)}}{2F^{(2)}} \right) \tilde{q} \right\} \frac{1}{(2F^{(2)})^{3/2}}, \tag{B.6}$$

where $p^* = p^*(x) > 0$, $\tilde{q}(k) = -\frac{1}{ik}$ and all derivatives of F and $\tilde{q}$ are evaluated at $k = ip^*(x)$.

Lemma B.2 *Assume the same notation as in Proposition B.1. Then, for $x \neq x_+$, we have the following asymptotic behaviour for X_t under $\mathbb{P}^*$:*

$$\mathbb{P}^*(X_t > xt) \sim 1_{\{x < x_+\}} + e^{-t(V^*(x) - x)} \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^{\infty} B_n^*(x) \frac{1}{t^n}, \qquad (t \to \infty), \tag{B.7}$$

for some coefficients $B_n^*(x)$. The formulae for the first two coefficients are

$$B_0^*(x) = \frac{1}{p_S^* \sqrt{V^{(2)}(p^*)}}, \qquad B_1^*(x) = \frac{2}{\sqrt{2\pi}} \Gamma(3/2) \left\{ 2\tilde{q}^{(2)} - \frac{2F^{(3)}\tilde{q}'}{F^{(2)}} + \left(\frac{5(F^{(3)})^2}{6(F^{(2)})^2} - \frac{F^{(4)}}{2F^{(2)}} \right) \tilde{q} \right\} \frac{1}{(2F^{(2)})^{3/2}}, \tag{B.8}$$

where $p_S^* = p_S^*(x) = p^*(x) - 1$, $\tilde{q}(k) = -\frac{1}{ik}$, all derivatives of F are evaluated at $k = ip^*(x)$, and all the derivatives of $\tilde{q}$ are evaluated at $k = ip_S^*(x)$.

Proof. We recall the following well-known formula for the tail probability:

$$\mathbb{P}^*(X_t > xt) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikxt} \frac{e^{t\psi_S(k)}}{ik} dk,$$

where $\psi_S(k) := \log \mathbb{E}^{\mathbb{P}^*}(e^{ikX_1}) = \log \mathbb{E}(e^{(1+ik)X_1}) = \psi(k-i)$. Then, applying similar arguments to those leading to (9) as in Appendix A, we have the following asymptotic behaviour for X_t under $\mathbb{P}^*$ for $x \neq x_+$

$$\mathbb{P}^*(X_t > xt) \sim \mathbf{1}_{\{x < x_+\}} + e^{-t(V^*(x)-x)} \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^{\infty} \tilde{B}_n^*(x) \frac{1}{t^n}, \qquad (t \to \infty), \tag{B.9}$$

for some coefficients $\tilde{B}_n^*(x)$, where $\tilde{B}_0^*(x) = \frac{1}{p_S^* \sqrt{V_S''(p_S^*)}}$ and

$$\tilde{B}_1^*(x) = \frac{2}{\sqrt{2\pi}} \Gamma(3/2) \left\{ 2\tilde{q}^{(2)} - \frac{2F_S^{(3)}\tilde{q}'}{F_S^{(2)}} + \left(\frac{5(F_S^{(3)})^2}{6(F_S^{(2)})^2} - \frac{F_S^{(4)}}{2F_S^{(2)}} \right) \tilde{q} \right\} \frac{1}{(2F_S^{(2)})^{3/2}}, \tag{B.10}$$

where $p_S^*(x) = p^*(x) - 1$, $F_S(k) = -ikx - \psi_S(-k)$ where all derivatives of F_S and $\tilde{q}$ are evaluated at $k = ip_S^*(x)$. $V_S''(p) = V''(p+1)$ so

$$V_S''(p_S^*(x)) = V_S''(p^*(x) - 1) = V''(p^*(x))$$

and $F_S''(k) = -\psi''(-k - i)$ which implies that

$$F_S''(ip_S^*(x)) = -\psi''(-ip_S^*(x) - i) = -\psi''(-i(p^*(x) - 1) - i) = F''(ip^*(x)).$$

From this we see that $\tilde{B}_1^*(x)$ in (B.10) reduces to the expression $B_1^*(x)$ in (B.8). Finally we use the identity in (B.4) to re-write the exponent in terms of $V^*(x)$. ∎

We now give the main proof for the case $x = x_+$. From page 479 in [LR80] or 26.2.48 in [AS72] with $\bar{y} = x_+$, $\phi = V_S$ and $N = t$, we have

$$\mathbb{P}^*(X_t > x_+ t) = \frac{1}{2} + \frac{1}{\sqrt{2\pi t}} \left[-\theta_3^* + \left(\frac{35}{2}\theta_3^{*3} - 15\theta_3^*\theta_4^* + 3\theta_5^* \right) \frac{1}{t} + O\left(\frac{1}{t^2}\right) \right] \tag{B.11}$$

where $\theta_n^* = \frac{1}{n!} \frac{V_S^{(n)}(0)}{[V_S''(0)]^{\frac{1}{2}n}} = \frac{1}{n!} \frac{V^{(n)}(1)}{[V''(1)]^{\frac{1}{2}n}}$. Combining (B.11) with Lemma B.1, we have

$$\begin{aligned}
\frac{1}{S_0}\mathbb{E}(S_t - S_0 e^{x_+ t})^+ &= \mathbb{P}^*(X_t > x_+ t) - e^{x_+ t}\mathbb{P}(X_t > x_+ t) \\
&= \frac{1}{2} + \frac{1}{\sqrt{2\pi t}} \left[-\theta_3^* + \left(\frac{35}{2}\theta_3^{*3} - 15\theta_3^*\theta_4^* + 3\theta_5^* \right) \frac{1}{t} + O\left(\frac{1}{t^2}\right) \right] \\
&\quad - e^{x_+ t} e^{-(p^* x_+ - V(p^*))t} \left[B_0(x_+) + B_1(x_+)\frac{1}{t} + O\left(\frac{1}{t^2}\right) \right] \\
&= \frac{1}{2} + \frac{1}{\sqrt{2\pi t}} \left[-\theta_3^* + \left(\frac{35}{2}\theta_3^{*3} - 15\theta_3^*\theta_4^* + 3\theta_5^* \right) \frac{1}{t} + O\left(\frac{1}{t^2}\right) \right] - \left[B_0(x_+) + B_1(x_+)\frac{1}{t} + O\left(\frac{1}{t^2}\right) \right] \\
&= \frac{1}{2} - \frac{1}{\sqrt{t}} \left[\frac{1}{\sqrt{2\pi V''(1)}} \left(1 + \frac{1}{6}\frac{V'''(1)}{V''(1)}\right) - \left(\frac{35}{2}\theta_3^{*3} - 15\theta_3^*\theta_4^* + 3\theta_5^* - B_1(x_+) \right) \frac{1}{t} + O\left(\frac{1}{t^2}\right) \right].
\end{aligned} \tag{B.12}$$

Similarly, for $x = x_-$ and fixing $\theta_n = \frac{1}{n!} \frac{V^{(n)}(0)}{[V''(0)]^{\frac{1}{2}n}}$, we have

$$\begin{aligned}
\frac{1}{S_0}\mathbb{E}(S_t - S_0 e^{x_- t})^+ &= \mathbb{P}^*(X_t > x_- t) - e^{x_- t}\mathbb{P}(X_t > x_- t) \\
&= 1 + \frac{e^{-V_S^*(x_-)t}}{\sqrt{2\pi t}} \left(B_0^*(x_-) + B_1^*(x)\frac{1}{t} + O\left(\frac{1}{t^2}\right) \right) \\
&\quad - e^{x_- t} \left[\frac{1}{2} + \frac{1}{\sqrt{2\pi t}} \left[-\theta_3 + \left(\frac{35}{2}\theta_3^3 - 15\theta_3\theta_4 + 3\theta_5 \right) \frac{1}{t} + O\left(\frac{1}{t^2}\right) \right] \right] \\
&= 1 + \frac{1}{\sqrt{2\pi t}} e^{x_- t} \left[(B_0^*(x_-) + \theta_3) + \left(B_1^*(x_-) - \frac{35}{2}\theta_3^3 + 15\theta_3\theta_4 - 3\theta_5 \right) \frac{1}{t} + O\left(\frac{1}{t^2}\right) \right].
\end{aligned} \tag{B.13}$$

This concludes the proof.

C Differentiability of call option prices and implied volatility

The following Lemma shows that the skew is well-defined whenever the log-return X_t is continuous (i.e. $\mathbb{P}(X_t = x) = 0$, for all x) for each t. In the case of Lévy processes, it is known that X_t is continuous if and only if $\sigma \neq 0$ or $\nu(\mathbb{R}\backslash\{0\}) = \infty$ (see Theorem 27.4 in [Sat99]).

Lemma C.1 *If the log-return X_t is continuous for any $t > 0$, then the skew $\frac{\partial \hat{\sigma}_t(x)}{\partial x}$ is well-defined.*

Proof. We first note that the Black-Scholes call price $C^{BS}(t, x, \sigma) := \mathbb{E}(e^{\sigma W_t - \frac{1}{2}\sigma^2} - e^x)^+$ is continuously differentiable in σ and x for each fixed $t > 0$, and the Vega

$$\mathcal{V} = \frac{\partial C^{BS}(t, x, \sigma)}{\partial \sigma} = \sqrt{\frac{t}{2\pi}} \exp\left\{-(\sigma^2 t/2 - x)^2/2\sigma^2 t\right\},$$

is strictly positive. Hence, for $\frac{\partial \hat{\sigma}_t(x)}{\partial x}$ to exist, it suffices to show that $C(t, x) = \mathbb{E}\left(e^{X_t} - e^x\right)^+$ is continuously differentiable in x (so that $F(t, x, \sigma) := C^{BS}(t, x, \sigma) - C(t, x)$ will be continuously differentiable in x and σ and the implicit function theorem can be applied). To check the differentiability of C, note that for fixed $h > 0$,

$$\frac{1}{h}\{C(t, x + h) - C(t, x)\} = -\frac{e^x}{h}\left(e^h - 1\right)\mathbb{P}(X_t \geq x + h) - \mathbb{E}\left\{1_{\{x < X_t < x+h\}}\frac{1}{h}\left(e^{X_t} - e^x\right)\right\}.$$

Given that $1_{\{x < X_t < x+h\}}\frac{1}{h}\left(e^{X_t} - e^x\right) \leq (e^{x+h} - e^x)/h = 1 + O(h)$, we can apply the dominated convergence theorem and obtain that

$$\frac{\partial C(t, x)}{\partial x} = -e^x \mathbb{P}(X_t \geq x), \tag{C.1}$$

which is continuous when $\mathbb{P}(X_t = x) = 0$, for all x. ∎

Remark C.1 As a corollary of the proof above, we recover the following formula for pricing digital call options

$$e^x \mathbb{P}\left(X_t \geq x\right) = e^x \mathbb{P}\left(\sigma_t(x)W_t - \frac{1}{2}\sigma_t(x)^2 \geq x\right) - \mathcal{V}\frac{\partial \hat{\sigma}_t(x)}{\partial x}, \tag{C.2}$$

by implicitly differentiating $F(t, x) = C^{BS}(t, x, \sigma_t(x)) - C(t, x)$.