Optimal trade execution for Gaussian signals with power-law resilience

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Abstract

We characterize the optimal signal-adaptive liquidation strategy for an agent subject to power-law resilience and zero temporary price impact with a Gaussian signal, which can include e.g an OU process or fractional Brownian motion. We show that the optimal selling speed u_t^* is a Gaussian Volterra process of the form $u^*(t) = u^0(t)+\bar{u}(t)+\int_0^t k(u,t)dW_u$ on $[0,T)$, where $k(.,.)$ and $\bar{u}$ satisfy a family of (linear) Fredholm integral equations of the first kind which can be solved in terms of fractional derivatives. $u^0(t)$ is the (deterministic) solution for the no-signal case given in Gatheral et al.[GSS12], and we give an explicit formula for $k(u,t)$ for the case of a Riemann-Liouville price process as a canonical example of a rough signal. With non-zero linear temporary price impact, the integral equation for $k(u,t)$ becomes a Fredholm equation of the second kind, and in this case we also outline how to compute $k(u,t)$ for the case of exponential resilience. These results build on the earlier work of Gatheral et al.[GSS12] for the no-signal case, and complement the recent work of Neuman&Voß[NV22]. Finally we show how to re-express the trading speed in terms of the price history using a new inversion formula for Gaussian Volterra processes of the form $\int_0^t g(t-s)dW_s$, and we calibrate the model to high frequency limit order book data for various NASDAQ stocks.[1]

1 Introduction

A critical problem for algorithmic traders is how to optimally split a large trade so as to minimize trading costs and market impact. The seminal article of Almgren&Chriss[AC01] formulates this problem as trade-off between expected execution cost and risk; more specifically, they assume the stock price is a martingale and execution costs are linear in the trading rate and the choice of risk criterion is variance. Under these assumptions, there is a well known closed-form analytical solution for the optimal selling speed which is deterministic.

More recently, authors have begun to relax the martingale assumption of Almgren-Chriss to incorporate the effect of signals. In particular, [CJ16] provide empirical evidence of the impact of order flow on NASDAQ stocks, and propose a model of order flow for an investor who executes a large order when market order-flow from all agents, including the investor's own trades, has a permanent price impact (see also section 7.3 in [CJP15]). [CJ16] derive a closed-form solution for the optimal strategy where the rate of trading depends on the expectation of future order flow. [CDJ18] show that volume imbalance is an effective predictor of the sign of future market orders, and how trading signals arising from order flow can be used to execute large orders and make markets. More recently [KLA20] and [CPS20] use signals as inputs to the signature of the market to devise trading algorithms.

For the case of zero signal with a general impact function G, the optimal trading strategy is deterministic and satisfies $\int_0^T G(|t-v|)dX_v = \lambda$, which is a Fredholm integral equation of the first kind. The constant λ has to be chosen so as to enforce the liqudation condition $X_T = 0$, and [GSS12] prove existence in this case if G is non-constant, non-decreasing, convex and integrable at zero. The Fredholm equation can be solved explicitly for the case of exponential and power law impact. For the former, the solution is well known from [OW13] and consists of a block (i.e. an impulse response) sell trade at time zero and at the final maturity, with continuous selling in between proportional to the resilience parameter ρ (see also example 2.12 in [GSS12]). For the case of power law impact, the integral equation reduces to the well known Abel integral equation which also has an explicit solution which is U-shaped and symmetric, c.f. section 2.2 in [CGL17]. The Fredholm equation becomes a weakly singular Urysohn equation of the first kind if the temporary price impact component is non-linear, i.e. the price paid per unit stock is $S_t + \int_0^t G(t-s)f(\dot{X}_s)dt$ for some non-linear impact function f, and X is assumed to be absolutely continuous (see [Dan14] and [CGL17] for more on this, and numerical schemes for solving such non-linear integral equations).

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[BMO20] derive the optimal trading strategy for a linear price impact model with a partial liquidation penalty of the form ΓX_T^2 for $\Gamma > 0$, when the stock price is a general unspecified semimartingale. Using a similar variational argument to [BSV17], they show that $(X_t, \dot{X}_t)$ satisfies a coupled linear FBSDE, which can be re-written in a matrix form and solved explicitly using the same trick that is used to compute the solution for a standard OU process. The [BMO20] argument can be very easily adapted to deal with the infinite penalty case $\Gamma = \infty$ by simply replacing the vector $(\frac{\Gamma}{\lambda}, -1)$ with $(1, 0)$, but one would need to verify admissibility of the solution.

More recently, [NV22] consider the problem of optimal trade execution under exponential resilience i.e. $G(t-s) = const. \times e^{-\rho(t-s)}$, with a general square integrable semi-martingale price process and (i) non-zero temporary price impact and a (ii) finite quadratic penalty for non-liquidation. The solution is shown to satisfy a system of four coupled linear FBSDEs (in X_t, u_t, $Y_t = \int_0^t e^{-\rho(t-s)} u_s ds$ and an auxiliary process Z_t) which can solved explicitly in terms of the matrix exponential function using similar arguments to [BMO20], and they find that the optimal selling speed (in feedback form) is affine-linear in the current inventory X_t and Y_t. The approach in [NV22] exploits the semimartingale structure of the price process and the fact that for the special case of exponential resilience, $dY_t = -\rho Y_t dt + u_t dt$ and thus is a Markov (OU) process driven by u. One can easily adapt the [NV22] solution to deal with the full liquidation case by just dividing the $(\varrho/\lambda, -\kappa/(2\lambda), -1, 0)$ vector in their Eq 3.3 by a factor ϱ (see also Eq 5.14 in [NV22] to see why we can do this) and then letting $\varrho \to \infty$ so the G vector in their Eq 3.3 is replaced with $(1, 0, 0, 0)S(t)$ (with the $S(t) = e^{At}$ matrix unchanged), but one would still need to check admissibility of the proposed optimal strategy for this $\varrho = \infty$ case.

Lorenz&Schied[LS13] show that for exponential resilience with zero temporary price impact and semimartingale price process, optimal trading strategies $(X_t)_{t \in [0,T]}$ with bounded variation do not exist in general, hence one has to enlarge the space of admissible strategies to the class of all semimartingales, which includes processes with non-zero quadratic variation. In this setting, Theorem 2.6 in [LS13] computes the optimal X_t (with the surprising result that if the drift is not absolutely continuous then the expected profit/loss is infinite, although such trading strategies with infinite variation will of course incur infinite transaction costs in the real world). For the well behaved case when the drift is absolutely continuous, they give an explicit formula for X_t which includes martingale terms, which minimizes the modified cost functional in Lemma 2.5 in [LS13] which involves quadratic variation terms. Moreover, the process X is Gaussian if the stock price process is Gaussian. Theorem 2.6 in [LS13] extends the classical [OW13] solution for the no-signal case (see above).

In this article we compute an explicit solution for the optimal signal-adaptive liquidation strategy for a trader subject to power-law resilience and a Gaussian signal with zero temporary price impact, which is obtained as the solution to a Forward-Backward Stochastic Integral Equation (FBSIE). The natural choice for the admissible space of strategies turns out to be intimately related to the Fractional Gaussian Field (FGF) with covariance equal to G which lives in the space of tempered distributions, and the optimal trading speed is a Gaussian Volterra process of the form $u^0(t) + \bar{u}(t) + \int_0^t k(u,t) dW_t$, where $u^0(t)$ is the (deterministic) solution for the non-signal case and k satisfies a family of Fredholm integral equations of the first kind (and $\bar{u}(t)$ also satisfies a single Fredholm equation of the first kind) all of which can be solved explicitly using the known solution given in e.g. [CG94], or more symbolically in terms of the adjoint of the square root of the linear operator associated with G. This generalizes the earlier work of Gatheral et al.[GSS12] for the no-signal case, and complements the recent work of Neuman et al.[NV22] and has the advantage over [NV22] that we impose the full liquidation constraint $X_T = 0$. We also outline how to adapt our arguments for the case of exponential resilience with non-zero temporary impact, and we look at the unconstrained problem with zero resilience with and without transaction costs (for the latter we show the existence of a *no-trade region* for the optimal policy).

The layout of the article is as follows: Section 2.1 derives the first order optimality condition for a general signal ξ_t, Section 2.2 contains the main Theorem 2.2 which specializes Section 2.1 to the case of Gaussian signals, Section 2.3 recalls the known solution for the special case of zero signal, and section 2.4 computes the expected P&L for Theorem 2.2. Section 3.1 gives the most important/interesting special case to consider for Theorem 2.2 (namely the Riemann-Liouville process) with numerical simulations, section 3.2 makes a minor addition to the setup in Section 2.2 with the addition of the usual temporary price impact term, Section 3.3 re-writes the optimal solution in Theorem 2.2 in a more natural/practical way in terms of the observable price process itself, Section 3.4 shows in principle how one can compute G from the trading history for Theorem 2.2, Section 3.5 considers the same problem but with exponential not power resilience as in [NV22] but with full liquidation constraint, Section 3.6 looks at the unconstrained version of the problem in Section 2.2 and Section 3.7 adds transactions costs to this unconstrained problem. Finally Section 4 calibrates the original model to real limit order book data for Apple, Cisco and Vodafone stocks using a discretized version of the model with difference equations.

2 The model setup

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ throughout, with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which satisfies the usual conditions, and $\mathbb{E}_t(.)$ will denote $\mathbb{E}(.|\mathcal{F}_t)$. We consider an agent subject to transient price impact where the execution price for an asset at time t is

$$S_t \;=\; P_t + \int_0^t G(t-s)dX_s\,, \tag{1}$$

where $X_t = X_0 - \int_0^t u_s ds$ is the number of shares held at time t, which we assume is absolutely continuous in t so u_t is the *selling speed*, and P is some $\mathcal{F}_t$-progressively measurable process P with $\mathbb{E}(P_t^2) < \infty$ for all $t \in [0,T]$ (which we refer to as the *unaffected* price process). $\int_0^t G(t-s)dX_s$ represents the cumulative effect of our trading activities on the current stock price, and G is the *decay kernel*, which characterizes resilience of price impact between trades.

From here on we assume that $G(t) = ct^{-\gamma}$ **for** $\gamma \in (0,1)$ **for some constant** $c > 0$.
We set

$$\xi_t \;:=\; \mathbb{E}_t(P_T - P_t)\,.$$

Then a natural criterion is to maximize the agent's expected profit/loss at T:

$$\begin{aligned} V(u) &= \mathbb{E}\Big(\int_0^T (P_t - \int_0^t G(t-s)u_s ds)u_t dt \,+\, P_T X_T\Big) \\ &= \mathbb{E}\Big(\int_0^T (P_t - \int_0^t G(t-s)u_s ds)u_t dt \,+\, P_T(X_0 - \int_0^T u_t dt)\Big) \\ &= \mathbb{E}(P_T X_0) \,+\, \mathbb{E}\Big(\int_0^T (P_t - P_T - \int_0^t G(t-s)u_s ds)u_t dt\Big) \end{aligned}$$

over $\mathcal{U}_0^{X_0}$, where $\mathcal{U}_0^x$ denote the space of $\mathcal{F}_t$-progressively measurable processes u such that $X_T = x - \int_0^T u_t dt = 0$ (i.e. we must liquidate all inventory by time T) such that $\mathbb{E}(\int_0^T |u_t(P_t - P_T)|dt) < \infty$ and $\mathbb{E}(\int_0^T \int_0^t |G(t-s)u_s u_t| ds dt) < \infty$.

One can in principle add additional penalty terms to our performance criterion (the most common being a quadratic inventory penalty of the form *const.* $\times \int_0^T X_t^2 dt$ to penalize large positions before T) but our optimal solution is already rather complicated to compute, so we leave the details of this for future works. We also remind the reader that since we are imposing full liquidation, we implicitly already have an infinite penalty here for non-liquidation.

Remark 2.1 From Fubini's theorem, we know that $u \in \mathcal{U}_0^x$ also implies that $\int_0^T \mathbb{E}(|u_t(P_t - P_T)|)dsdt < \infty$ and $\int_0^T \int_0^t \mathbb{E}(|G(t-s)u_s u_t|)dsdt < \infty$.

From Fubini's theorem and the definition of $\mathcal{U}_0^{X_0}$, we can re-write $V(u)$ as

$$\begin{aligned} V(u) &= \mathbb{E}(P_T X_0) + \int_0^T \mathbb{E}((P_t - P_T)u_t)dt \,-\, \mathbb{E}\Big(\int_0^t G(t-s)u_s ds u_t dt\Big) \\ &= \mathbb{E}(P_T X_0) - \int_0^T \mathbb{E}(u_t \xi_t)dt \,-\, \mathbb{E}\Big(\int_0^t G(t-s)u_s ds\, u_t dt\Big) \end{aligned}$$

(using the optional projection for the second term, see also [BSV17] and [NV22] for similar arguments)

$$= X_0 \mathbb{E}(P_T) \,-\, \mathbb{E}\Big(\int_0^T (\xi_t \,+\, \int_0^t G(t-s)u_s ds)u_t dt\Big)\,, \tag{2}$$

where we have used Fubini again in the final line, since

$$\begin{aligned} \int_0^T \mathbb{E}(|u_t \xi_t|)dt &= \int_0^T \mathbb{E}(|u_t \mathbb{E}_t(P_T - P_t)|)dt \;=\; \int_0^T \mathbb{E}(|\mathbb{E}_t(u_t(P_T - P_t))|)dt \\ &\qquad \text{(by conditional Jensen)} \\ &\leq \int_0^T \mathbb{E}(\mathbb{E}_t(|u_t(P_T - P_t)|))dt \\ &= \int_0^T \mathbb{E}(|u_t(P_T - P_t)|)dt\,, \end{aligned}$$

which is finite for $u \in \mathcal{U}_0^{X_0}$ (see Remark 2.1). Since $X_0 \mathbb{E}(P_T)$ is independent of u, for convenience we henceforth work with the modified functional:

$$\tilde{V}(u) \quad = \quad -\mathbb{E}(\int_0^T (\xi_t + \int_0^t G(t-s) u_s ds) u_t dt) \,. \tag{3}$$

Note that we do not assume that S is a semimartingale (as is usually assumed in the literature).

2.1 The first order condition for the optimizer

We now establish the first order optimality condition for an optimal trading strategy using variational and convexity arguments, similar to section 5 in [BSV17].

Theorem 2.1 *A sufficient condition for $u \in \mathcal{U}_0^{X_0}$ to be an optimal trading strategy is that u satisfies the Forward-Backward Stochastic Integral equation (FBSIE):*

$$\xi_t + \mathbb{E}_t(\int_0^T G(|t-v|) u_v dv) \quad = \quad M_t \qquad \text{a.s.} \tag{4}$$

for $t \in [0,T]$ for some martingale M such that $X_T = 0$.

$$0 + \int_0^T G(|T-v|) u_v dv \quad = \quad M_T \qquad \text{a.s.,}$$

Remark 2.2 Note that (4) by itself does not uniquely determine the optimal u, we need the additional terminal condition $X_T = 0$ as well (see e.g. Lemma 5.2 ii) in [BSV17] and Eq 3.5 in [BMO20] for qualitatively similar results for different problems).

Proof. Let $\mathcal{L} = \{u \in \mathcal{A} : \langle u, u \rangle_G < \infty\}$, where $\langle u, v \rangle_G := \mathbb{E}(\int_0^T u_t \int_0^T v_s G(|t-s|) ds dt)$ and $\mathcal{A}$ is the space of $\mathcal{F}_t$-progressively measurable processes.

Perturbing u to $u + \varepsilon u^1$ with $u^1 \in \mathcal{U}_0^0$ (i.e. a round trip so $\int_0^T u_t^1 dt = 0$) we find that

$$\begin{aligned}
\tilde{V}(u + \varepsilon u^1) &= -\mathbb{E}(\int_0^T (\xi_t + \int_0^t (u_s + \varepsilon u_s^1) G(t-s) ds)\, (u_t + \varepsilon u_t^1) dt) \\
&= \tilde{V}(u) - \varepsilon \mathbb{E}\big(\int_0^T \xi_t u_t^1 dt + \int_0^T u_t^1 \int_0^t u_s G(t-s) ds dt + \int_0^T u_t \int_0^t u_s^1 G(t-s) ds dt)\big) \\
&- \varepsilon^2 \mathbb{E}(\int_0^T u_t^1 \int_0^t u_s^1 G(t-s) ds dt) \\
&= \tilde{V}(u) + \tilde{V}(\varepsilon u^1) - \varepsilon \mathbb{E}(\int_0^T u_t^1 \int_0^t u_s G(t-s) ds dt) - \varepsilon \mathbb{E}(\int_0^T u_t \int_0^t u_s^1 G(t-s) ds dt) \\
&= \tilde{V}(u) + \tilde{V}(\varepsilon u^1) - \varepsilon \mathbb{E}(\int_0^T u_t^1 \int_0^t u_s G(t-s) ds dt) - \varepsilon \mathbb{E}(\int_0^T u_s \int_0^s u_t^1 G(s-t) ds dt) \\
&= \tilde{V}(u) + \tilde{V}(\varepsilon u^1) - \varepsilon \mathbb{E}(\int_0^T u_t^1 \int_0^t u_s G(t-s) ds dt) - \varepsilon \mathbb{E}(\int_0^T u_t^1 \int_t^T u_s G(s-t) ds dt) \\
&= \tilde{V}(u) + \tilde{V}(\varepsilon u^1) - \varepsilon \mathbb{E}(\int_0^T u_t^1 \int_0^T u_s G(|t-s|) ds dt) \\
&= \tilde{V}(u) + \tilde{V}(\varepsilon u^1) - \varepsilon \langle u^1, u \rangle_G \,.
\end{aligned} \tag{5}$$

From the definition of $\mathcal{U}_0^{X_0}$ above, we know that $u \in \mathcal{U}_0^{X_0}$ implies that $\mathbb{E}(\int_0^T \int_0^t G(t-s) u_s u_t ds dt) = \|u\|_G^2 < \infty$.

The $O(\varepsilon)$ component of (5) can be re-written as

$$\begin{aligned}
&-\mathbb{E}(\int_0^T \xi_t u_t^1 dt + \int_0^T u_t^1 \int_0^t u_s G(t-s) ds\, dt + \int_0^T u_t \int_0^t u_s^1 G(t-s) ds dt) \\
= &-\mathbb{E}(\int_0^T \xi_t u_t^1 dt + \int_0^T u_t^1 \int_0^t u_s G(t-s) ds\, dt + \int_0^T u_s^1 \int_s^T G(t-s) u_t dt ds) \\
= &-\mathbb{E}(\int_0^T \xi_t u_t^1 dt + \int_0^T u_t^1 [\int_0^t u_s G(t-s) ds\, dt + \int_t^T u_s G(s-t) ds] dt) \\
= &-\mathbb{E}(\int_0^T u_t^1 (\xi_t + \int_0^T u_s G(|t-s|) ds) dt) \\
= &-\mathbb{E}(\int_0^T u_t^1 [\xi_t + \mathbb{E}_t(\int_0^T u_s G(|t-s|) ds)] dt) \,.
\end{aligned} \tag{6}$$

Now assume that (4) is satisfied which implies $M_t := \xi_t + \mathbb{E}_t(\int_0^T G(|t-s|)u_s ds) = \mathbb{E}_t(\int_0^T G(|T-v|)u_v dv)$. Then we see that

$$\mathbb{E}(\int_0^T u_t^1 M_t dt) \quad = \quad \mathbb{E}(\int_0^T u_t^1 (\xi_t + \mathbb{E}_t(\int_0^T u_s G(|t-s|)ds))dt)\,. \tag{7}$$

The second term on the right in (7) is just $\langle u, u_1 \rangle_G$, which we know is finite from Lemma A.1, and the first term on the right is also finite from the definition of $\mathcal{U}_0^0$. The following observations will be needed in what follows:

- $\int_0^T \mathbb{E}(|u_t^1 \xi_t|)dt = \int_0^T \mathbb{E}(|\mathbb{E}_t(u_t^1(P_T - P_t))|)dt \leq \int_0^T \mathbb{E}(\mathbb{E}_t(|u_t^1(P_T - P_t)|))dt = \int_0^T \mathbb{E}(|u_t^1(P_T - P_t)|)dt$, which is finite for $u^1 \in \mathcal{U}_0^0$ (see the definition of $\mathcal{U}_0^x$ and Remark 2.1)
- Similarly $\int_0^T \mathbb{E}(|u_t^1 \mathbb{E}_t(\int_0^T G(|t-v|)u_v dv)|)dt = \int_0^T \mathbb{E}(|\mathbb{E}_t(u_t^1(\int_0^T G(|t-v|)u_v dv))|)dt \leq \int_0^T \mathbb{E}(\mathbb{E}_t(|u_t^1(\int_0^T G(|t-v|)u_v dv)|))dt = \int_0^T \mathbb{E}(|u_t^1(\int_0^T G(|t-v|)u_v dv)|)dt \leq \langle |u^1|, |u| \rangle_G$, which is finite by Lemma A.1 since $|u|$ and $|u^1|$ are in $\mathcal{U}_0^{X_0}$ and $\mathcal{U}_0^0$ respectively, which implies they are also in $\mathcal{L}$.

Then using that $M_t = \xi_t + \mathbb{E}_t(\int_0^T u_s G(|t-s|)ds)$ and the two bullet points immediately above, we can apply Fubini and the tower property to say that

$$\begin{aligned}
\mathbb{E}(\int_0^T u_t^1 M_t dt) \quad &= \quad \mathbb{E}(\int_0^T u_t^1 \mathbb{E}_t(M_T)dt) \quad = \quad \mathbb{E}(\int_0^T \mathbb{E}_t(u_t^1 M_T)dt) \quad = \quad \int_0^T \mathbb{E}(\mathbb{E}_t(u_t^1 M_T))dt \\
&= \quad \int_0^T \mathbb{E}(u_t^1 M_T)dt \\
&= \quad \mathbb{E}(M_T \int_0^T u_t^1 dt) \\
&= \quad 0\,,
\end{aligned}$$

since u^1 is a round trip. Thus (6) is zero, so (4) is a sufficient condition for u to be a local optimizer. Moreover, using the Plancherel identity, we can re-write the expectation in the $O(\varepsilon^2)$ term in (5) (up to a minus sign) as

$$\begin{aligned}
\mathbb{E}(\int_0^T u_t^1 \int_0^T u_s^1 G(|t-s|)dsdt) \quad &= \quad \mathbb{E}(\int_{-\infty}^{\infty} u_t^1 \int_{-\infty}^{\infty} u_s^1 G(|t-s|)dsdt) \\
&= \quad \mathbb{E}(\int_{-\infty}^{\infty} \hat{u}^1(k)\overline{\widehat{u^1}(k)}\hat{G}(k)dk) \\
&= \quad \mathbb{E}(\int_{-\infty}^{\infty} |\hat{u}^1(k)|^2 \hat{G}(k)dk) \quad \geq \quad 0\,,
\end{aligned}$$

where we are setting $u^1 \equiv 0$ outside $[0,T]$, and $\hat{G}(k) = c_\gamma |k|^{\gamma-1}$ for some constant c_γ; hence $\tilde{V}(u+\varepsilon u^1)$ is concave in ε, so any local optimizer is a global optimizer. ■

2.2 Gaussian signals

We now assume that ξ_t is a Gaussian Volterra process of the form

$$\xi_t \quad = \quad \bar{\xi}(t) + \int_0^t K_\xi(u,t)dW_u \tag{8}$$

for some deterministic function $\bar{\xi}(t)$, where W is a standard Brownian motion and $\int_0^t K_\xi(u,t)^2 du < \infty$ for all $t \in [0,T]$ and $\mathcal{F}_t = \mathcal{F}_t^W$. Given that $\xi_T = \mathbb{E}_T(P_T - P_T) = 0$ is a Normal random variable with zero mean and zero variance, we see that

$$\bar{\xi}(T) \quad = \quad K_\xi(u,T) \quad = \quad 0 \tag{9}$$

for all $u \in [0,T]$. Let

$$\begin{aligned}
k(u,t) \quad &= \quad \frac{1}{c|T-u|^{1-\gamma}} G_1^{-1}(-K_\xi(u, u+(T-u)(.)) - \lambda_1(u))(\frac{t-u}{T-u}) \\
\text{and} \quad \lambda_1(u) \quad &= \quad -\frac{1}{\bar{c}_\gamma}\int_0^1 G_1^{-1}(K_\xi(u, u+(T-u)(.)))(s)ds
\end{aligned} \tag{10}$$

where $\bar{c}_\gamma = \frac{2^{\frac{1}{2}(3-\Gamma)}\pi^{\frac{5}{4}}(T-u)\Gamma(\frac{1}{2}(3+\gamma))\sec(\frac{1}{2}\pi\gamma)}{(1+\gamma)\Gamma(\frac{1}{2}(1-\gamma))^{\frac{3}{2}}\sqrt{\Gamma(\frac{1}{2}\gamma)}\,\Gamma(1+\gamma)}$, and the operator G_1 is defined by

$$(G_1\phi)(t) \quad := \quad \int_0^1 \phi(s)G(t-s)ds\,. \tag{11}$$

$G_1^{-1}(f)$ for a general function f has an explicit form which is stated and used in the proof of Theorem 2.2.

We let $X^0(t) = X_0 - \int_0^t u^0(s)ds$ denote the (deterministic) solution to the same problem but with no signal (see Subsection 2.3 for the explicit solution for X^0).

We now state the main result of the article:

Theorem 2.2 *If K_ξ is such that $\int_0^{\cdot} k(v,.)dW_v \in \mathcal{U}_0^0$, then the optimal trading strategy X^* is given by $dX_t^* = dX^0(t) - \hat{u}(t)dt$, where $\hat{u}(t) = \bar{u}(t) + \int_0^t k(v,t)dW_v$ is a Gaussian Volterra process on $[0,T)$ and $k(u,.)$ and $\bar{u}(t)$ are the unique solutions to the following Fredholm integral equations of the first kind:*

$$-K_\xi(u,t) \quad = \quad \int_u^T G(|t-v|)k(u,v)dv \,+\, \lambda(u) \tag{12}$$

$$-\bar{\xi}(t) \quad = \quad \int_0^T G(|t-v|)\bar{u}(v)dv \,+\, \lambda_2 \tag{13}$$

where the first equation holds for each $u \in [0,T]$ fixed and all $t \in [u,T]$, and the function $\lambda(u)$ and the constant λ_2 are chosen (uniquely) to ensure that $\mathbb{E}(X_T^2) = 0$, for which the following two conditions are necessary and sufficient:

$$\int_u^T k(u,t)dt \quad = \quad 0 \quad \textit{for all } u \in [0,T] \quad , \quad \int_0^T \bar{u}(v)dv \quad = \quad 0\,. \tag{14}$$

$d\hat{X}(t) = -\hat{u}(t)dt$ is the optimal solution to the round trip problem, i.e. for the case $X_0 = 0$.

Remark 2.3 Note that $\bar{u} \equiv 0$ if $\bar{\xi} \equiv 0$, since from the uniqueness part at the end of the proof, we know the solution to the Fredholm equation is unique.

Proof. We break up the proof into multiple parts.

- **Deriving the Fredholm equation**. We first assume $X_0 = 0$ (at the end of the proof we show how to extend to the general case with case $X_0 \neq 0$). Since $\hat{u}$ has to be adapted, we guess that $\hat{u}_t = \bar{u}(t) + \int_0^t k(v,t)dW_v$, so $\mathbb{E}_t(\hat{u}_v) = \bar{u}(v) + \int_0^{t\wedge v} k(u,v)dW_u$. Then from (4) we see that

$$\begin{aligned} 0 \quad &= \quad \xi_t \,+\, \mathbb{E}_t(\int_0^T (G(|t-v|) - G(|T-v|))\hat{u}_v dv) \\ &= \quad \bar{\xi}(t) \,+\, \int_0^t K_\xi(u,t)dW_u \,+\, \int_0^T (G(|t-v|) - G(|T-v|))\bar{u}(v)dv \\ &+ \quad \int_0^T (G(|t-v|) - G(T-v)) \int_0^{t\wedge v} k(u,v)dW_u)dv \\ &= \quad \int_0^t [K_\xi(u,t) + \int_u^T k(u,v)(G(|t-v|) - G(T-v))dv]dW_u \\ &+ \quad \bar{\xi}(t) \,+\, \int_0^T (G(|t-v|) - G(|T-v|))\bar{u}(v)dv\,. \end{aligned}$$

  Then we see that this is zero for all $t \in [0,T]$ a.s. if and only if

$$-K_\xi(u,t) \quad = \quad \int_u^T k(u,v)(G(|t-v|) - G(T-v))dv \tag{15}$$

$$-\bar{\xi}(t) \quad = \quad \int_0^T (G(|t-v|) - G(|T-v|))\bar{u}(v)dv \tag{16}$$

  are satisfied for all u,t with $0 \le u \le t \le T$.

- **Enforcing the liquidation condition**. Now consider a solution $k(u,.)$ to (12) for all $u\in[0,T]$, where $\lambda(u)$ will be chosen to ensure that $\mathbb{E}(X_T^2)=0$, and we will see that this implies that $k(u,.)$ satisfies (15) and (16) for all $u\in[0,T]$ as well. Setting $\hat{u}(t)=\bar{u}(t)+\int_0^t k(v,t)dW_v$ we see that

$$X_t = -\int_0^t \bar{u}(v)dv - \int_0^t\int_0^s k(v,s)dW_v ds = -\int_0^t \bar{u}(v)dv - \int_0^t\int_v^t k(v,s)dsdW_v$$

so in particular

$$X_T = -\int_0^T \bar{u}(v)dv - \int_0^T\int_0^t k(v,t)dW_v dt = -\int_0^T \bar{u}(v)dv - \int_0^T\int_v^T k(v,t)dtdW_v\,. \quad (17)$$

Consequently, to impose that $\mathbb{E}(X_T^2)=0$, we see that both equations in (14) must hold, the first of which determines $\lambda(u)$ and second determines the constant λ_2 (below we will show that $\lambda(u)$ and λ_2 are uniquely determined using operator formalism and we give an explicit formula in (21)). Then setting $t=T$ in (12) and using that $K_\xi(u,T)=0$ (from (9)), we see that

$$0 = \int_u^T G(|T-v|)k(u,v)dv + \lambda(u)$$

so (15) is indeed satisfied. Similarly using that $\bar{\xi}(T)=0$ (from (9)) we find that $\int_0^T G(|T-v|)\bar{u}(v)dv + \lambda_2=0$, so (12) implies (16).

- **Explicit computation of $\lambda(u)$ and λ_2**. We now transform (12) so the range of integration is $[0,1]$. To this end, we first re-write (12) in the form

$$c\int_u^T \frac{g(v)}{|x-v|^\gamma}dv = \tilde{f}(x)$$

where $g(v)=k(u,v)$ and $\tilde{f}(x)=-K_\xi(u,x)-\lambda(u)$ and let $w=\frac{v-u}{T-u}$, so $dw=\frac{dv}{T-u}$, then we can re-write this as

$$c(T-u)\int_0^1 \frac{g((T-u)w+u)}{|x-(T-u)w-u|^\gamma}dw = c(T-u)\int_0^1 \frac{g_1(w)}{|x-(T-u)w-u|^\gamma}dw = \tilde{f}(x)$$

where $g_1(w)=g((T-u)w+u)$, where our notation is chosen so as to be consistent with that used in [CG94]. Now let $x-u=(T-u)x'$ to obtain

$$c(T-u)\int_0^1 \frac{g_1(w)}{|(T-u)x'-(T-u)w|^\gamma}dw = c|T-u|^{1-\gamma}\int_0^1 \frac{g_1(w)}{|x'-w|^\gamma}dw = \tilde{f}(u+(T-u)x') \quad (18)$$

which we can re-write more succinctly as

$$G_1 g_1 = \frac{\tilde{f}(u+(T-u)(.))}{c|T-u|^{1-\gamma}}\,, \quad (19)$$

where G_1 is the operator defined in (11). Then from (12) and the linearity of G_1^{-1}, we see that

$$\begin{aligned} k(u,t) = g(t) &= \frac{1}{c|T-u|^{1-\gamma}}G_1^{-1}\tilde{f}(u,u+(T-u)(.))(\frac{t-u}{T-u}) \\ &= \frac{1}{c|T-u|^{1-\gamma}}G_1^{-1}(-K_\xi(u,u+(T-u)(.))-\lambda(u))(\frac{t-u}{T-u})\,. \end{aligned} \quad (20)$$

Integrating from $t=u$ to T and using that $\int_u^T k(u,t)dt=0$ for all $u\in[0,T]$ and moving the $\lambda(u)$ term to the other side and cancelling terms, we see that

$$\int_u^T G_1^{-1}(-K_\xi(u,u+(T-u)(.)))(\frac{t-u}{T-u})dt = \int_u^T G_1^{-1}(\lambda(u))(\frac{t-u}{T-u})dt\,,$$

so by the linearity of G_1^{-1}, we see that

$$\lambda(u) = -\frac{\int_u^T G_1^{-1}(K_\xi(u,u+(T-u)(.)))(\frac{t-u}{T-u})dt}{\int_u^T G_1^{-1}(1)(\frac{t-u}{T-u})dt}\,. \quad (21)$$

Moreover, from Example 2.30 in [GSS12], we know that

$$G_1^{-1}(1)(s) \quad = \quad \frac{c_\gamma}{(s(1-s))^{\frac{1}{2}(1-\gamma)}}\,, \tag{22}$$

where $c_\gamma = [2^{\gamma-1}\Gamma(\frac{1}{2}-\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\gamma)/\sqrt{\pi}]^{-\frac{1}{2}}$. Then

$$\int_u^T G_1^{-1}(1)(\frac{t-u}{T-u})dt \quad = \quad \bar{c}_\gamma(T-u)$$

(where $\bar{c}_\gamma$ is defined in the statement of the Theorem), so $\lambda(u)$ simplifies to

$$\begin{aligned}\lambda(u) \quad &= \quad -\frac{1}{\bar{c}_\gamma}\frac{1}{T-u}\int_u^T G_1^{-1}(K_\xi(u,u+(T-u)(.)))(\frac{t-u}{T-u})dt \\ &= \quad -\frac{1}{\bar{c}_\gamma}\int_0^1 G_1^{-1}(K_\xi(u,u+(T-u)(.)))(s)ds\,.\end{aligned}$$

Similarly we find that

$$\lambda_2 \quad = \quad -\frac{1}{\bar{c}_\gamma}\int_0^1 G_1^{-1}(\bar{\xi}(T(.)))(s)ds$$

and

$$\bar{u}(t) \quad = \quad \frac{1}{cT^{1-\gamma}}G_1^{-1}(-\bar{\xi}(T(.))-\lambda_2)(\frac{t}{T})$$

and note that $u=0$ in these last two formulae.

- **Decomposing G_1 and explicit computation of G_1^{-1}.** From Example 9.2 (see also Example 6.2) in [PS90], setting $\nu=\gamma$ we know that G_1 can be decomposed as $G_1=\mathcal{T}\mathcal{T}^*$, where $\mathcal{T}$ is the Volterra-type operator defined by

$$(\mathcal{T}\phi)(t) \quad = \quad \int_0^t \kappa(s,t)\phi(s)ds$$

and $\kappa(s,t) = c_\nu(\frac{t}{s})^{(1-\gamma)/2}(t-s)^{-\frac{1}{2}(1+\gamma)}$ for some constant c_ν depending on ν, and $\mathcal{T}^*$ is its adjoint given by $(T^*\phi)(t)=\int_s^T\kappa(s,t)\phi(t)dt$ (see e.g. the start of Appendix A of [FZ17] to see why $\mathcal{T}^*$ takes this form). Then we can further re-write $\mathcal{T}$ as $\mathcal{T}=B^{-1}I_\nu B$, where B is the bounded operator on L^2 which multiplies functions by $t^{-(1-\nu)/2}$ and I_ν is the Riemann-Liouville operator $(I_\nu\phi)(t) := \int_0^t(t-s)^{-\frac{1}{2}(1+\gamma)}\phi(s)ds = \frac{1}{\Gamma(1-r)}I^r$ where $r=\frac{1}{2}-\frac{1}{2}\gamma$ so $I_\nu^{-1}=\Gamma(1-r)D^r$, where I^r and D^r are the fractional derivative operators of order r. Summing this up, we can re-write (18) as

$$\mathcal{T}\mathcal{T}^*g_1 \quad = \quad h_1$$

for some function h_1, which has solution

$$g_1 \quad = \quad \mathcal{T}^{*-1}(\mathcal{T}^{-1}h_1)\,.$$

To compute $(\mathcal{T}^*)^{-1}$, we note that $(\phi,T\psi)=(\phi,B^{-1}I_\nu B\psi)=(B^{-1}\phi,I_\nu B\psi)=(I_\nu^*B^{-1}\phi,B\psi)=(BI_\nu^*B^{-1}\phi,\psi)$, so $\mathcal{T}^*=BI_\nu^*B^{-1}$, and we know how to invert B and I_ν^*.

- **Practical computation of $k(u,t)$.** We can read off the solution to (18) more explicitly from [CG94], with $f(x_1)=\frac{\tilde{f}(x')}{|T-u|^{1-\gamma}}$ and their $a=b=c$, for which the explicit solution is given in Eqs 3.14a and 3.14b in [CG94] which we can re-write in our variables as

$$\begin{aligned}k(u,t) \quad &= \quad -t^{\bar{\gamma}+\mu-1}\frac{\sin^2(\pi\bar{\gamma})}{\pi^2}\frac{d}{dt}\int_t^1\frac{1}{(s-t)^{\bar{\gamma}}}\int_0^s\frac{v^{-\bar{\gamma}}h(v)}{(s-v)^{1-\bar{\gamma}}}dv \\ \text{where} \quad h(t) \quad &= \quad \frac{t^{1-\gamma}}{b}\frac{d}{dt}\int_0^t\frac{f(y)}{(x-y)^{1-\gamma}}dy\end{aligned}$$

and $\mu=\gamma$, $\alpha+\gamma=1$, $-\lambda=\frac{\pi}{\sin(\pi(1-\gamma))}+\pi\cot(\pi(1-\gamma))$ and $\bar{\gamma}$ satisfies $|\lambda|=\pi\cot(\pi\bar{\gamma})$ with $0<\bar{\gamma}<\frac{1}{2}$ (note $\bar{\gamma}$ here is the γ parameter in [CG94] and our γ is the μ parameter in [CG94].

Remark 2.4 For the case commonly considered where $\gamma = \frac{1}{2}$, the α-parameter in [CG94] is $1-\gamma = \frac{1}{2}$ and their λ parameter is $-(a\pi/(b\sin(\pi\alpha)-\pi\cot(\pi\alpha)-=-\pi$ so their γ parameter is $\frac{1}{4}$ (which we call γ_1 to distinguish from our γ parameter).

If two distinct solutions exist to (20), then we must have a non-zero solution ϕ to $G_1\phi = 0$, so in particular $\int_{[0,1]}\int_{[0,1]}\phi(s)\phi(t)G(|t-s|)dsdt = \langle\phi, G_1\phi\rangle_{L^2} = 0$. But from Plancherel's theorem we know this quantity is equal to

$$\int_{[0,T]}\int_{[0,T]}\phi(s)\phi(t)G(|t-s|)dsdt \quad = \quad \int_{-\infty}^{\infty}|\hat{\phi}(k)|^2\hat{G}(k)dk \quad = \quad const. \times \|\phi\|^2_{H^{-\frac{1}{2}\gamma}},$$

where $\hat{G}(k) = c_\gamma|k|^{\gamma-1} > 0$ is the Fourier transform of G (see Appendix A for the exact formula) for some constant $c_\gamma > 0$, and $\|.\|_{H^{-s}}$ denotes the norm on the homogenous fractional Sobolev space of order $-s < 0$ (see Appendix A for details, and references on this). Hence we cannot have two distinct solutions to (20) in $H^{-\gamma/2}$. Finally for the general case with $X_0 \neq 0$, we can easily verify that $X^0(t) + \hat{X}_t$ satisfies (4), i.e. we can decompose the general solution as the (deterministic) no-signal solution plus the round trip solution (again see next subsection for details of how to compute X^0).

■

Remark 2.5 If we replace W with an Itô process of the form $M_t = \int_0^t \sigma_s^2 dW_s$ then the stochastic integral part of (17) will be replaced by $\int_0^T\int_v^T k(v,t)dt\,\sigma_v dW_v$, whose variance is $\mathbb{E}(\int_0^T(\int_v^T k(v,t)dt)^2\,\sigma_v^2 dv) = \int_0^T(\int_v^T k(v,t)dt)^2\,\mathbb{E}(\sigma_v^2)dv$. Then if $\mathbb{E}(\sigma_v^2) > 0$ for all v we still require that $\int_v^T k(v,t)dt = 0$ and (formally at least) Theorem 2.3 still holds if the proposed trading strategy is admissible. A potentially interesting example which falls in this framework is an affine driftless Rough-Heston model-type process for P of the form $P_t = P_0 + c\int_0^t(t-s)^{H-\frac{1}{2}}\sqrt{P_s}dW_s$, which also has the advantage that P is non-negative (we defer the details for future research).

2.3 The zero-signal case

For the case of power-law impact where $G(t) = ct^{-\gamma}$ for $\gamma \in (0,1)$, the optimal selling speed with no-signal satisfies

$$\int_0^T G(|t-v|)u^0(v)dv \quad = \quad \lambda\,, \tag{23}$$

where λ is the unique constant which ensures that $X_T = X_0 - \int_0^T u^0(t)dt = 0$, and setting $t = T$ we see that

$$\int_0^T (G(|t-v|) - G(|T-v|))u^0(v)dv \quad = \quad 0$$

which is consistent with (4) for the case of zero signal. We can re-write (23) using operator formalism as $Gu^0 = \lambda$ where $G\phi(.) := \int_0^T G(|(.)-v|)\phi(v)dv$, so λ satisfies

$$X_0 - \lambda\int_0^T G^{-1}(1)(t)dt \quad = \quad 0$$

and the solution is given by

$$u^0(t) \quad = \quad \frac{c_1}{(t(T-t))^{\frac{1}{2}(1-\gamma)}}$$

for some constant c_1 (see Example 2.30 in [GSS12], and [CGL17]).

2.4 Computing the expected optimal profit/loss

If $\bar{\xi}(t) \equiv 0$, the expected profit/loss from the optimal trading strategy in Theorem 2.2 is

$$\begin{aligned}
V(\hat{u}) \quad &= \quad \mathbb{E}(P_T X_0) \;-\; \mathbb{E}(\int_0^T (\xi_t + \int_0^t G(t-s)\hat{u}_s ds)\hat{u}_t dt) \\
&= \quad \mathbb{E}(P_T X_0) \;-\; \mathbb{E}(\int_0^T\int_0^t K_\xi(s,t)dW_s\int_0^t k(u,t)dW_u dt) \\
&\quad -\mathbb{E}(\int_0^T\int_0^t G(t-s)(\int_0^s k(u,s)dW_u)(\int_0^t k(v,t)dW_v)ds\,dt) \\
&= \quad \mathbb{E}(P_T X_0) \;-\; \int_0^T\int_0^t K_\xi(u,t)k(u,t)dudt) \;-\; \int_0^T\int_0^t G(t-s)\int_0^s k(u,s)k(u,t)dudsdt
\end{aligned} \tag{24}$$

We can easily adapt this expression to include the case of a general non-zero $\bar{\xi}(t)$ but the expression will be a lot messier due to the squared terms. We have found Monte Carlo to be the most efficient way to compute this triple integral in practice, which is what was used to compute the right plot in Figure 1.

2.5 Re-expressing the trading speed in terms of the price history

At the moment our optimal selling speed is expressed as $u_t = \int_0^t k(u,t)dW_u$, but it is more natural and useful to re-express u_t in terms of P itself. To this end, let $Z_t = \int_0^t g(s,t)dW_s$, and we seek a function $h(.,.)$ such that $h(t,t)Z_t - \int_0^t h_s(s,t)Z_s ds = W_t$. Then we see that

$$\begin{aligned} h(t,t)Z_t - \int_0^t h_s(s,t)Z_s ds &= h(t,t)\int_0^t g(u,t)dW_u - \int_0^t h_s(s,t)\int_0^s g(u,s)dW_u ds \\ &= h(t,t)\int_0^t g(u,t)dW_u - \int_0^t \int_u^t h_s(s,t)g(u,s)ds dW_u\,, \end{aligned}$$

where $h_s(.,.)$ denotes the partial derivative of h with respect to the first argument. Hence to find an inversion formula, we need to solve the integral equation

$$h(t,t)g(u,t) - \int_u^t h_s(s,t)g(u,s)ds = 1\,.$$

If $g(s,t) = g(t-s)$ with $g \in L^2$ and we guess that $h(s,t) = h(t-s)$, then the equation takes the special form

$$h(0)g(t-u) + \int_u^t h'(t-s)g(s-u)ds = 1\,.$$

Setting $\tilde{s} = s-u$, we can re-write this as

$$h(0)g(t-u) + \int_0^{t-u} h'(t-(u+\tilde{s}))g(\tilde{s})d\tilde{s} = 1\,,$$

and replacing $t-u$ with t we can further re-write as

$$h(0)g(t) + \int_0^t h'(t-\tilde{s})g(\tilde{s})d\tilde{s} = h(0)g(t) + h' * g = 1\,.$$

Then taking the Laplace transform, we have

$$h(0)\hat{g} + \widehat{(h')}\hat{g} = h(0)\hat{g} + (\lambda\hat{h} - h(0))\hat{g} = \frac{1}{\lambda}\,,$$

so we see that

$$\hat{h} = \frac{1}{\lambda^2\hat{g}}\,. \tag{25}$$

Hence if $P_t = \int_0^t g(t-u)dW_u$ for some $g \in L^2$ then $\xi_t = \int_0^t K_\xi(u,t)dW_u$ with $K_\xi(u,t) = g(T-u) - g(t-u)$, and from the preceding computations we have the inversion formula

$$W_t = h(t,t)P_t - \int_0^t h_s(s,t)P_s ds$$

and recall that $\hat{u}_t = \int_0^t k(u,t)dW_u$ (where $k(.,.)$ depends on K_ξ via the Fredholm eq (12), and hence on g itself) so we now see how $\hat{u}$ depends solely on the (unaffected) stock price history $(P_u)_{0\le u\le t}$, which gives us our signal-adaptive optimal selling speed.

We can compute h explicitly for the case when $g(t) = t^{H-\frac{1}{2}}e^{-\theta t}$ for $H \in (0,1)$, $\theta > 0$ for which we find that

$$h(t) = \frac{e^{-\theta t}t^{-\frac{1}{2}-H}\left[2 - e^{t\theta}(1+2H+2t\theta)(E_{\frac{3}{2}+H}(t\theta) - (t\theta)^{\frac{1}{2}+H}\Gamma(-\frac{1}{2}-H))\right]}{2\theta\Gamma(-\frac{1}{2}-H)\Gamma(\frac{1}{2}+H)}\,. \tag{26}$$

where $E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n}dt$. $H = \frac{1}{2}$ corresponds to the OU process for which $h(t) = 1 + \theta t$, and $\theta = 0$ corresponds to the Riemann-Liouville process for which $h(t) = \frac{t^{\frac{1}{2}-H}}{\Gamma(\frac{3}{2}-H)\Gamma(\frac{1}{2}+H)}$ (see next section).

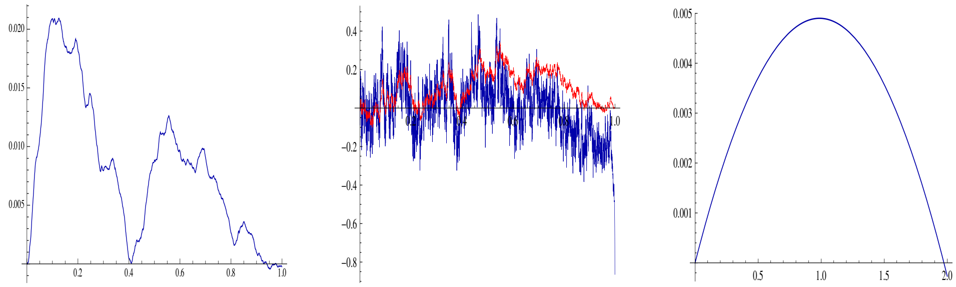

Figure 1: On the left we have plotted the optimal inventory X^*_t in Theorem 2.2 when $P_t = \sigma \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ is a Riemann-Liouville process using (27) with $H = \frac{2}{3}$, $\sigma = 1$, $c = 1$ and $\gamma = .5$ and $X_0 = 0$, and in the middle we have plotted u^*_t (blue) and ξ_t (in red). On the right, as a sanity check, we have plotted the expected profit/loss for α times the optimal trading speed, as a function of α (which we see is correctly maximized close to $\alpha = 1$, the small numerical error is there because we have to estimate the triple integral in (24) with Monte Carlo)

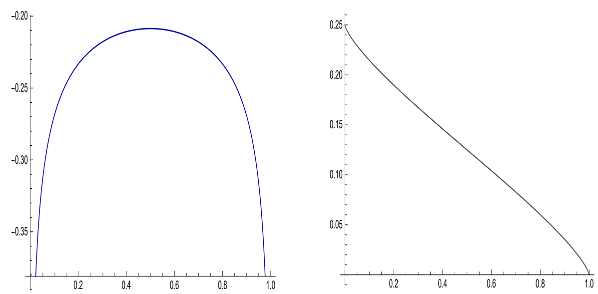

Figure 2: Non-Round trip case: from left to right (with $X_0 = .25$ and the same parameters as above) we see (i) the optimal buying speed with no-signal (ii) X^*_t with no signal

3 Examples and variations of the main model

3.1 Rough signals

If $P_t = \sigma \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ (i.e. a Riemann-Liouville process) for $H \in (0,1)$ and $\gamma = \frac{1}{2}$ and $\bar{\xi}(t) \equiv 0$ for simplicity, then clearly $\xi_t = \mathbb{E}_t(P_T - P_t) = \int_0^t ((T-s)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}}) dW_s$ and (after some lengthy Mathematica computations) we find that

$$\begin{aligned} k(u,t) &= -(2c\pi^{\frac{3}{2}} \tau^{\frac{3}{2}} \bar{u}^{\frac{1}{4}} \Gamma(H))^{-1} \cdot \Big[\frac{\tau^{\frac{3}{2}+H} \sigma \Gamma(\frac{1}{4}) \Gamma(H_{\frac{1}{4}})}{w^{\frac{1}{4}} (u-t)} \\ &+ H_{\frac{1}{4}} \tau^{\frac{1}{2}+H} \bar{u}^{-\frac{3}{4}+H} \sigma \Gamma(\frac{1}{4}) (-B(\bar{u}, -H_{\frac{1}{4}}, \frac{3}{4}) + \frac{\Gamma(\frac{3}{4}) \Gamma(-H_{\frac{1}{4}})}{\Gamma(\frac{1}{2} - H)}) \Gamma(H_{\frac{1}{4}}) \\ &+ \frac{\sqrt{2\pi}\, \Gamma(H) (\tau^{\frac{1}{2}+H} \sigma + \tau \lambda_1(u))}{w^{\frac{1}{4}}} \Big], \end{aligned} \tag{27}$$

where $H_{\frac{1}{4}} = H + \frac{1}{4}$, $\tau = T - u$, $w = \frac{T-t}{T-u}$, $\bar{u} = \frac{t-u}{T-u}$ and $B(z,a,b) = \int_0^z t^{a-1}(1-t)^{b-1} dt$ denotes the incomplete Beta function, and enforcing the liquidation condition $\int_u^T k(u,t) dt = 0$ we find that

$$\lambda(u) = -\Upsilon \tau^{H-\frac{1}{2}}$$

where Υ is given by

$$\sigma \frac{\pi^2 \csc(\theta\pi) + \Gamma(\omega_-) \big[2H\Gamma(\frac{3}{4})^2 \Gamma(H) - H\sqrt{\pi} \Gamma(-\frac{1}{4}) \Gamma(\theta) - \pi \cos(H\pi) \csc(\theta\pi) \Gamma(\omega_+) + \sqrt{\pi} \Gamma(-\frac{1}{4}) \Gamma(\frac{5}{4} + H) \big]}{2H\Gamma(\frac{3}{4})^2 \Gamma(\omega_-) \Gamma(H)}$$

with $\theta = \frac{1}{4} + H$ and $\omega_\pm = \frac{1}{2} \pm H$ (see numerical simulations above and overleaf). Note that we have not rigourously verified that this strategy is admissible which would be extremely difficult to check.

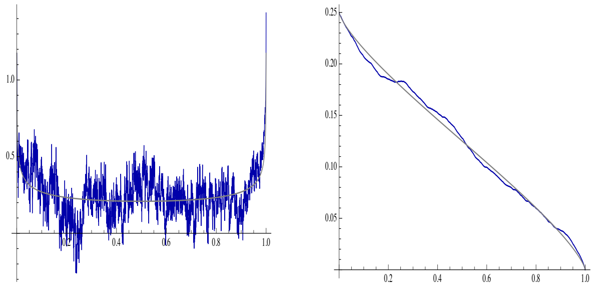

Figure 3: On the left we see the optimal selling speed with non-zero signal (blue) and the no-signal optimal speed (grey) and on the right we see X_t^* with non-zero signal (blue) and zero signal (grey), for the same parameters and simulated Brownian motion as Figure 2.

3.2 Computing G from the trading history

Recall that

$$S_t - P_t \quad = \quad \int_0^t G(t-s)dX_s \quad = \quad - \int_0^t u_{t-s}G(s)ds\,, \tag{28}$$

which we can interpret as a Volterra integral equation for the unknown G, if the trading history is known. More specifically, assume we have our trading history $(X_t)_{t\in[0,T)}$ and we just set $dX_t = 0$ for $t \geq T$. Taking the Laplace transform of (28) we get

$$\hat{S}(\lambda) - \hat{P}(\lambda) \quad = \quad \hat{G}(\lambda)\widehat{dX}(\lambda)\,,$$

where $\hat{\cdot}$ denotes the Laplace transform operator. From this, in principle, we can then compute $(G(t))_{t\in[0,T]}$ as the inverse Laplace transform of

$$\hat{G}(\lambda) \quad = \quad \frac{\hat{S}(\lambda) - \hat{P}(\lambda)}{\widehat{dX}(\lambda)}\,.$$

Note that if the G computed in this way is complex-valued or negative, it means the true model is not of the form in (1).

3.3 Adding Temporary price impact

If we add a temporary price impact term $\eta\dot{X}_t = -\eta u_t$ on the right hand side of (1), then we incur an additional ηu_t^2 term in (3), and a standard first order variational analysis of this expression leads to the following modified (4):

$$\xi_t + 2\eta u_t^* + \mathbb{E}_t(\int_0^T G(|t-v|)u_v^* dv) \quad = \quad M_t$$

for some martingale M to be determined such that $X_T = 0$ as before. Then using the same ansatz $u_t = \int_0^t k(u,t)dW_u$, we can readily verify that (12) and (13) and change to

$$\begin{aligned} -K_\xi(u,t) \quad &= \quad 2\eta k(u,t) + \int_u^T G(|t-v|)k(u,v)dv + \lambda(u) \\ -\bar{\xi}(t) \quad &= \quad 2\eta\bar{u}(t) + \int_0^T G(|t-v|)\bar{u}(v)dv + \lambda_2 \end{aligned}$$

where $\lambda(u)$ and λ_2 are again chosen to ensure that $X_T = 0$, and this is now a Fredholm equation of the *second kind*, for $u \in [0,T]$ fixed.

3.4 Exponential resilience with temporary impact

In this subsection, we assume $G(t) = ce^{-\rho|t|}$ for $\rho > 0$ with $\bar{\xi}(t) = 0$, and assume non-zero temporary price impact as in the previous subsection and (unlike [NV22]) we impose full liquidation. Since $G(|t-s|)$ is bounded from above strictly positive $[0,T]^2$, for this problem $\mathcal{L} = \{u : \mathbb{E}(\int_0^T \int_0^T u_t u_s dsdt) = \mathbb{E}((\int_0^T u_t dt)^2) < \infty\}$. Then the integral equation for $k(u,t)$ becomes

$$K_\xi(u,t) + \eta k(u,t) + \int_u^T k(u,v)G(|t-v|)dv \quad = \quad -\lambda_{\eta,\rho}(u)$$

for some $\lambda_{\eta,\rho}(u)$ which has to be chosen to ensure liquidation (see below for explicit computation of $\lambda_{\eta,\rho}(u)$), and setting $t=T$ we see that

$$\eta k(u,T) \,+\, \int_u^T k(u,v)G(|T-v|)dv \;=\; -\lambda_{\eta,\rho}(u)\,.$$

We can re-write the integral equation as

$$K_\xi(u,t) \,+\, G_{\rho,\eta}k(u,.)(t) \;=\; -\lambda_{\eta,\rho}(u)\,, \tag{29}$$

where the operator $G_{\rho,\eta}$ is defined by $G_{\rho,\eta}\phi(t) := \eta\phi(t) + \int_u^T ce^{-\rho|t-s|}\phi(s)ds$ for $\eta > 0$.

$G^{-1}_{\rho,\eta}$ can be computed explicitly (see item 15, pg 324 in [PM08]) and note that the A parameter in [PM08] will be $\frac{c}{\eta}$ for our problem here, so we need $\eta > 0$ and their $\lambda = -\rho < 0$. More specifically, if y satisfies

$$y(x) \,+\, A\int_a^b e^{\lambda|x-t|}y(t)dt \;=\; f(x)$$

then

$$\begin{aligned} y'(x) \,+\, \lambda A\int_a^b \operatorname{sgn}(x-t)e^{\lambda|x-t|}y(t)dt &= f'(x) \\ y''(x) \,+\, 2\lambda A y(x) \,+\, \lambda^2 A\int_a^b e^{\lambda|x-t|}y(t)dt &= f''(x) \end{aligned}$$

so we see that y satisfies the ODE

$$y''(x) + \lambda(2A-\lambda)y(x) \;=\; f''(x) - \lambda^2 f(x) \tag{30}$$

and

$$\begin{aligned} y(a) \,+\, A\int_a^b e^{\lambda(t-a)}y(t)dt = f(a) \quad &, \quad y'(a) \,-\, \lambda A\int_a^b e^{\lambda(t-a)}y(t)dt = f'(a) \\ y(b) \,+\, A\int_a^b e^{\lambda(b-t)}y(t)dt = f(b) \quad &, \quad y'(b) \,+\, \lambda A\int_a^b e^{\lambda(t-a)}y(t)dt = f'(b) \end{aligned}$$

so the boundary conditions for the ODE in (30) are given by

$$y'(a) + \lambda y(a) = f'(a) + \lambda f(a) \quad , \quad y'(b) - \lambda y(b) = f'(b) - \lambda f(b)\,.$$

The explicit solution for $y(x)$ in Eq 3 on 324 in [PM08] is incorrect but can be computed (the expression is rather long so we omit the details here for the sake of brevity).

Then from (29) we see that

$$k(u,t) \;=\; -G^{-1}_{\rho,\eta}(K_\xi(u,.) + \lambda_{\eta,\rho}(u))(t)$$

and combining this with the constraint that $\int_u^T k(u,t)dt = 0$ for all $u \in [0,T]$, we see that $\int_u^T G^{-1}_{\rho,\eta}(K_\xi(u,.) + \lambda_{\eta,\rho}(u))(t)dt = 0$ so

$$\lambda_{\eta,\rho}(u) \;=\; -\frac{\int_u^T G^{-1}_{\rho,\eta}(K_\xi(u,.))(t)dt}{\int_u^T G^{-1}_{\rho,\eta}(1)(t)dt}$$

and this solution is valid so long as K_ξ is such that the resulting u process is in $\mathcal{U}_0^0$. To compute the non round-trip solution for the no-signal case, we have to solve

$$G_{\rho,\eta}u(t) \;=\; \lambda$$

for some constant λ such that $X_0 - \int_0^T u(s)ds = 0$, so λ here satisfies

$$X_0 - \lambda\int_0^T G^{-1}_{\rho,\eta}(1)(t)dt \;=\; 0\,.$$

3.5 The unconstrained problem with zero resilience

We now consider the case when

$$S_t = P_t + f(\phi_t) \tag{31}$$

for some general increasing function f with $f(0) = 0$, where $\phi_t = -u_t$ and we remove the restriction that $X_T = 0$. Then when $f(x) = kx$, using the optional projection argument as in (2), our performance criterion simplifies to $V(\phi) = \mathbb{E}(\int_0^T (\xi_t\phi_t - \eta\phi_t^2)dt)$, and we can then maximize the integrand $\xi_t\phi_t - \eta\phi_t^2$ *pointwise* to obtain the optimal ϕ_t as $\phi_t^* = \frac{1}{2\eta}\xi_t$, and note that $\phi_T^* = \xi_T = 0$, as in Lemma 5.2 in [BSV17].

Remark 3.1 For a general non-linear impact function f (as in e.g. Guasoni et al.[GNR19]) if $g(x) = xf(x)$ and we assume g is e.g. smooth and convex, then pointwise optimization yields that $\phi_t^* = (g')^{-1}(\xi_t)$.

3.6 Transactions costs and the no-trade region

Setting $f(x) = \eta x + \varepsilon\,\mathrm{sgn}(x)$ in (31) for $k, \varepsilon > 0$ corresponds to linear temporary price impact plus fixed transaction costs, i.e. a fixed bid/offer spread of ε. In this case, using that $\xi x - \eta x^2 - \varepsilon|x|$ is maximized at $x = \frac{\xi-\varepsilon}{2\eta}$ if $\xi - \varepsilon \geq 0$, at $x = \frac{\xi+\varepsilon}{2\eta}$ if $\xi + \varepsilon \leq 0$ and at $x = 0$ otherwise (combined with a pointwise optimization argument as above), we find that

$$\phi_t^* = \frac{1}{2\eta}(\xi_t - \varepsilon\,\mathrm{sgn}(\xi_t))\, 1_{|\xi_t|\geq\varepsilon}\,,$$

and we see that there is now a *no-trade region* defined by $|\mathrm{sgn}(\xi_t)| < \varepsilon$.

4 Calibrating the model to real limit order book data

To calibrate the price impact model in equation (1) we employ the order flow of all market participants, the transaction prices weighted by volume, and the unaffected price process. We then look for parameters that best fit the data. In (1) we refer to P_t as the unaffected price process, and $dX_t = -u_t\,dt$ is the instantaneous trading of the agent. Let $\tilde{P}_t = P_t - \int_0^t G(t-s)dY_s$ be the "observable unaffected price", where $dY_t = v_t dt$ and Y is the cumulative instantaneous trading of all other market participants excluding the agent. Then (1) changes to

$$S_t = \tilde{P}_t + \int_0^t G(t-s)dZ_s \tag{32}$$

where $dZ_s = dX_s + dY_s = (u_s + v_s)\,ds$ captures the order flow of the entire market – see [CJ16].

Given the previous decomposition, we show how to estimate the parameters that appear in the decay kernel G. Let Θ be the parameter space associated with G. For example, in the power-law impact case, in which $\theta = (c, \gamma)$, the parameter space is $\Theta = \mathbb{R}^+ \times (0, 1)$. Take $\theta \in \Theta$ and consider a discretized version of (32) given by

$$S_{t_n} \approx \tilde{P}_{t_{n-1}} + \sum_{i=1}^{n} G^\theta(t_n - t_{i-1})\,(u_{t_i} + v_{t_i})\,\Delta$$

where $0 = t_0 < t_1 < \ldots < t_n$, and $\Delta = t_i - t_{i-1}$ for $i \in \{1, 2, \ldots, n\}$. The quantity $(u_{t_i} + v_{t_i})\,\Delta$ represents the volume traded in $[t_{i-1}, t_i)$ by all market participants. The observable unaffected price $\tilde{P}_{t_{n-1}}$ can be taken to be the mid-price of the asset at time t_{n-1}, and S_{t_n} is the volume-weighted average price of all transactions in $[t_{i-1}, t_i)$.

Fix a given calibration horizon T (for example, one day of trading), let $t_0 < t_1 < \ldots < t_N$ be a fixed time grid, where $t_0 = 0$ and $t_N = T$ (for example, one minute intervals throughout the day), let $(S_{t_i})_{1\leq i\leq N}$ be the observed volume-weighted transaction prices[2], and let $(V_{t_i})_{0\leq i\leq N}$ be the volume traded by all market participants. For instance, for $i \in \{1, 2, \ldots, N\}$, $V_{t_i} = (u_{t_i} + v_{t_i})\,\Delta$. Finally, let $(\tilde{P}_{t_i})_{0\leq i\leq N-1}$ be the mid-price sampled at times $t_0 < t_1 < \ldots < t_{N-1}$. We assume our observations have noise, that is to say

$$S_{t_n} = \tilde{P}_{t_{n-1}} + \sum_{i=1}^{n} G^\theta(t_n - t_{i-1})\,V_{t_i} + \epsilon_n\,,$$

[2] We define $S_0 = P_0$. If there are no transactions in a given interval $[t_{i-1}, t_i)$ for $i \in \{1, 2, \ldots, N\}$, we define $S_{t_i} = S_{t_{i-1}}$. Otherwise, S_{t_i} is the volume-weighted trade price over all trading carried in $[t_{i-1}, t_i)$.

where $(\epsilon_n)_{n\in\mathbb{N}}$ is a collection of independent and identically distributed normal random variables. We take the estimator $\hat{\theta}$ of θ to be the parameters that minimize the residual sum of squares, in other words,

$$\hat{\theta} \quad = \quad \underset{\theta\in\Theta}{\operatorname{argmin}} \sum_{n=1}^{N} (S_{t_n} - \tilde{P}_{t_n} \; - \sum_{i=1}^{n} G^{\theta}(t_n - t_i)\, V_{t_i})^2 \,. \tag{33}$$

Next, we test the calibration method in (33). We employ limit order book (LOB) data from VOD, AAPL, and CSCO trading in NASDAQ from 2 December 2019 to 31 January 2020. The data comprise all of the updates in the best prices, quantities, and trades. We take the time intervals to be spaced by one minute, and we set $[0,T]$ to be from 10:00am to 2:00pm. We calibrate the parameters (c,γ) in $\mathbb{R}^{+}\times(0,1)$ for the power-law impact case $G^{\theta}(t)=ct^{-\gamma}$, and we refer to the estimates as $\hat{c}$ and $\hat{\gamma}$. We observe that over the two months of data, the mean value (and standard deviation) of the estimate $\hat{\gamma}$ was 0.384 (0.104) for VOD, 0.440 (0.125) for AAPL, and 0.493 (0.104) for CSCO. Similarly, the mean value (and standard deviation) of the estimate $\hat{c}$ was 0.0015 (0.0004) for VOD, 0.0028 (0.0007) for AAPL, and 0.0009 (0.0004) for CSCO. For an alternate approach to the calibration of parameters under transient market impact, see [BL12].

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A Appendix

Recall that $\langle u, v\rangle_G = \mathbb{E}(\int_0^T \int_0^T u_s v_t G(|t-s|)dsdt)$.

Lemma A.1 *Let $u, v \in \mathcal{U}$ such that $\|u\|_G$ and $\|v\|_G$ are finite. Then $\langle u, v\rangle_G < \infty$.*

Proof. We first consider a deterministic function ϕ in the Schwarz space $\mathcal{S}$ with $\text{supp}(\phi) \subseteq [0,T]$ (ϕ will be replaced with a random $u \in \mathcal{U}_0^{X_0}$ below once we have the required machinery in place). Using Plancherel's theorem, we see that

$$\begin{aligned}\langle \phi, \phi\rangle_G &= \int_0^T \phi(t) \int_0^T \phi(s)\, G(|t-s|)dsdt = \int_{-\infty}^{\infty} \phi(t) \int_{-\infty}^{\infty} \phi(s)\, G(|t-s|)dsdt \\ &= \int_{-\infty}^{\infty} \hat{\phi}(k)\overline{\hat{\phi}(k)}\hat{G}(k)dk \\ &= \int_{-\infty}^{\infty} |\hat{\phi}(k)|^2 \hat{G}(k)dk \geq 0\,,\end{aligned}$$

where $\hat{G}(k) = c_\gamma |k|^{\gamma-1}$ is the Fourier transform of G, for some constant $c_\gamma > 0$. Thus $\langle .,.\rangle_G$ is a positive semi-definite bilinear form on $\mathcal{S}$. Using similar arguments to Eq 8 in [FS20], we can also show $\langle .,.\rangle_G$ is continuous on the Schwarz space $\mathcal{S}(\mathbb{R})$. Hence by Minlos's theorem,

$$e^{-\frac{1}{2}\langle \phi,\phi\rangle_G} = \mathbb{E}(e^{i\langle \phi, Z\rangle})$$

is the characteristic functional of the Fractional Gaussian Field (FGF) Z with covariance function $G(|t-s|) = c|t-s|^{-\gamma}$ which lives in the space of tempered distributions $\mathcal{S}'$ (see e.g. pg 8 of Janson[Jan09], and [DRSV17] and Appendix A in [FFGS20] for more details) which is the dual of the Schwartz space $\mathcal{S}$ (see e.g. section 2.2 in [DRSV14] and Theorem 2.1 in [BDW17]). Moreover, $\mathcal{S}$ is a Montel space and thus is reflexive, i.e. $(\mathcal{S}')'$ is isomorphic to $\mathcal{S}$ using the canonical embedding of S into its bi-dual $(S')'$.

Proceeding as in [FS20], we now let $\bar{F}$ denote the Hilbert space equal to the $L^2(\mathcal{S}, \mathcal{F}_T, \mathbb{P})$ closure of

$$F = \{Z(\phi) : \phi \in \mathcal{S},\, \text{supp}(\phi) \subseteq [0,T]\}$$

where $\mathcal{F}_T = \sigma((Z_u)_{0\leq u\leq T})$.

In order to characterize $\bar{F}$, we first note that

$$\mathbb{E}((Z,\phi)^2) = \int_0^T \int_0^T G(|t-s|)\phi(s)\phi(t)dsdt\,.$$

We also know that

$$\int_0^T \int_0^T G(|t-s|)\phi(s)\psi(t)dsdt = \mathbb{E}(\int_{-\infty}^{\infty} \hat{\phi}(k)\bar{\hat{\psi}}(k)\hat{G}(k)dk) = c_\gamma\langle \phi, \psi\rangle_{H^{-\frac{1}{2}(1-\gamma)}}$$

where $\hat{G}(k) = c_\gamma |k|^{\gamma-1}$ for some constant c_γ, and H^s denotes the homogenous fractional Sobolev space of order s (see e.g. page 5 in [DRV12] for definitions). Thus, setting $s = \frac{1}{2}(1-\gamma)$, the following two inner products on the linear space $\mathcal{S}$ of Schwarz functions are equivalent and hence generate the same topologies on $\mathcal{S}$:

1. $\langle \phi, \psi\rangle_{H^{-s}} := \int_{-\infty}^{\infty} |k|^{-2s}\hat{\phi}(k)\bar{\hat{\psi}}(k)dk$ (i.e. the standard inner product on H^{-s})
2. $\langle \phi, \psi\rangle := \mathbb{E}[Z(\phi)Z(\psi)] = \int_0^T \int_0^T \phi(s)\psi(t)G(|t-s|)dsdt$.

We now make the following observations:

- Let $\phi \in H^{-s}$, with $\text{supp}(\phi) \subseteq [0,T]$. $\mathcal{S}$ is dense in H^{-s}, so there exists a sequence $\phi_n \in \mathcal{S}$ with $\text{supp}(\phi_n) \subseteq [0,T]$ such that $\|\phi_n - \phi\|_{H^{-s}} \to 0$, and ϕ is a Cauchy sequence in H^{-s} so (by the equivalence of norms) $Z(\phi_n)$ is a Cauchy sequence in $\bar{F}$, and thus converges to some Y in $\bar{F}$. This defines $Z(\phi) := Y$ as a continuous linear extension of Z from $\mathcal{S}$ to the larger space H^{-s}, which we will also often write as $\int \phi(t)Z_t dt$. To check that $Z(\phi)$ is uniquely specified, consider two such sequences ϕ_n and ϕ_n'. Then from the triangle inequality

  $$\|\phi_n - \phi_n'\|_{H^{-s}} \leq \|\phi_n - \phi\|_{H^{-s}} + \|\phi - \phi_n'\|_{H^{-s}} \to 0$$

  and thus (by the equivalence of norms) we have $\|Z(\phi_n) - Z(\phi_n')\|_{L^2(\mathcal{S},\mathcal{F}_T,\mathbb{P})} = \|Z(\phi_n) - Z(\phi_n')\|_{\bar{F}} \to 0\,.$

- Conversely, for any $Z \in \bar{F}$, there exists a sequence $\phi_n \in \mathcal{S}$ such that $Z(\phi_n)$ converges to $Z \in L^2(\mathcal{S}, \mathcal{F}_T, \mathbb{P})$, so ϕ_n is a Cauchy sequence with respect to the second norm defined above, and hence also a Cauchy sequence with respect to the H^{-s} norm (by the equivalence of the two norms). H^{-s} is a Hilbert space so Cauchy sequences in H^{-s} converge i.e. there exists a ϕ in H^{-s} such that $\phi_n \to \phi \in H^{-s}$.

Thus we have shown that

$$\bar{F} \quad = \quad \{Z(\phi) : \phi \in H^{-s}, \, \mathrm{supp}(\phi) \subseteq [0, T]\},$$

where we are using the extension of Z to H^{-s} on the right hand side here as defined in the first bullet point above. Moreover, we can now extend the inner product to H^{-s} as

$$\langle \phi, \psi \rangle \quad = \quad \lim_{n\to\infty} \mathbb{E}[Z(\phi_n)Z(\psi_n)] \quad = \quad \lim_{n\to\infty} \int_0^T \int_0^T \phi_n(s)\psi_n(t)G(|t-s|)dsdt$$

where $\phi_n, \phi_n \in \mathcal{S}$ and $\phi_n \to \phi$ in H^{-s} and $\psi_n \to \psi$ in H^{-s}.

Finally, to prove the lemma, if $u \in \mathcal{U}_0^{X_0}$ and $\mathbb{E}(\int_0^T \int_0^T u_s u_t G(|t-s|)dsdt) < \infty$, then $\int_0^T \int_0^T u_s u_t G(|t-s|)dsdt < \infty$ a.s., so $u \in H^{-s}$ a.s. Then if we assume the field Z is independent of u then

$$\begin{aligned}
\langle u, v \rangle_G \quad = \quad \mathbb{E}((Z,u)(Z,v)) \quad & \leq \quad \mathbb{E}((Z,u)^2)^{\frac{1}{2}} \, \mathbb{E}((Z,v)^2)^{\frac{1}{2}} \\
& = \quad \mathbb{E}(\mathbb{E}((Z,u)^2|u))^{\frac{1}{2}} \, \mathbb{E}(\mathbb{E}((Z,v)^2)^{\frac{1}{2}}) \\
& = \quad \mathbb{E}(\int_0^T \int_0^T u_s u_t G(|t-s|)dsdt)^{\frac{1}{2}} \, \mathbb{E}(\int_0^T \int_0^T v_s v_t G(|t-s|)dsdt)^{\frac{1}{2}} \\
& < \quad \infty
\end{aligned}$$

as required. ■