Large deviations for the boundary local time of doubly reflected Brownian motion[☆]

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Abstract

We compute a closed-form expression for the moment generating function $\hat{f}(x;\lambda,\alpha)=\frac{1}{\lambda}\mathbb{E}_x(e^{\alpha L_\tau})$, where L_t is the local time at zero for standard Brownian motion with reflecting barriers at 0 and b, and $\tau\sim\mathrm{Exp}(\lambda)$ is independent of W. By analyzing how and where $\hat{f}(x;\cdot,\alpha)$ blows up in λ, a large-time large deviation principle (LDP) for L_t/t is established using a Tauberian result and the Gärtner-Ellis Theorem.

Keywords: Brownian motion, Large deviation, Local time.

1. Introduction

Diffusion processes with reflecting barriers have found many applications in finance, economics, biology, queueing theory, and electrical engineering. In a financial context, we recall the currency exchange rate target-zone models in [KRU91] (see also [SVE91, BER92, DJ94], and [BAL98]), where the exchange rate is allowed the float within two barriers; asset pricing models with price caps (see [HAN99]); interest rate models with targeting by the monetary authority (e.g.[FAR03]); short rate models with reflection at zero (e.g. [GOL97, GOR04]); and stochastic volatility models (most notably the Heston and Schöbel-Zhu models). In queueing theory, diffusions with reflecting barriers arise as heavy-traffic approximations of queueing systems and reflected Brownian motions is ubiquitous in queueing models [HAR85, ABA87a, ABA87b]. More recently, reflected Ornstein-Uhlenbeck(OU) and reflected affine processes have been studied as approximations of queueing systems with reneging or balking [WAR03a, WAR03b]. Applications of reflected OU processes in mathematical biology are discussed in [RIC87]. Doubly reflected Brownian motion also arises naturally in the solution for the optimal trading strategy in the large-time limit for an investor who is permitted to trade a safe and a risky asset under the Black-Scholes model, subject to proportional transaction costs with exponential or power utility (see [GM13] and [GGMS12] respectively).

The asymptotics in this article are obtained using a Tauberian theorem. Tauberian results typically allow us to deduce the large-time or tail behavior of a quantity of interest based on the behavior of its Laplace transform (see Feller[FEL71] or the excellent monograph of Bingham et al.[BGT87] for details or [BF08] for applications to tail asymptotics for time-changed exponential Lévy models). In this article, we compute a closed-form expression for the moment generating function (mgf) $\hat{f}(x;\lambda,\alpha)=\frac{1}{\lambda}\mathbb{E}_x(e^{\alpha L_\tau})$, where L_t is the local time at zero for standard Brownian motion with reflecting barriers at 0 and b, and τ is an independent exponential random variable with parameter λ. We do this by first deriving the relevant ODE and boundary conditions for $\hat{f}(x;\lambda,\alpha)$ using an augmented filtration and computing the optional projection, and we then solve this ODE in closed form. $\hat{f}(x;\lambda,\alpha)$ does not appear amenable to Laplace inversion; however from an analysis of the location of the pole of $\hat{f}(x;\cdot,\alpha)$, we can compute the re-scaled log mgf limit $V(\alpha)=\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x(e^{\alpha L_t})$ for $\alpha\in\mathbb{R}$ using the Tauberian result in Proposition 4.3 in [KOR02] via the so-called Fejér kernel. From this we then establish a large deviation principle for L_t/t as $t\to\infty$ using the Gärtner-Ellis Theorem from large deviations theory,

Throughout the paper, we let $\mathbb{P}_x(\cdot)=\mathbb{P}(\cdot|X_0=x)$ denote the law of X given its initial value at time 0 for any $x\in[0,b]$, and by $\mathbb{E}_x(\cdot)$ the expectation under $\mathbb{P}_x$. Further, we let $\mathbb{E}\equiv\mathbb{E}_0$.

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2. The modelling set up

We begin by defining the Brownian motion X with two reflecting boundaries. Let W_t be standard Brownian motion starting at 0. Then for any $x \in [0,b]$, there is a unique pair of non-decreasing, continuous adapted processes (L,U), starting at 0, such that

$$X_t = x + W_t + L_t - U_t \quad \in \quad [0,b], \qquad \forall t \geq 0.$$

such that L can only increase when $X=0$ and U_t can only increase when $X=b$. Existence and uniqueness follow easily from the more general work of Lions&Sznitman[LS84] the earlier work of Skorokhod[SKO62], or a bare-hands proof can be given by successive applications of the standard one-sided reflection mapping using a sequence of stopping times (see [WIL92].)

It can be shown that

$$\lim_{t\to\infty} L_t/t = \mathbb{E}(L_{\tau^b+\tau'})/\mathbb{E}(\tau^b+\tau'), \qquad \lim_{t\to\infty} U_t/t = \mathbb{E}(U_{\tau^b+\tau'})/\mathbb{E}(\tau^b+\tau'),$$
$$\lim_{t\to\infty} \frac{1}{t}\mathrm{Var}(L_t) = \sigma_L^2, \qquad \lim_{t\to\infty} \frac{1}{t}\mathrm{Var}(U_t) = \sigma_U^2,$$

where $\tau^b = \inf\{t : X_t = b\}$, $\tau' = \inf\{t \geq \tau^b : X_t = 0\}$(see [WIL92]) for some non-negative constants σ_L, σ_U.

Proposition 2.1. *Let τ denote an independent exponential random variable with parameter λ. Then for $\alpha < 0$,*

$$\hat{f}(x) \equiv \hat{f}(x;\lambda,\alpha) := \frac{1}{\lambda}\mathbb{E}_x(e^{\alpha L_\tau}) = \int_0^\infty e^{-\lambda t}\,\mathbb{E}_x(e^{\alpha L_t})dt$$

is smooth on $(0,b)$ and satisfies the following ODE

$$\frac{1}{2}\hat{f}_{xx} = \lambda\hat{f} - 1, \; \hat{f}_x(0) + \alpha\hat{f}(0) = \hat{f}_x(b) = 0. \tag{1}$$

Proof. We first show that $\hat{f} \in C^\infty(0,b)$. To this end, note that for $x \in [0,b]$,

$$\mathbb{E}_x(e^{\alpha L_\tau}) = \mathbb{P}_x(\tau > H_0)\,\mathbb{E}_0(e^{\alpha L_\tau}) + \mathbb{P}_x(\tau \leq H_0)$$

where $H_x = \inf\{t : X_t = x\}$ is the first hitting time to x. The law of $(b - X_t; t \in [0,H_0])$ given $X_t = x$ is the same as that of $(|W_t|; t \in [0,H_b])$ given $|W_0| = b - x$. Thus by Eq. 2.0.1 on page 355 of [BS02] we have

$$\mathbb{P}_x(\tau > H_0) = \mathbb{E}_x(e^{-\lambda H_0}) = \frac{\cosh((b-x)\sqrt{2\lambda})}{\cosh(b\sqrt{2\lambda})}.$$

It follows that

$$\mathbb{E}_x(e^{\alpha L_\tau}) = \frac{\cosh((b-x)\sqrt{2\lambda})}{\cosh(b\sqrt{2\lambda})}\,[\mathbb{E}_0(e^{\alpha L_\tau}) - 1] + 1\,.$$

That is,

$$\hat{f}(x) = \frac{\cosh((b-x)\sqrt{2\lambda})}{\cosh(b\sqrt{2\lambda})}\,(\hat{f}(0) - \frac{1}{\lambda}) + \frac{1}{\lambda}\,, \;\; \forall x \in [0,b]\,. \tag{2}$$

It can then be easily seen from (2) that $\hat{f} \in C^\infty(0,b)$.

To show that $\hat{f}$ satisfies (1) and the boundary conditions, we construct a martingale that is adapted to the filtration generated by X. More specifically, we introduce the natural filtration $\mathcal{F}_t = \sigma(X_s; s \leq t)$ and the augmented filtration $\overline{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(\mathbf{1}_{\{\tau<t\}})$, where $\sigma(\mathbf{1}_{\{\tau<t\}})$ is the sigma algebra generated by $\mathbf{1}_{\{\tau<t\}}$. Then we have a uniformly bounded, and hence uniformly integrable $\overline{\mathcal{F}}_t$-martingale:

$$\overline{M}_t := \mathbb{E}(e^{\alpha L_\tau}|\overline{\mathcal{F}}_t) = \mathbf{1}_{\{\tau<t\}}e^{\alpha L_\tau} + \mathbf{1}_{\{\tau\geq t\}}e^{\alpha L_t}\mathbb{E}_{X_t}(e^{\alpha L_\tau}) = \mathbf{1}_{\{\tau<t\}}e^{\alpha L_\tau} + \mathbf{1}_{\{\tau\geq t\}}e^{\alpha L_t}\lambda\hat{f}(X_t)\,.$$

We now define the optional projection of $\overline{M}_t$: using the fact that X and τ are independent, we have

$$M_t = \mathbb{E}(\overline{M}_t|\mathcal{F}_t) = \lambda\int_0^t e^{\alpha L_s - \lambda s}ds + e^{\alpha L_t - \lambda t}\lambda\hat{f}(X_t)\,.$$

Further, M_t is a $\mathcal{F}_t$-martingale, in that for all $t > s$ we have

$$\mathbb{E}(M_t|\mathcal{F}_s) = \mathbb{E}(\mathbb{E}(\overline{M}_t|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}(\overline{M}_t|\mathcal{F}_s) = \mathbb{E}(\mathbb{E}(\overline{M}_t|\overline{\mathcal{F}}_s)|\mathcal{F}_s) = \mathbb{E}(\overline{M}_s|\mathcal{F}_s) = M_s.$$

Applying Itō's lemma to M_t, we have that

$$dM_t = e^{\alpha L_t - \lambda t}\left[\lambda dt + \lambda \hat{f}(X_t)(\alpha dL_t - \lambda dt) + \frac{1}{2}\lambda \hat{f}_{xx}(X_t)dt + \lambda \hat{f}_x(X_t)(dW_t + dL_t - dU_t)\right].$$

But for M_t to be a martingale, we must have

$$\frac{1}{2}\hat{f}_{xx}(x) - \lambda \hat{f}(x) + 1 = 0, \;\; \hat{f}_x(0) + \alpha \hat{f}(0) = 0, \;\; \hat{f}_x(b) = 0.$$

This completes the proof. □

Solving the ODE in Proposition 2.1 we obtain the following result:

Proposition 2.2.

$$\hat{f}(x;\lambda,\alpha) = \frac{1}{\lambda} + e^{x\sqrt{2\lambda}}A_\lambda(\alpha) \; + \; e^{-x\sqrt{2\lambda}}B_\lambda(\alpha) \tag{3}$$

for $\lambda > 0, \alpha < 0$, where

$$A_\lambda(\alpha) = \frac{\alpha e^{-b\sqrt{2\lambda}}/\cosh(b\sqrt{2\lambda})}{2\lambda\left[\alpha^*(\lambda) - \alpha\right]}, \; B_\lambda(\alpha) = e^{2\sqrt{2\lambda}\,b}A_\lambda(\alpha), \; \alpha^*(\lambda) = \sqrt{2\lambda}\tanh(b\sqrt{2\lambda}). \tag{4}$$

Remark 2.3. *Observe that the expression for $\hat{f}(x)$ involves $\sqrt{\lambda}$, which has a branch point at $\lambda = 0$. However, $\hat{f}$ remains a continuous function across the branch cut at $\lambda = 0$; thus $\hat{f}$ is an analytic function of λ in some punctured disc about $\lambda = 0$. As $\lim_{\lambda\to 0}\lambda \cdot \hat{f}(\lambda) = 0$, we conclude that $\lambda = 0$ is a removable singularity.*

Remark 2.4. *It can be verified that $\alpha^*(\cdot)$ in (4) is a strictly increasing mapping from $[0,\infty)$ onto $[0,\infty)$. Further, we may analytically extend $\alpha^*(\cdot)$ to get a strictly increasing, strictly concave, smooth real-valued function that maps $(-\frac{\pi^2}{8b^2},\infty)$ onto $\mathbb{R}$.*

3. Large-time asymptotics

In this section, we characterize the large-time behaviour of L_t. To this end, let us consider the inverse of α^*, $V(\alpha) := (\alpha^*)^{-1}(\alpha)$ for $\alpha \in \mathbb{R}$. From Remark 2.4, we know that V is a strictly increasing, strictly convex smooth function, with range $(-\frac{\pi^2}{8b^2},\infty)$.

Lemma 3.1. *The equality (3) also holds for all $\alpha \in \mathbb{R}, \lambda \in \mathbb{C}$ such that $\Re(\lambda) > V(\alpha)$.*

Proof. See Appendix A. □

Proposition 3.2. *We have the following large-time behaviour for the moment generating function of L_t:*

$$\lim_{t\to\infty}\frac{1}{t}\log \mathbb{E}_x(e^{\alpha L_t}) = V(\alpha) < \infty \;\; \forall \alpha \in \mathbb{R}.$$

Proof. See Appendix B. □

Remark 3.3. *Note that $V(\cdot)$ does not depend on the starting value x, due to the ergodicity of X.*

The following lemma will be needed in the statement of the large deviation principle in the theorem that follows.

Lemma 3.4. *(a) Define $V^*(x) := \sup_{\alpha\in\mathbb{R}}[\alpha x - V(\alpha)]$ for all $x \geq 0$. Then we have*

$$V^*(x) = \begin{cases} x\alpha^*(\lambda^*) - \lambda^*, & \text{for } x > 0 \\ \pi^2/(8b^2), & \text{for } x = 0 \end{cases}, \tag{5}$$

where $\lambda^* = \lambda^*(x)$ is the unique solution of $(\alpha^*)'(\lambda) = 1/x$ for fixed $x > 0$.

(b) $V^* \in C([0,\infty)) \cap C^1((0,\infty))$ and V^* is a strictly convex function on $(0,\infty)$.

(c) V^* attains its minimum value of zero uniquely at $x^* = \frac{1}{2b}$.

Proof. See Appendix C. □

Theorem 3.5. *L_t/t satisfies a large deviation principle on $[0,\infty)$ as $t\to\infty$ with a strictly convex rate function $V^*(x)$.*

Proof. From Lemma 3.4, we know that V^* is a strictly convex function on $(0,\infty)$. Hence the set of exposed points of V^* is $(0,\infty)$ (see Definition 2.3.3 in [DZ98]), and since $D_V^0=(-\infty,\infty)$, the exposing hyperplane will always lie in D_V^0. Therefore, by the Gärtner-Ellis Theorem (see Theorem 2.3.6 in [DZ98]), L_t/t satisfies the LDP with convex rate function $V^*(x)$. □

Appendix A. Proof of Lemma 3.1

Recall from Propositions 2.1 and 2.2 that, for $\alpha<0$ and $\lambda>0$,

$$\int_0^\infty e^{-\lambda t}\,\mathbb{E}_x(e^{\alpha L_t})dt=\hat{f}(x;\lambda,\alpha)=\frac{1}{\lambda}\,+\,e^{x\sqrt{2\lambda}}A_\lambda(\alpha)\,+\,e^{-x\sqrt{2\lambda}}B_\lambda(\alpha) \tag{B-1}$$

where $A_\lambda(\alpha)=\frac{\alpha e^{-b\sqrt{2\lambda}}/\cosh(b\sqrt{2\lambda})}{2\lambda\,[\alpha^*(\lambda)-\alpha]}$ and $B_\lambda(\alpha)=e^{2\sqrt{2\lambda}\,b}A_\lambda(\alpha)$. We wish to show that (B-1) still holds for a wider range of α and λ values using analytic continuation. We first note that $\hat{f}$ has a singularity when $\alpha=\alpha^*(\lambda)$, and by Theorems 5a and 5b on page 57 in [WID46], we know that the abscissa of convergence for a Laplace transform is a point of singularity and the Laplace transform is analytic in its region of convergence.

We are interested in the values of $\alpha\in\mathbb{R}$ and $\lambda\in\mathbb{C}$ such that the Laplace transform $\hat{f}(x;\lambda,\alpha)=\int_0^\infty e^{-\lambda t}\,\mathbb{E}_x(e^{\alpha L_t})dt$ is finite. We recall the following fact: for any fixed $x\in[0,b]$,
(†) $\hat{f}(x;\lambda,\alpha)<\infty$ for $\alpha<0$ and $\lambda>0$.
We now proceed in three stages:

- Fix $\lambda>0$ (so $\lambda\in\mathbb{R}$). We apply the Widder results with α as the Laplace variable, i.e. we consider

  $$\mathbb{E}(e^{\alpha L_\tau})=\int_0^\infty e^{\alpha y}dF(y)\,,$$

  where $F(y)$ is the distribution function of L_τ. By (†), the region of convergence is non-empty. We can then extend the region of convergence up to $\alpha^*(\lambda)>0$, as $\alpha^*(\lambda)$ is the point of singularity.

- Fix $\alpha<0$. We apply the Widder results again, but we now take λ as the Laplace variable. By (†), the region of convergence is non-empty. According to Widder, the abscissa of convergence (say λ_c) is a point of singularity and $\hat{f}(x;\lambda,\alpha)$ is analytic in λ when $\Re(\lambda)>\lambda_c$. So we are looking at a point of singularity on the real line, and this is the value of λ_c that satisfies $\alpha^*(\lambda_c)=\alpha$. Or, in other words, $\lambda_c=(\alpha^*)^{-1}(\alpha)=V(\alpha)$. Thus, by Widder, $\hat{f}(x;\lambda,\alpha)$ is finite when $\Re(\lambda)>V(\alpha)$.

- Fix $\alpha>0$. We apply Widder's theorem using λ as the Laplace variable. By the first bullet point, we know that there exists some $\lambda\in\mathbb{R}$ (such that $\alpha<\alpha^*(\lambda)$), for which $\hat{f}(x;\lambda,\alpha)$ is finite. Hence, the region of convergence of $\hat{f}(x;\lambda,\alpha)$ is non-empty for this α. Then, by Widder, the abscissa of convergence λ_c is a point of singularity and $\hat{f}(x;\lambda,\alpha)$ is analytic for $\Re(\lambda)>\lambda_c$. The singularity is at $\alpha=\alpha^*(\lambda)$. Solving for points of singularity on the real line i.e. solving for λ_c in $\alpha=\alpha^*(\lambda_c)$, gives us $\lambda_c=V(\alpha)$ and so $\hat{f}(x;\lambda,\alpha)$ converges when $\Re(\lambda)>\lambda_c=V(\alpha)$.

This gives the region of λ and α for which $\hat{f}(x;\lambda,\alpha)$ converges: for every $\alpha\in\mathbb{R}$ and $\lambda\in\mathbb{C}$ such that $\Re(\lambda)>V(\alpha)$, and $\hat{f}(x;\lambda,\alpha)$ is analytic in this region.

Appendix B. Proof of Proposition 3.2

From the known large-time behaviour of the local time of standard Brownian motion, we expect that $\mathbb{E}_x(e^{\alpha L_t})\sim$ *const.* $\times\,e^{U(\alpha)t}$ as $t\to\infty$, for some non-decreasing function $U(\alpha)$ to be determined. Then as $t\to\infty$,

$$\hat{f}(x;\lambda,\alpha)=\int_0^\infty e^{-\lambda t}\,\mathbb{E}_x(e^{\alpha L_t})dt\sim\int_0^\infty e^{-\lambda t}\,\mathit{const.}\times e^{U(\alpha)t}dt, \tag{C-1}$$

and $\hat{f}(x;\lambda,\alpha)$ blows up when $\lambda = U(\alpha)$ (for α fixed). But we know that $\hat{f}(x;\lambda,\alpha)$ blows up at $\alpha = \alpha^*(\lambda)$; thus we expect that $\lambda = U(\alpha^*(\lambda))$, i.e. $U(\alpha) = (\alpha^*)^{-1}(\alpha) = V(\alpha)$. We now make this statement rigorous using a variant of Ikehara's Tauberian Theorem (see e.g. Theorem 17 on page 233 in Widder[WID46]).

We first define a positive function v on $\mathbb{R}$:

$$v(t) \equiv v(t;x,\alpha) := \mathbf{1}_{t\geq 0}\, e^{-V(\alpha)t}\mathbb{E}_x(e^{\alpha L_t}).$$

Then the Laplace transform of v is given by

$$\hat{v}(\lambda) = \int_0^\infty e^{-\lambda t} v(t)dt = \int_0^\infty e^{-(\lambda+V(\alpha))t}\mathbb{E}_x(e^{\alpha L_t})dt = \hat{f}(x;\lambda+V(\alpha),\alpha)\,,$$

which, by Lemma 3.1 is analytic for all $\lambda \in \mathbb{C}$ such that $\Re(\lambda) > 0$. We now need to characterize how $\hat{v}(\lambda)$ blows up as $\Re(\lambda) \downarrow 0$. To this end, looking at the expression for $A_\lambda(\alpha)$, we notice that $A_\lambda(\alpha)$ has a pole at $\lambda = V(\alpha) \in (-\frac{\pi^2}{8b^2},\infty)$, and is analytic elsewhere for $\Re(\lambda) > -\frac{\pi^2}{8b^2}$ (see Remarks 2.3 and 2.4). It is also easily seen that, $\alpha^{*\prime}(\lambda) > 0$ for all $\lambda \in (-\frac{\pi^2}{8b^2},\infty)$. Hence, by the Laurent expansion of $\hat{v}(\lambda)$ at 0, there exists a function $g(\lambda)$, which is analytic for all $\lambda \in \mathbb{C}$ with $\Re(\lambda) > -\varepsilon$ and $|\Im(\lambda)| \leq c$ for some constants $\varepsilon, c > 0$, such that

$$\hat{v}(\lambda) = \frac{C}{\lambda} + g(\lambda)$$

for some constant C which we find to be positive (C is the residue of $\hat{v}$ at $\lambda = 0$). $g(x+iy)$ is continuous on $\mathcal{D} := \{(x,y) : |x| \leq \varepsilon, |y| \leq c\}$, thus $g(x+iy)$ is uniformly continuous on $\mathcal{D}$, so $g(x+iy) \to g(iy)$ uniformly as $x \downarrow 0$ for any fixed $y \in [-c,c]$. Moreover, for any $x > 0$

$$\int_{-c}^{c} \left|\hat{v}(x+iy) - \frac{C}{x+iy} - g(iy)\right| dy = \int_{-c}^{c} |g(x+iy) - g(iy)|dy$$

Since g is analytic everywhere and uniformly continuous, if we take the limit as $x \to 0$, the above integral converges to 0, so the function $g(x+i\cdot)$ also converges to $g(i\cdot)$ in $\mathbb{L}^1([-c,c])$, as $x \downarrow 0$.

We can now apply Proposition 4.3 in [KOR02] to obtain that for the "Fejér kernel" $K(t) = \frac{1-\cos t}{\pi t^2}$,

$$\lim_{t\to\infty} \int_{-\infty}^{ct} v(t-\frac{s}{c}) \cdot K(s)ds \;=\; C. \tag{C-2}$$

We now proceed as in the proof of Theorem 4.2 in [KOR02] to show that $v(t) = O(1)$ as $t \to \infty$.

1. $\alpha > 0$. In this case we know that $\mathbb{E}_x(e^{\alpha L_t})$ is non-decreasing, so $v(t) \geq v(s)e^{V(\alpha)(s-t)}$ for all $t \geq s \geq 0$. For any fixed $a > 0$, using (C-2) we have that

$$C = \lim_{t\to\infty}\int_{-\infty}^{ct} v(t-\frac{s}{c})\cdot K(s)ds \geq \limsup_{t\to\infty}\int_{-a}^{a} v(t-\frac{s}{c})\cdot K(s)ds \geq \limsup_{t\to\infty} v(t-\frac{a}{c})\, e^{-2V(\alpha)\frac{a}{c}} \int_{-a}^{a} K(s)ds\,,$$

which implies that

$$\limsup_{t\to\infty} v(t) \leq \frac{e^{2V(\alpha)\frac{a}{c}}}{\int_{-a}^{a} K(s)ds}\, C \;<\; \infty\,.$$

Hence, there exists a constant $M > 0$ such that $v(t) \leq M$ for all t. Similarly, for any fixed $a > 0$, we have

$$\begin{aligned}\liminf_{t\to\infty} v(t+\frac{a}{c})\, e^{2V(\alpha)\frac{a}{c}} \int_{-a}^{a} K(s)ds &\geq \liminf_{t\to\infty}\int_{-a}^{a} v(t-\frac{s}{c})K(s)ds \\ = \liminf_{t\to\infty}\Big(\int_{-\infty}^{ct} + \int_{ct}^{\infty} - \int_{-\infty}^{-a} - \int_{a}^{\infty}\Big)v(t-\frac{s}{c})K(s)ds &\geq \liminf_{t\to\infty}\Big(\int_{-\infty}^{ct} + \int_{ct}^{\infty}\Big)v(t-\frac{s}{c})K(s) \\ - \limsup_{t\to\infty}\int_{-\infty}^{-a} v(t-\frac{s}{c})K(s) \;-\; \limsup_{t\to\infty}\int_{a}^{\infty} v(t-\frac{s}{c})K(s) &\geq C \;-\; \frac{4M}{\pi}\int_a^\infty \frac{1}{s^2}ds = C \;-\; \frac{4M}{\pi a}\,,\end{aligned}$$

where we have used (C-2) and the fact that $0 \leq K(t) \leq \frac{2}{\pi t^2}$ in the last inequality. Hence, for $a > 0$ sufficiently large, we have

$$\liminf_{t\to\infty} v(t) \geq \frac{e^{-2V(\alpha)\frac{a}{c}}}{\int_{-a}^{a} K(s)ds}\,(C - 4M/\pi a) \;\;>\;\; 0.$$

2. $\alpha < 0$. In this case we know that $\mathbb{E}_x(e^{\alpha L_t})$ is non-increasing, so $v(t) \le v(s)e^{V(\alpha)(s-t)}$ for all $t \ge s \ge 0$. Using the same argument as above, we have, for any fixed $a > 0$,

$$\begin{aligned} Ce^{2V(\alpha)\frac{a}{c}} &\ge \limsup_{t\to\infty} v(t + \frac{a}{c}) \int_{-a}^{a} K(s)ds, \\ (C - \frac{4M}{\pi a})\, e^{-2V(\alpha)\frac{a}{c}} &\le \liminf_{t\to\infty} v(t - \frac{a}{c}) \int_{-a}^{a} K(s)ds\,. \end{aligned}$$

Hence for $a > 0$ sufficiently large, we have

$$0 \le \frac{e^{-2V(\alpha)\frac{a}{c}}}{\int_{-a}^{a} K(s)ds}(C - 4M/\pi a) \le \liminf_{t\to\infty} v(t) \le \limsup_{t\to\infty} v(t) \le \frac{e^{2V(\alpha)\frac{a}{c}}}{\int_{-a}^{a} K(s)ds} C < \infty\,.$$

Hence, by Proposition 4.3 in [KOR02], the result follows.

Appendix C. Proof of Lemma 3.4

We break the proof into three parts:

(a) Computing the Legendre transform of V boils down to solving $V'(\alpha) = x$. But this is the same as solving $(V^{-1})'(\lambda) = \frac{1}{x}$ for λ, when $x > 0$. Recall that $V^{-1}(\cdot) = \alpha^*(\cdot)$ is known in closed form. Since $(\alpha^*)''(\lambda) < 0$ for all λ in the domain of α^*, i.e. $\lambda > -\frac{\pi^2}{8b^2}$ (from Remark 2.4), by the Inverse function theorem, $\lambda^*(x) := ((\alpha^*)')^{-1}(1/x)$ is well-defined and $\lambda^* \in C^1((0,\infty))$. Using the fact that $\alpha^*(\lambda^*) = V^{-1}(\lambda^*)$, we have

$$V^*(x) = x\alpha^* - V(\alpha^*) = x\alpha^*(\lambda^*(x)) - \lambda^*(x)\,.$$

When $x = 0$, the definition of V^* in Lemma 3.4 gives us $V^*(0) = \sup_{\alpha\in\mathbb{R}}\{-V(\alpha)\} = -\inf_{\alpha\in\mathbb{R}}\{V(\alpha)\} = -\lim_{\alpha\to-\infty} V(\alpha) = \pi^2/(8b^2)$, where the last two equalities hold because V is a monotonically increasing function with range $(-\pi^2/(8b^2), \infty)$.

(b) By the Inverse function theorem, we know that $\lambda^* \in C^1((0,\infty))$ and so is α^*, thus $V^* \in C^1((0,\infty))$. It is easy to check that $\lim_{x\downarrow 0}\{x\alpha^*(\lambda^*(x)) - \lambda^*(x)\} = \pi^2/(8b^2) = V^*(0)$, which gives continuity of V^* up to the boundary $x = 0$. Using (5), we obtain

$$\begin{aligned} (V^*)'(x) &= \alpha^*(\lambda^*(x)) + x\cdot(\alpha^*)'(\lambda^*(x))\cdot(\lambda^*)'(x) - (\lambda^*)'(x) \\ &= \alpha^*(\lambda^*(x)) + x\cdot\frac{1}{x}\cdot(\lambda^*)'(x) - (\lambda^*)'(x) = \alpha^*(\lambda^*(x)) \end{aligned}$$

$(V^*)'(x) = \alpha^*(\lambda^*(x))$. Thus we have (using again $(\alpha^*)'' < 0$)

$$(V^*)''(x) = (\alpha^*)'(\lambda^*(x))\cdot(\lambda^*)'(x) = \frac{1}{x}\cdot((\alpha^*)'^{-1})'(\frac{1}{x})\cdot\left(-\frac{1}{x^2}\right) = -\frac{1}{x^3}\cdot\frac{1}{(\alpha^*)''(\lambda^*(x))} > 0\,.$$

(c) Since V^* is strictly convex, it has a unique minimum. The unique minimum of V^* occurs at $x^* = ((V^*)')^{-1}(0) = V'(0) = 1/\alpha^{*\prime}(0) = \frac{1}{2b}$

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