THE DERIVATION OF THE FREE ENERGY OF THE ISING MODEL FROM THAT OF THE EIGHT-VERTEX MODEL*

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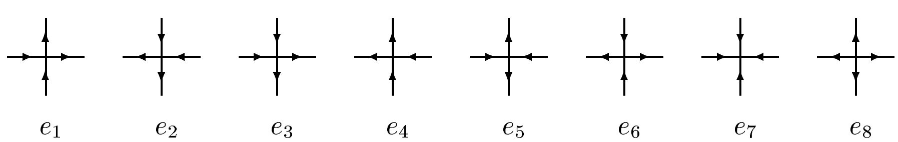
Abstract

The zero-field eight-vertex model is equivalent to a square lattice Ising model, with a four-spin coupling and second neighbour coupling but no nearest neighbour coupling. When the four-spin coupling is zero the model reduces to two decoupled nearest neighbour Ising models. The standard formula for the free energy of the Ising model is, therefore, implicit in Baxter's expression for the free energy of the zero-field eight-vertex model. The detailed derivation of this result has not appeared in print. As a footnote to Baxter's work this paper provides the necessary analysis.

1 Introduction

The name *vertex model* is used to denote a lattice model in which the microstates are represented by putting an arrow on each line connecting a pair of nearest-neighbour sites. Such models can be constructed on any lattice, but those for the plain square lattice have received the greatest attention. The most general model of this type is the sixteen-vertex model, where the different vertex types correspond to all possible directions of the arrows on the four edges meeting at a vertex. This model, which can be shown [1] to be equivalent to an Ising model with two, three and four-site interactions and with an external field, is unsolved. The eight-vertex model corresponds to the case where the vertex types are restricted to those with an even number of arrows pointing in and out. The vertices in this case, with their corresponding energies, are:



The six-vertex model, where the vertices 7 and 8 are eliminated by setting $e_7 = e_8 = \infty$, is the situation when the same number of arrows point in and out. This restriction is called the *ice rule*. The ground-state entropy of this model was obtained

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by Lieb [2], who also derived the free energy [3, 4, 5]. The conditions

$$e_1 = e_2, \qquad e_3 = e_4, \qquad e_5 = e_6, \qquad e_7 = e_8 \tag{1}$$

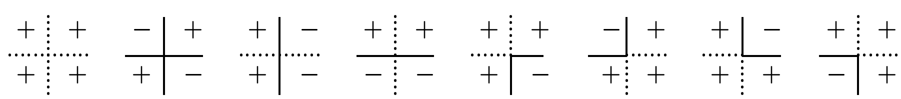
correspond to to the situation when the vertex energies are unaltered when all the arrows are reversed. By analogy with a ferroelectric model this is referred to as the *zero-field* case. Now the model has four independent Boltzmann factors

$$\begin{aligned} \exp(-e_1/k_{\mathrm{B}}T) &= \exp(-e_2/k_{\mathrm{B}}T) = \mathrm{a}, \\ \exp(-e_3/k_{\mathrm{B}}T) &= \exp(-e_4/k_{\mathrm{B}}T) = \mathrm{b}, \\ \exp(-e_5/k_{\mathrm{B}}T) &= \exp(-e_6/k_{\mathrm{B}}T) = \mathrm{c}, \\ \exp(-e_7/k_{\mathrm{B}}T) &= \exp(-e_8/k_{\mathrm{B}}T) = \mathrm{d}. \end{aligned} \tag{2}$$

In section 2 the spin formulation of the zero-field eight-vertex model is described and in section 3 Baxter's derivation of the free energy is summarized. The reduction of that formula to that of the free energy of the Ising model is presented in section 4. Since this work is dependent on the properties of elliptic functions the relevant formulae are given in an appendix together with the derivation of some crucial nome series.

2 The equivalent spin model

Given a particular arrow configuration of the eight-vertex model, configuration graphs are drawn consisting of lines on all bonds with arrows pointing to the left or downwards. The restriction of vertex types to those of the eight-vertex model means that an even number of lines are incident at each vertex; the configuration graphs are polygons. If an Ising ($s = \frac{1}{2}$) spin is placed at the centre of each face then the spin sites form another plane square lattice, which is the dual of the original lattice. If the spins contained within each polygon graph are aligned in the same direction, which is different from that of the neighbouring regions, then to every vertex configuration there are two spin configurations. One of the two equivalent relationships between vertices, bonds and spins is



1 2 3 4 5 6 7 8

With a four-spin coupling $-K_4$, a bottom-left top-right coupling $-K-K'$, a bottom-right top-left coupling $-K+K'$ and a trivial coupling $-K_0$ we have the identification

$$\begin{aligned} \mathrm{a} &= \exp\left(K_0 + 2K + K_4\right), & \mathrm{b} &= \exp\left(K_0 - 2K + K_4\right), \\ \mathrm{c} &= \exp\left(K_0 + 2K' - K_4\right), & \mathrm{d} &= \exp\left(K_0 - 2K' - K_4\right) \end{aligned} \tag{3}$$

between vertex and spin Boltzmann factors [6]. Because of the two spin configurations corresponding to each vertex configuration the spin partition function is twice the partition function of the zero-field eight-vertex model.

When $K > K'$ and $K_4 \geq 0$ the spin model is ferromagnetic and $\mathrm{a} \geq \mathrm{b}, \mathrm{c}, \mathrm{d}$. Using the weak-graph transformation [7], it can be shown that, when these conditions apply, the zero-field eight-vertex model has a transition surface

$$\mathrm{a} = \mathrm{b} + \mathrm{c} + \mathrm{d}. \tag{4}$$

In terms of the spin couplings this takes the form

$$\Theta(K, K', K_4) = 1, \tag{5}$$

where

$$\Theta(K,K',K_4) = \tanh(2K_4)\cosh^2(2K) + \sinh^2(2K) + \sinh^2(2K')[\tanh(2K_4)-1]. \quad (6)$$

When $K' = K_4 = 0$ (5) and (6) give the well-known formula $\sinh(2K_c) = 1$ for the critical coupling of the simple Ising model. When $K' = 0$ the system is isotropic; the critical curve in the K–K_4 plane cuts the K axis at K_c. When $K \to 0$, $K_4 \to \infty$, showing that there is no phase transition in a spin-$\frac{1}{2}$ model with a purely four-spin interaction. The critical exponents α and β, given by

$$\alpha = 2 - \frac{\pi}{\pi - \arccos\{\tanh(2K_4)\}}, \qquad \beta = \frac{\pi}{16[\pi - \arccos\{\tanh(2K_4)\}]}, \quad (7)$$

[8] vary as functions of the four-spin coupling. This result is consistent with scaling theory only if K_4 is a marginal coupling. That this is indeed the case follows from the work of Kadanoff and Wegner [6] who established that K_4 scales as $1/r^2 \equiv 1/r^d$. (The scaling dimension of K_4 is the physical dimension, so the scaling exponent is zero.)

3 The free energy of the eight-vertex model

An exact expression for the free energy of the zero-field eight-vertex model was derived by Baxter [8, 9, 10] from the largest eigenvalue of the appropriate transfer matrix. The first step of Baxter's analysis was to obtain conditions under which transfer matrices with different values of a, b, c and d commute. This procedure is most easily represented in terms of a set of new variables w_1, w_2, w_3 and w_4 which satisfy the conditions

$$w_1 \geq w_2 \geq w_3 \geq |w_4| \quad (8)$$

The case $a > b, c, d$ has the spin representation described above and the critical temperature T_c, with $K = J/k_B T$, $K' = J'/k_B T$, $K_4 = J_4/k_B T$, is given as the solution of (4) or (5). Now the variables w_1, w_2, w_3 and w_4 are defined by

$$w_1 = \tfrac{1}{2}(a+b), \qquad w_2 = \begin{cases} \tfrac{1}{2}(a-b), & T < T_c, \\ \tfrac{1}{2}(c+d), & T > T_c, \end{cases}$$

$$w_3 = \begin{cases} \tfrac{1}{2}(c+d), & T < T_c, \\ \tfrac{1}{2}(a-b), & T > T_c, \end{cases} \qquad w_4 = \tfrac{1}{2}(c-d). \quad (9)$$

and the critical temperature is given by $w_2 = w_3$. Baxter [8, 9, 10] expressed the transfer matrix in terms of Pauli matrices and showed that two transfer matrices, corresponding to two different sets of w_i values, commute when the ratios $(w_j^2 - w_k^2)/(w_l^2 - w_m^2)$, where (j,k,l,m) is any permutation of $(1,2,3,4)$, are the same for both sets of w_i. In fact it is not difficult to see that only two ratios of this form can be chosen independently. Let

$$\frac{w_1^2 - w_4^2}{w_3^2 - w_2^2} = \frac{1+\Delta}{1-\Delta}, \qquad \frac{w_1^2 - w_2^2}{w_3^2 - w_4^2} = \frac{1+\Gamma}{1-\Gamma}. \quad (10)$$

From (8) and (10)

$$\Delta \geq 1, \qquad -1 \leq \Gamma \leq 1. \quad (11)$$

Transfer matrices with the same values of Δ and Γ commute. For fixed values of Δ and Γ there remains one degree of freedom in the ratios of the values of $w_1, \ldots, w_4$.

Using the Jacobian elliptic functions $\mathrm{cn}(\cdot|\cdot)$, $\mathrm{sn}(\cdot|\cdot)$ and $\mathrm{dn}(\cdot|\cdot)$ (see Gradshteyn and Ryzhik [11] and the appendix below), let

$$w_1 : w_2 : w_3 : w_4 = \frac{\mathrm{cn}(\mathsf{U}|\ell)}{\mathrm{cn}(\zeta|\ell)} : \frac{\mathrm{dn}(\mathsf{U}|\ell)}{\mathrm{dn}(\zeta|\ell)} : 1 : \frac{\mathrm{sn}(\mathsf{U}|\ell)}{\mathrm{sn}(\zeta|\ell)}, \tag{12}$$

with ζ, U and ℓ real numbers and $0 \le \ell \le 1$. It can now be shown using the formulae (A5) that all the ratios $(w_j^2 - w_k^2)/(w_l^2 - w_m^2)$ are functions of ζ and ℓ, but not of U and that

$$\Delta = \frac{1}{\mathrm{dn}(2\zeta|\ell)}, \qquad\qquad \Gamma = -\frac{\mathrm{cn}(2\zeta|\ell)}{\mathrm{dn}(2\zeta|\ell)}, \tag{13}$$

with the inverse relations

$$\mathrm{sn}^2(\zeta|\ell) = \frac{\Delta + \Gamma}{\Delta + 1}, \qquad\qquad \ell^2 = \frac{\Delta^2 - 1}{\Delta^2 - \Gamma^2}. \tag{14}$$

So when ζ and ℓ are fixed, transfer matrices with different values of U commute. In terms of these variables the dimensionless free energy ϕ per lattice site, is given [8, 9, 10] by

$$\phi(w_1, w_2, w_3, w_4) = -\ln(w_1 + w_2) - \sum_{n=1}^{\infty} \frac{(x^{2n} - q^n)^2 (x^n + x^{-n} - z^n - z^{-n})}{n x^n (1 - q^{2n})(1 + x^{2n})}, \tag{15}$$

where

$$x = \exp\{-\pi\zeta/\mathcal{K}(\ell')\}, \qquad q = \exp\{-2\pi\mathcal{K}(\ell)/\mathcal{K}(\ell')\}, \quad z = \exp\{-\pi\mathsf{U}/\mathcal{K}(\ell')\}. \tag{16}$$

The variable q is the nome of ℓ' defined by (A9).

4 The free energy of the Ising model

From (3), (6), (9) and (10)

$$\Delta = \begin{cases} \Theta(K, K', K_4) & T < T_\mathrm{c}, \\ 1/\Theta(K, K', K_4) & T > T_\mathrm{c}, \end{cases} \tag{17}$$

$$\Gamma = \begin{cases} \tanh(2K_4) & T < T_\mathrm{c}, \\ \tanh(2K_4)/\Theta(K, K', K_4) & T > T_\mathrm{c}. \end{cases} \tag{18}$$

The transition surface corresponds to $\Delta = 1$. We now consider the case where $K' = K_4 = 0$, when, from (3), (9), (12), (13) and (18),

$$\begin{array}{llll} w_4 &= 0, & \mathsf{U} &= 0, \\ \Gamma &= 0, & \zeta &= \frac{1}{2}\mathcal{K}(\ell). \end{array} \tag{19}$$

From (6), (14) and (17)

$$\ell' = \begin{cases} \sinh^2(2K), & T < T_\mathrm{c}, \\ \sinh^{-2}(2K), & T > T_\mathrm{c}, \end{cases} \tag{20}$$

and from (16) and (19)

$$z = 1, \qquad x = q^{\frac{1}{4}} = \exp\{-\pi\mathcal{K}(\ell)/2\mathcal{K}(\ell')\}. \tag{21}$$

Using the transformation

$$k_1 = \frac{2\sqrt{\ell'}}{1 + \ell'}, \qquad k_1' = \frac{1 - \ell'}{1 + \ell'}, \tag{22}$$

it follows, from (A15) and (A16), that

$$\mathcal{K}(k_1) = (1+\ell')\mathcal{K}(\ell'), \qquad \mathcal{K}(k_1') = \tfrac{1}{2}(1+\ell')\mathcal{K}(\ell). \tag{23}$$

From (21)

$$x = \exp\{-\pi\mathcal{K}(k_1')/\mathcal{K}(k_1)\} \tag{24}$$

is the nome of k_1. From (3), (9), (15), (21) and (24)

$$\phi(K) = \begin{cases} -2K - \displaystyle\sum_{n=1}^{\infty} h_n(x), & T < T_{\mathrm{c}},\ (K > K_{\mathrm{c}}), \\ -\ln[\cosh(2K)+1] - \displaystyle\sum_{n=1}^{\infty} h_n(x), & T > T_{\mathrm{c}},\ (K < K_{\mathrm{c}}), \end{cases} \tag{25}$$

where

$$h_n(x) = \frac{x^{2n}(1-x^{2n})^2(1-x^n)^2}{n(1-x^{8n})(1+x^{2n})}. \tag{26}$$

Since

$$\frac{u}{J} = \frac{\partial\phi}{\partial K}, \tag{27}$$

it follows from (20), (25) and (A10) that

$$\frac{u}{J} = \begin{cases} -2 + \dfrac{B(K)}{\mathcal{K}^2(k_1)} \displaystyle\sum_{n=1}^{\infty} x\frac{\mathrm{d}h_n(x)}{\mathrm{d}x}, & T < T_{\mathrm{c}}, \\ -\dfrac{2\sinh(2K)}{1+\cosh(2K)} + \dfrac{B(K)}{\mathcal{K}^2(k_1)} \displaystyle\sum_{n=1}^{\infty} x\frac{\mathrm{d}h_n(x)}{\mathrm{d}x}, & T > T_{\mathrm{c}}, \end{cases} \tag{28}$$

where

$$B(K) = \frac{\pi^2\cosh^3(2K)}{\sinh(2K)[\sinh^2(2K)-1]} \tag{29}$$

From (26)

$$\begin{aligned} x\frac{\mathrm{d}h_n(x)}{\mathrm{d}x} \quad = \quad & \frac{x^n}{1+x^{2n}} + \frac{2x^{2n}}{(1+x^{2n})^2} - \frac{8x^{3n}}{(1+x^{2n})^3} \\ & -\frac{x^n(1+x^{2n})}{1+x^{4n}} - \frac{4x^{4n}}{(1+x^{4n})^2} + \frac{4x^{3n}(1+x^{2n})}{(1+x^{2n})^3} \end{aligned} \tag{30}$$

and hence, from equations (A12)–(A14), (A17)–(A19),

$$\sum_{n=1}^{\infty} x\frac{\mathrm{d}h_n(x)}{\mathrm{d}x} = \frac{k_1'\mathcal{K}^2(k_1)}{\pi^2}\left\{\sqrt{2(1+k_1')}-1\right\} - \frac{2{k_1'}^2\mathcal{K}^3(k_1)}{\pi^3}. \tag{31}$$

Since, from (20) and (22),

$$k_1 \quad = \quad 2\sinh(2K)\mathrm{sech}^2(2K), \tag{32}$$

$$k_1' \quad = \quad \pm\left\{\frac{1-\sinh^2(2K)}{\cosh^2(2K)}\right\}, \qquad T \lessgtr T_{\mathrm{c}}, \tag{33}$$

by substituting from (32) and (33) into (31) and then into (28) we obtain, after substitution from (A4) for the integral form of $\mathcal{K}(k_1)$,

$$u = -J\coth(2K)\left[1+\frac{2}{\pi}\{2\tanh^2(2K)-1\}\int_0^{\frac{1}{2}\pi}\frac{\mathrm{d}\psi}{\sqrt{1-k_1^2\sin^2\psi}}\right], \tag{34}$$

which is the formula for the internal energy of the Ising model on the plane square lattice at all temperatures [5]. Using (32) it is now straightforward to show that

$$\phi(K) = \ln[\cosh(2K)] + \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\ln\left\{\tfrac{1}{2}\left(1+\sqrt{1-k_1^2\sin^2\psi}\right)\right\}\mathrm{d}\psi \tag{35}$$

satisfies (27) and (34). The expression (35) is one of the many equivalent forms [5] for the free energy of the plane square lattice Ising model.

Appendix

An advanced treatment of elliptic functions is given by Baxter [8] and a comprehensive list of formulae is given by Gradshteyn and Ryzhik [11]. Complete elliptic integrals of the first and second kind are denoted by $\mathcal{K}(m)$ and $\mathcal{E}(m)$ respectively. They satisfy the formulae

$$\mathcal{K}(0) = \mathcal{E}(0) = \tfrac{1}{2}\pi, \qquad \mathcal{E}(1) = 1, \tag{A1}$$

$$\mathcal{E}(m)\mathcal{K}(m') + \mathcal{E}(m')\mathcal{H}(m) - \mathcal{K}(m)\mathcal{K}(m') = \tfrac{1}{2}\pi, \tag{A2}$$

$$\frac{\mathrm{d}\mathcal{K}(m)}{\mathrm{d}m} = \frac{\mathcal{E}(m)-m'^2\mathcal{K}(m)}{mm'^2}, \qquad \frac{\mathrm{d}\mathcal{K}(m')}{\mathrm{d}m} = \frac{m^2\mathcal{K}(m')-\mathcal{E}(m')}{mm'^2} \tag{A3}$$

and we shall also need the integral form

$$\mathcal{K}(m) = \int_0^{\frac{1}{2}\pi}\frac{\mathrm{d}\psi}{\sqrt{1-m^2\sin^2\psi}}. \tag{A4}$$

The Jacobian elliptic functions $\mathrm{sn}(\mathfrak{u}|m)$, $\mathrm{cn}(\mathfrak{u}|m)$ and $\mathrm{dn}(|m)$ are related by

$$\mathrm{cn}(\mathfrak{u}|m) = \sqrt{1-\mathrm{sn}^2(\mathfrak{u}|m)}, \qquad \mathrm{dn}(u|m) = \sqrt{1-m^2\mathrm{sn}^2(u|m)}. \tag{A5}$$

The elliptic functions are doubly-periodic ([11], p.909). For real $\mathfrak{u}$, $\mathrm{sn}(\mathfrak{u}|m)$ and $\mathrm{cn}(\mathfrak{u}|m)$ have period $4\mathcal{K}(m)$ and $\mathrm{dn}(\mathfrak{u}|m)$ has period $2\mathcal{K}(m)$. For present purposes it is the elliptic function $\mathrm{dn}(\mathfrak{u}|m)$ which is of particular importance. We shall need the following results

$$\int_0^{2\mathcal{K}(m)}\mathrm{d}\mathfrak{u}\,\mathrm{dn}^2(\mathfrak{u}|m) = 2\mathcal{E}(m), \tag{A6}$$

$$\int_0^{2\mathcal{K}(m)}\mathrm{d}\mathfrak{u}\int_0^{2\mathcal{K}(m)}\mathrm{d}\mathfrak{v}\,\mathrm{dn}(\mathfrak{u}|m)\mathrm{dn}(\mathfrak{u}-\mathfrak{v}|m)\mathrm{dn}(\mathfrak{v}|m) = 2m'^2\mathcal{K}^2(m) + \tfrac{1}{2}\pi^2, \tag{A7}$$

$$\int_0^{2\mathcal{K}(m)}\mathrm{d}\mathfrak{u}\,\mathrm{dn}(\mathfrak{u}|m)\mathrm{dn}(\mathfrak{u}+\tfrac{1}{2}\imath\mathcal{K}(m')|m) = \tfrac{1}{2}\sqrt{1+m}\{\pi+2(1-m)\mathcal{K}(m)\}. \tag{A8}$$

Formula (A6) is straightforward to prove using (A5), the substitution $x = \mathrm{sn}(\mathfrak{u}|m)$ and the definition of an elliptic integral of the second kind. The results (A7) and

(A8) are more difficult to prove and need the use of the formulae for $\mathrm{dn}(\mathfrak{u} \pm \mathfrak{v}|m)$, ([11], p.916). The *nome* x of m is defined by

$$x = \exp\{-\pi\mathcal{K}(m')/\mathcal{K}(m)\}. \tag{A9}$$

From (A2) and (A3),

$$\frac{\mathrm{d}x}{\mathrm{d}m} = \frac{x\pi^2}{2\mathcal{K}^2(m)mm'^2}. \tag{A10}$$

In terms of x,

$$\mathrm{dn}(\mathfrak{u}|m) = \frac{\pi}{2\mathcal{K}(m)} + \frac{2\pi}{\mathcal{K}(m)}\sum_{n=1}^{\infty}\frac{x^n}{1+x^{2n}}\cos\left\{\frac{n\pi\mathfrak{u}}{\mathcal{K}(m)}\right\}, \tag{A11}$$

([11], p.911). Substituting $\mathfrak{u} = 0$ into (A11) and using $\mathrm{dn}(0|m) = 1$ gives

$$\sum_{n=1}^{\infty}\frac{x^n}{1+x^{2n}} = \frac{\mathcal{K}(m)}{2\pi} - \frac{1}{4}. \tag{A12}$$

By substituting from (A11) into (A6) and noting that the cosines in (A11) are orthogonal over the range of this integral, it follows that

$$\sum_{n=1}^{\infty}\frac{x^{2n}}{(1+x^{2n})^2} = \frac{\mathcal{E}(m)\mathcal{K}(m)}{2\pi^2} - \frac{1}{8}. \tag{A13}$$

In a similar way, by substituting from (A11) into (A7),

$$\sum_{n=1}^{\infty}\frac{x^{3n}}{(1+x^{3n})^3} = \frac{m'^2\mathcal{K}^3(m)}{4\pi^3} + \frac{\mathcal{K}(m)}{16\pi} - \frac{1}{16}. \tag{A14}$$

We now use a modified version

$$m_1 = \frac{1-m'}{1+m'}, \qquad m_1' = \sqrt{1-m_1^2} = \frac{2\sqrt{m'}}{1+m'} \tag{A15}$$

of the Landen transformation, for which

$$\mathcal{K}(m_1) = \tfrac{1}{2}(1+m')\mathcal{K}(m), \qquad \mathcal{E}(m_1) = \frac{\mathcal{E}(m) + m'\mathcal{K}(m)}{1+m'}. \tag{A16}$$

([11], p.908). It follows from (A9) that x^2 is the nome of m_1. Replacing m by m_1 in (A11), setting $\mathfrak{u} = \frac{1}{2}\imath\mathcal{K}(m_1')$ and using $\mathrm{dn}(\frac{1}{2}\imath\mathcal{K}(m_1')|m_1) = \sqrt{1+m_1}$, gives

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{x^n(1+x^{2n})}{1+x^{4n}} &= \frac{\mathcal{K}(m_1)\sqrt{1+m_1}}{\pi} - \frac{1}{2}, \\ &= \sqrt{\frac{1+m'}{2}}\frac{\mathcal{K}(m)}{\pi} - \frac{1}{2}\end{aligned} \tag{A17}$$

and replacing m by m_1 in (A13) gives

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{x^{4n}}{(1+x^{4n})^2} &= \frac{\mathcal{E}(m_1)\mathcal{K}(m_1)}{2\pi^2} - \frac{1}{8}, \\ &= \frac{1}{4\pi^2}\mathcal{K}(m)\{\mathcal{E}(m) + m'\mathcal{K}(m)\} - \frac{1}{8}.\end{aligned} \tag{A18}$$

Final, by replacing m by m_1 in (A8), substituting from (A11) and performing the integration, it can be shown that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{x^{3n}(1+x^{2n})}{(1+x^{4n})^2} &= \frac{1}{4\pi^2}\mathcal{K}(m_1)\sqrt{1+m_1}\{\pi + 2(1-m_1)\mathcal{K}(m_1)\} - \frac{1}{4} \\ &= \frac{1}{4\pi^2}\mathcal{K}(m)\sqrt{\frac{1+m'}{2}}\{\pi + 2m'\mathcal{K}(m)\} - \frac{1}{4}.\end{aligned} \tag{A19}$$

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