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On the geometry of homogeneous real hypersurfaces in the complex quadric

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Abstract. In this note we investigate the geometry of some homogeneous real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, $m \geq 2$.

1 Introduction

The m-dimensional complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, $m \geq 2$, is a Hermitian symmetric space of rank 2. We normalize the Riemannian metric on Q^m such that the maximum of the sectional curvature is 4.

In this note we consider two cohomogeneity one actions on Q^m, namely by $SO_{m+1} \subset SO_{m+2}$ and by $U_{k+1} \subset SO_{2k+2}$, $m = 2k$. The action of SO_{m+1} has two singular orbits, a totally geodesic m-dimensional sphere S^m which is embedded in Q^m as a real form and an $(m-1)$-dimensional complex quadric $Q^{m-1} = SO_{m+1}/SO_{m-1}SO_2$ which is embedded in Q^m as a totally geodesic complex hypersurface. The action of U_{k+1} has two singular orbits both of which are k-dimensional complex projective spaces $\mathbb{C}P^k = U_{k+1}/U_kU_1$ embedded in Q^{2k} as totally geodesic complex submanifolds.

The distance between the two singular orbits of the SO_{m+1}-action is $\pi/2\sqrt{2}$ and

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the distance between the two singular orbits of the U_{k+1}-action is $\pi/2$. The principal orbits of the SO_{m+1}-action are therefore the tubes of radius $0 < r < \pi/2\sqrt{2}$ around the totally geodesic $Q^{m-1} \subset Q^m$, and the principal orbits of the U_{k+1}-action are the tubes of radius $0 < r < \pi/2$ around the totally geodesic $\mathbb{C}P^k \subset Q^{2k}$. Each of these tubes is a homogeneous real hypersurface of the complex quadric. In this note we determine the principal curvatures and a few geometric properties of these homogeneous real hypersurfaces.

2 The complex quadric Q^m

The homogeneous quadratic equation

$$z_1^2 + \ldots + z_{m+2}^2 = 0$$

on $\mathbb{C}^{m+2}$ defines a complex hypersurface Q^m in the $(m+1)$-dimensional complex projective space $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. The hypersurface Q^m is known as the m-dimensional complex quadric. The complex structure J on $\mathbb{C}P^{m+1}$ naturally induces a complex structure on Q^m which we will denote by J as well. We equip Q^m with the Riemannian metric g which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. Then (Q^m, J, g) is a Hermitian symmetric space and the maximal sectional curvature is equal to 4.

We denote by $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$ on $\mathbb{C}P^{m+1}$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$.

For a nonzero vector $z \in \mathbb{C}^{m+1}$ we denote by $[z]$ the complex span of z, that is,

$$[z] = \{\lambda z \mid \lambda \in \mathbb{C}\}.$$

Note that by definition $[z]$ is a point in $\mathbb{C}P^{m+1}$. As usual, for each $[z] \in Q^m$ we identify $T_{[z]}\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus [z]$ of $[z]$ in $\mathbb{C}^{m+2}$. The tangent space $T_{[z]}Q^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus ([z] \oplus [\xi])$ of $[z] \oplus [\xi]$ in $\mathbb{C}^{m+2}$, where $\xi \in \nu_{[z]}Q^m$ is a normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$.

For a unit normal vector ξ of Q^m at a point $[z] \in Q^m$ we denote by A_ξ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to ξ. The shape operator is an involution on the tangent space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_\xi) \oplus JV(A_\xi),$$

where $V(A_\xi)$ is the $(+1)$-eigenspace and $JV(A_\xi)$ is the (-1)-eigenspace of A_ξ. Geometrically this means that the shape operator A_ξ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, is a complex conjugation on $T_{[z]}Q^m$.

Since the normal space $\nu_{[z]}Q^m$ of Q^m in $\mathbb{C}P^{m+1}$ at $[z]$ is a complex subspace of complex dimension one, every unit normal vector $\xi' \in \nu_{[z]}Q^m$ can be written as $\xi' = \lambda\xi$, where ξ is a fixed unit normal vector in $\nu_{[z]}Q^m$ and $\lambda \in S^1 \subset \mathbb{C}$. We then have $V(A_{\xi'}) = \lambda V(A_\xi)$. We thus have an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $[z] \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under the action of the center SO_2 of the isotropy subgroup of SO_{m+2} at $[z]$. The geodesic reflection of Q^m in such a real form S^m is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way the family $\mathfrak{A}_{[z]}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms S^m of Q^m containing $[z]$, and the subspaces $V(A_\xi) \subset T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms S^m of Q^m containing $[z]$.

The Gauss equation for the complex hypersurface $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor R of Q^m can be expressed in terms of the Riemannian metric g, the complex structure J and a generic complex conjugation $A \in \mathfrak{A}$:

$$
\begin{aligned}
&R(X,Y)Z = \\
&g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\
&+g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY.
\end{aligned}
$$

Note that J anti-commutes with each conjugation A, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a conjugation $A \in \mathfrak{A}_{[z]}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.

2. If there exist a conjugation $A \in \mathfrak{A}_{[z]}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

For every unit tangent vector $W \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}_{[z]}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$.

For more details about the complex quadric we refer to the papers by Reckziegel [4] and Klein [3].

3 The tubes around $\mathbb{C}P^k \subset Q^{2k}$

We now assume that m is even, say $m = 2k$. The map

$$\mathbb{C}P^k \to Q^{2k} \subset \mathbb{C}P^{2k+1} \ , \ [z_1, \dots, z_{k+1}] \mapsto [z_1, \dots, z_{k+1}, iz_1, \dots, iz_{k+1}]$$

provides an embedding of $\mathbb{C}P^k$ into Q^{2k} as a totally geodesic complex submanifold. Consider the standard embedding of U_{k+1} into SO_{2k+2} which is determined by the Lie algebra embedding

$$\mathfrak{u}_{k+1} \to \mathfrak{so}_{2k+2} \ , \ C + iD \mapsto \begin{pmatrix} C & -D \\ D & C \end{pmatrix},$$

where $C, D \in M_{k+1,k+1}(\mathbb{R})$. The action of U_{k+1} on Q^{2k} is of cohomogeneity one and $\mathbb{C}P^k = U_{k+1}/U_1U_k$ is the orbit of this action containing the point $[z] = [1, 0, \dots, 0, i, 0, \dots, 0] \in Q^{2k}$, where the i sits in the $(k+2)$-nd component. A principal orbit of this action is isomorphic to the homogeneous space U_{k+1}/U_1U_{k-1}, which is an $S^{2k-1} = U_k/U_{k-1}$-bundle over $\mathbb{C}P^k$.

We define a complex structure j on $\mathbb{C}^{2k+2}$ by

$$j(z_1, \dots, z_{k+1}, z_{k+2}, \dots, z_{2k+2}) = (-z_{k+2}, \dots, -z_{2k+2}, z_1, \dots, z_{k+1}).$$

Note that $ij = ji$. We can then identify $\mathbb{C}^{2k+2}$ with $\mathbb{C}^{k+1} \oplus j\mathbb{C}^{k+1}$ and get

$$T_{[z]}\mathbb{C}P^k = \{X + jiX \mid X \in \mathbb{C}^{k+1} \ominus [z]\}.$$

Now fix a unit normal vector ξ of Q^{2k} at $[z]$ and let $A_\xi \in \mathfrak{A}_{[z]}$ be the corresponding complex conjugation. Then we can write alternatively

$$T_{[z]}\mathbb{C}P^k = \{X + jiX \mid X \in V(A_\xi)\}.$$

Note that the complex structure i on $\mathbb{C}^{2k+2}$ corresponds to the complex structure J on $T_{[z]}Q^{2k}$ via the obvious identifications. For the normal space $\nu_{[z]}\mathbb{C}P^k$ of $\mathbb{C}P^k$ at $[z]$ we have

$$\nu_{[z]}\mathbb{C}P^k = \{Y + jiY \mid Y \in JV(A_\xi)\} = \{X + jiX \mid X \in V(A_{J\xi})\}.$$

It is easy to see that both the tangent bundle and the normal bundle of $\mathbb{C}P^k$ consist of $\mathfrak{A}$-isotropic singular tangent vectors of Q^{2k}.

We will now calculate the principal curvatures and principal curvature spaces of the tube with radius $0 < r < \pi/2$ around $\mathbb{C}P^k$ in Q^{2k}. Let ξ be a unit normal vector of $\mathbb{C}P^k$ in Q^{2k} at $[z] \in \mathbb{C}P^k$. Since ξ is $\mathfrak{A}$-isotropic, the four vectors $\xi, J\xi, A\xi, JA\xi$ are pairwise orthonormal for each $A \in \mathfrak{A}_{[z]}$. We fix a conjugation $A \in \mathfrak{A}_{[z]}$. Then the normal Jacobi operator R_ξ is given by

$$R_\xi Z = R(Z,\xi)\xi = Z - g(Z,\xi)\xi + 3g(Z,J\xi)J\xi - g(Z,A\xi)A\xi - g(Z,JA\xi)JA\xi.$$

This implies readily that R_ξ has the three eigenvalues $0, 1, 4$ with corresponding eigenspaces $\mathbb{R}\xi \oplus [A\xi]$, $T_{[z]}Q^{2k} \ominus ([\xi] \oplus [A\xi])$ and $\mathbb{R}J\xi$. Since $[\xi] \subset \nu_{[z]}\mathbb{C}P^k$ and

$[A\xi] \subset T_{[z]}\mathbb{C}P^k$, we conclude that both $T_{[z]}\mathbb{C}P^k$ and $\nu_{[z]}\mathbb{C}P^k$ are invariant under R_ξ. Note that $[A\xi] = [A'\xi]$ for all $A, A' \in \mathfrak{A}_{[z]}$ since $A' = \lambda A$ for some $\lambda \in S^1 \subset \mathbb{C}$.

To calculate the principal curvatures of the tube of radius $0 < r < \pi/2$ around $\mathbb{C}P^k$ we use the standard Jacobi field method as described in Section 8.2 of [1]. Let γ be the geodesic in Q^{2k} with $\gamma(0) = [z]$ and $\dot\gamma(0) = \xi$ and denote by $\gamma^\perp$ the parallel subbundle of TQ^{2k} along γ defined by $\gamma^\perp_{\gamma(t)} = T_{[\gamma(t)]}Q^{2k} \ominus \mathbb{R}\dot\gamma(t)$. Moreover, define the $\gamma^\perp$-valued tensor field $R^\perp_\gamma$ along γ by $R^\perp_{\gamma(t)}X = R(X, \dot\gamma(t))\dot\gamma(t)$. Now consider the $\mathrm{End}(\gamma^\perp)$-valued differential equation

$$Y'' + R^\perp_\gamma \circ Y = 0.$$

Let D be the unique solution of this differential equation with initial values

$$D(0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \ , \ D'(0) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

where the decomposition of the matrices is with respect to

$$\gamma^\perp_{[z]} = T_{[z]}\mathbb{C}P^k \oplus (\nu_{[z]}\mathbb{C}P^k \ominus \mathbb{R}\xi)$$

and I denotes the identity transformation on the corresponding space. Then the shape operator $S(r)$ of the tube of radius $0 < r < \pi/2$ around $\mathbb{C}P^k$ with respect to $\dot\gamma(r)$ is given by

$$S(r) = -D'(r) \circ D^{-1}(r).$$

If we decompose $\gamma^\perp_{[z]}$ further into

$$\gamma^\perp_{[z]} = (T_{[z]}\mathbb{C}P^k \ominus [A\xi]) \oplus [A\xi] \oplus (\nu_{[z]}\mathbb{C}P^k \ominus [\xi]) \oplus \mathbb{R}J\xi,$$

we get by explicit computation that

$$S(r) = \begin{pmatrix} \tan(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\cot(r) & 0 \\ 0 & 0 & 0 & -2\cot(2r) \end{pmatrix}$$

with respect to that decomposition. Therefore the tube of radius $0 < r < \pi/2$ around $\mathbb{C}P^k$ has four distinct constant principal curvatures $\tan(r)$, 0, $-\cot(r)$ and $-2\cot(2r)$ (unless $m = 2$ in which case there are only two distinct constant principal curvatures 0 and $-2\cot(2r)$). The corresponding principal curvature spaces are $T_{[z]}\mathbb{C}P^k \ominus [A\xi]$, $[A\xi]$, $\nu_{[z]}\mathbb{C}P^k \ominus [\xi]$ and $\mathbb{R}J\xi$ respectively, where we identify the subspaces obtained by parallel translation along γ from $[z]$ to $\gamma(r)$.

Since $J\xi$ is a principal curvature vector we also conclude that every tube around $\mathbb{C}P^k$ is a Hopf hypersurface. We also see that all principal curvature spaces orthogonal to $\mathbb{R}J\xi$ are J-invariant. Thus, if ϕ denotes the structure tensor field on the tube which is induced by J, we get $S\phi = \phi S$. It is known that this condition is equivalent for the Reeb flow to be an isometric flow [2].

For $r = \pi/2$ this process leads to focal points of $\mathbb{C}P^k$. In fact, the focal set of $\mathbb{C}P^k$ is the second singular orbit of the cohomogeneity one action by U_{k+1} on Q^{2k}. It is another k-dimensional complex projective space which is embedded in Q^{2k} as a totally geodesic complex submanifold.

We summarize the previous discussion in the following proposition.

Proposition 3.1. *Let M be the tube of radius $0 < r < \pi/2$ around the totally geodesic $\mathbb{C}P^k$ in Q^{2k}. Then the following statements hold:*

1. *M is a Hopf hypersurface.*
2. *The normal bundle of M consists of $\mathfrak{A}$-isotropic singular tangent vectors of Q^{2k}.*
3. *M has four distinct constant principal curvatures (unless $m = 2$ in which case M has two distinct constant principal curvatures). Their values and corresponding principal curvature spaces and multiplicities are given in the following table:*

principal curvature	*eigenspace*	*multiplicity*
0	$[A\xi]$	2
$\tan(r)$	$T_{[z]}\mathbb{C}P^k \ominus [A\xi]$	$2k-2$
$-\cot(r)$	$\nu_{[z]}\mathbb{C}P^k \ominus [\xi]$	$2k-2$
$-2\cot(2r)$	$\mathbb{R}J\xi$	1

.

4. *Each of the two focal sets of M is a k-dimensional complex projective space which is embedded in Q^{2k} as a totally geodesic complex submanifold.*
5. *The shape operator S of M and the structure tensor field ϕ of M satisfy the equation $S\phi = \phi S$.*
6. *The Reeb flow on M is an isometric flow.*
7. *M is a homogeneous hypersurface of Q^{2k}. More precisely, it is an orbit of the U_{k+1}-action on Q^{2k} isomorphic to $U_{k+1}/U_{k-1}U_1$, an S^{2k-1}-bundle over $\mathbb{C}P^k$.*

4 The tubes around $Q^{m-1} \subset Q^m$

The map

$$Q^{m-1} \to Q^m \subset \mathbb{C}P^{m+1}\ ,\ [z_1, \ldots, z_{m+1}] \mapsto [z_1, \ldots, z_{m+1}, 0]$$

provides an embedding of Q^{m-1} into Q^m as a totally geodesic complex hypersurface. From the construction of $\mathfrak{A}$ it is clear that $T_{[z]}Q^{m-1}$ and $\nu_{[z]}Q^{m-1}$ are A-invariant for each conjugation $A \in \mathfrak{A}_{[z]}$. Moreover, since the real codimension of Q^{m-1} in Q^m is 2, there exists for each unit normal vector ξ of Q^{m-1} at $[z] \in Q^{m-1}$ a conjugation $A \in \mathfrak{A}_{[z]}$ such that $A\xi = \xi$. We then have

$$T_{[z]}Q^{m-1} = (V(A) \ominus \mathbb{R}\xi) \oplus J(V(A) \ominus \mathbb{R}\xi).$$

The subgroup SO_{m+1} of SO_{m+2} leaving the quadric Q^{m-1} invariant acts on Q^m with cohomogeneity one. The second singular orbit of this action is an m-dimensional sphere $S^m = SO_{m+1}/SO_m$ which is a real form of Q^m, that is, a totally geodesic real submanifold of Q^m of half dimension. A principal orbit of the action is a homogeneous space of the form SO_{m+1}/SO_{m-1}, which is an S^1-bundle over the singular orbit Q^{m-1} and an S^{m-1}-bundle over the singular orbit S^m. One can show that the distance between the two singular orbits Q^{m-1} and S^m is equal to $\pi/2\sqrt{2}$.

We will now calculate the principal curvatures and principal curvature spaces of the tube with radius $0 < r < \pi/2\sqrt{2}$ around Q^{m-1} in Q^m. Let ξ be a unit normal vector of Q^{m-1} in Q^m at a point $[z] \in Q^{m-1}$. Then there exists a complex conjugation $A \in \mathfrak{A}_{[z]}$ such that $A\xi = \xi$. We then have $AJ\xi = -JA\xi = -J\xi$. Therefore the normal Jacobi operator R_ξ is given by

$$R_\xi Z = R(Z,\xi)\xi = Z + AZ - 2g(Z,\xi)\xi + 2g(Z,J\xi)J\xi.$$

This implies that R_ξ has the two eigenvalues 0 and 2 with corresponding eigenspaces $J(V(A) \ominus \mathbb{R}\xi) \oplus \mathbb{R}\xi$ and $(V(A) \ominus \mathbb{R}\xi) \oplus \mathbb{R}J\xi$ respectively.

We use again the standard Jacobi field method as described in Section 8.2 of [1]. Let γ be the geodesic in Q^m with $\gamma(0) = [z]$ and $\dot\gamma(0) = \xi$ and denote by $\gamma^\perp$ the parallel subbundle of TQ^m along γ defined by $\gamma^\perp_{\gamma(t)} = T_{[\gamma(t)]}Q^m \ominus \mathbb{R}\dot\gamma(t)$. Moreover, define the $\gamma^\perp$-valued tensor field $R^\perp_\gamma$ along γ by $R^\perp_{\gamma(t)}X = R(X,\dot\gamma(t))\dot\gamma(t)$. We consider again the $\mathrm{End}(\gamma^\perp)$-valued differential equation

$$Y'' + R^\perp_\gamma \circ Y = 0,$$

but with different initial values. Let D be the unique solution of this differential equation with initial values

$$D(0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \ , \ D'(0) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

where the decomposition of the matrices is with respect to

$$\gamma^\perp_{[z]} = T_{[z]}Q^{m-1} \oplus \mathbb{R}J\xi$$

and I denotes the identity transformation on the respective space. Then the shape operator $S(r)$ of the tube of radius $0 < r < \pi/2\sqrt{2}$ around Q^{m-1} with respect to $\dot\gamma(r)$ is given by

$$S(r) = -D'(r) \circ D^{-1}(r).$$

We decompose $\gamma_{[z]}^{\perp}$ further into

$$\gamma_{[z]}^{\perp} = J(V(A) \ominus \mathbb{R}\xi) \oplus (V(A) \ominus \mathbb{R}\xi) \oplus \mathbb{R}J\xi$$

and get by explicit computation that

$$S(r) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2}\tan(\sqrt{2}r) & \\ 0 & 0 & -\sqrt{2}\cot(\sqrt{2}r) \end{pmatrix}$$

with respect to that decomposition. Therefore the tube of radius $0 < r < \pi/2\sqrt{2}$ around Q^{m-1} has three distinct constant principal curvatures 0, $\sqrt{2}\tan(\sqrt{2}r)$ and $-\sqrt{2}\cot(\sqrt{2}r)$. The corresponding principal curvature spaces are $J(V(A) \ominus \mathbb{R}\xi)$, $V(A) \ominus \mathbb{R}\xi$ and $\mathbb{R}J\xi$ respectively, where we identify the subspaces obtained by parallel translation along γ from $[z]$ to $\gamma(r)$.

Since $J\xi$ is a principal curvature vector we also conclude that every tube around Q^{m-1} is a Hopf hypersurface. We also see now that the shape operator S and the structure tensor field ϕ of the tube of radius $0 < r < \pi/2\sqrt{2}$ around Q^{m-1} satisfy the equation $S\phi + \phi S = \sqrt{2}\tan(\sqrt{2}r)\phi$. This means that the tube is a contact real hypersurface.

We summarize the previous discussion in the following proposition.

Proposition 4.1. *Let M be the tube of radius $0 < r < \pi/2\sqrt{2}$ around the totally geodesic Q^{m-1} in Q^m. Then the following statements hold:*

1. *M is a Hopf hypersurface.*

2. *The normal bundle of M consists of $\mathfrak{A}$-principal singular tangent vectors of Q^m.*

3. *M has three distinct constant principal curvatures. Their values and corresponding principal curvature spaces and multiplicities are given in the following table:*

principal curvature	*eigenspace*	*multiplicity*
0	$J(V(A) \ominus \mathbb{R}\xi)$	$m-1$
$\sqrt{2}\tan(\sqrt{2}r)$	$V(A) \ominus \mathbb{R}\xi$	$m-1$
$-\sqrt{2}\cot(\sqrt{2}r)$	$\mathbb{R}J\xi$	1

   Here, ξ is a unit normal vector of M and $A \in \mathfrak{A}$ such that $A\xi = \xi$.

4. *The shape operator S and the structure tensor field ϕ of M satisfy the equation*

$$S\phi + \phi S = \sqrt{2}\tan(\sqrt{2}r)\phi,$$

   that is, M is a contact real hypersurface.

5. *M is a homogeneous hypersurface of Q^m. More precisely, it is an orbit of the SO_{m+1}-action on Q^m and isomorphic to SO_{m+1}/SO_{m-1}, an S^1-bundle over $Q^{m-1} = SO_{m+1}/SO_{m-1}SO_2$ and an S^{m-1}-bundle over $S^m = SO_{m+1}/SO_m$.*

5 Some open problems

We determined some geometric properties of the homogeneous hypersurfaces in the complex quadric Q^m. It would be interesting to see whether some of these geometric properties actually characterize these homogeneous hypersurfaces. More specifically, we propose to investigate the following questions:

1. Are the tubes around $\mathbb{C}P^k$ in Q^{2k} characterized among all real hypersurfaces in complex quadrics by the property that their Reeb flow is an isometric flow?
2. Are the tubes around Q^{m-1} the only contact real hypersurfaces in Q^m?

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