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A Note on Hypersurfaces in Symmetric Spaces

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Abstract. This note contains a leisurely written summary of the author's lectures at the 14th International Workshop on Differential Geometry and the 3rd KNUGRG-OCAMI Differential Geometry Workshop held at Kyungpook National University, Taegu, South Korea, 4-6 November 2010.

1 Hypersurfaces in real space forms

The problem of classifying hypersurfaces with constant principal curvatures has a long and interesting history. Some of the earlier papers related to this problem were written by Somigliana [44], Levi-Civita [35] and Segre [43], and are to some extent related to geometrical optics. It follows from their work that a hypersurface M in an n-dimensional Euclidean space $\mathbb{E}^n$ has constant principal curvatures if and only if M is an open part of a round sphere, or of an affine hyperplane, or of a tube around a k-dimensional affine subspace, $k \in \{1, \ldots, n-2\}$.

Élie Cartan [15] obtained the corresponding classification in real hyperbolic space $\mathbb{R}H^n$. A hypersurface M in $\mathbb{R}H^n$ has constant principal curvatures if and only if M is congruent to a horosphere in $\mathbb{R}H^n$, to a totally geodesic hyperplane $\mathbb{R}H^{n-1} \subset \mathbb{R}H^n$ or one its equidistant hypersurfaces, or to a tube around a totally geodesic $\mathbb{R}H^k \subset \mathbb{R}H^n$, $k \in \{0, \ldots, n-2\}$. The first step in the proof is to show, using the Gauss-Codazzi equations, that the number g of distinct principal curvatures of M is either 1 or 2. If $g = 1$, then M is an umbilical hypersurface, and the classification of such hypersurfaces in $\mathbb{R}H^n$ is rather elementary. If $g = 2$, one can show that M has a focal manifold at some fixed distance, and using methods from focal set theory one can prove that this focal set must be totally geodesic.

A remarkable consequence of the previous two results is that every hypersurface with constant principal curvatures in $\mathbb{E}^n$ or $\mathbb{R}H^n$ is an open set of a homogeneous hypersurface. In other words, every complete hypersurface with constant princi-

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pal curvatures in $\mathbb{E}^n$ or $\mathbb{R}H^n$ is homogeneous. This is no longer true for hypersurfaces with constant principal curvatures in the sphere S^n. The first complete inhomogeneous hypersurfaces with constant principal curvatures in a sphere were constructed by Ozeki and Takeuchi [42]. Using representations of Clifford algebras, Ferus, Karcher and Münzner [24] generalized these examples and obtained a large number of complete inhomogeneous hypersurfaces with constant principal curvatures in spheres. Homogeneous hypersurfaces in spheres were classified by Hsiang and Lawson [25]. Basically, every such hypersurface is isometrically congruent to an orbit of the isotropy representation of a Riemannian symmetric space of rank 2.

The story about the classification of isoparametric hypersurfaces in spheres is quite interesting. It started with several papers by Ëlie Cartan ([15], [16], [17], [18]). Cartan observed that the level sets of isoparametric functions on spheres have constant principal curvatures, and conversely, every hypersurface in S^n with constant principal curvatures is an open part of a level set of an isoparametric function on S^n. He also classified all isoparametric hypersurfaces in spheres with at most three distinct principal curvatures. Roughly, these are the umbilical hyperspheres, the Riemannian products $S^a \times S^b \subset S^n$ with $a + b = n - 1$, and the tubes around the four Cartan-Veronese embeddings of the projective planes over the normed real division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. Using methods from algebraic topology, Münzner [40] proved that the number g of distinct principal curvatures of an isoparametric hypersurface in a sphere satisfies $g \in \{1, 2, 3, 4, 6\}$. Abresch [1] then obtained for the cases $g \in \{4, 6\}$ some restrictions on the multiplicities of the principal curvatures. In particular, when $g = 6$, all the multiplicities must be equal and $n \in \{7, 13\}$. Dorfmeister and Neher [23] settled the case for S^7. Every isoparametric hypersurface in S^7 with six distinct principal curvatures is an open part of a principal orbit of the isotropy representation of the symmetric space G_2/SO_4. Miyaoka [38] recently proved that every isoparametric hypersurface in S^{13} with six distinct principal curvatures is an open part of a principal orbit of the adjoint representation of the exceptional compact Lie group G_2. All these results together imply that every complete isoparametric hypersurface in a sphere with $g \in \{1, 2, 3, 6\}$ is homogeneous. The remaining case $g = 4$ is still open, but there has been some significant recent progress. Abresch's result about the multiplicities for $g = 4$ states that they come in pairs (m_1, m_2), where m_1 is the multiplicity of two principal curvatures, and m_2 is the multiplicity of the remaining two principal curvatures. Cecil, Chi and Jensen [19] proved that every inhomogeneous isoparametric hypersurface in S^n is congruent to one of the examples constructed by Ferus, Karcher and Münzner with the possible exception of the pairs $(4, 5)$, $(3, 4)$, $(7, 8)$ and $(6, 9)$. Recently Chi [20] presented a simpler proof of this result and also settled the case $(3, 4)$, showing that there are no further new examples.

It is remarkable that apart from spheres no other symmetric space is known which admit inhomogeneous hypersurfaces with constant principal curvatures (apart from trivial constructions on products using known examples).

2 Hypersurfaces in complex space forms

The homogeneous hypersurfaces in complex projective space $\mathbb{C}P^n$ were classified by Takagi [45]. Consider a Hermitian symmetric space G/K of rank 2 and with real dimension $2n+2$. The isotropy representation of G/K induces a cohomogeneity one action on S^{2n+1} which is compatible with the Hopf fibration $S^{2n+1} \to \mathbb{C}P^n$. The principal orbits of the induced cohomogeneity one action on $\mathbb{C}P^n$ are of course homogeneous hypersurfaces. Takagi proved that each homogeneous hypersurfaces in $\mathbb{C}P^n$ is holomorphically congruent to such an orbit. The classification of Hermitian symmetric spaces therefore readily leads to the classification of homogeneous hypersurfaces in $\mathbb{C}P^n$.

The classification of homogeneous hypersurfaces in the complex hyperbolic space $\mathbb{C}H^n$ is more complicated due to the noncompactness of the isometry group. The final classification was obtained by Berndt and Tamaru [12]. Consider an Iwasawa decomposition $G = KAN$ of $G = SU_{1,n}$. Then the solvable Lie group AN acts simply transitively on $\mathbb{C}H^n$. The orbits of N are horospheres in $\mathbb{C}H^n$, and they are all holomorphically congruent to each other. Now consider the Lie algebra $\mathfrak{a}+\mathfrak{n}$ of AN. The nilpotent subalgebra $\mathfrak{n}$ is isomorphic to the $(2n-1)$-dimensional Heisenberg algebra and can be realized algebraically as the sum of two root spaces $\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ of a suitable restricted root space decomposition of $\mathfrak{g} = \mathfrak{su}_{1,n}$. The root space $\mathfrak{g}_\alpha$ is isomorphic to $\mathbb{C}^{n-1}$ with respect to the complex structure on $\mathbb{C}H^n$. Let $\mathfrak{v}$ be a linear subspace of $\mathfrak{g}_\alpha$ with constant Kähler angle, and denote by $\mathfrak{g}_\alpha \ominus \mathfrak{v}$ the orthogonal complement of $\mathfrak{v}$ in $\mathfrak{g}_\alpha$. Then $\mathfrak{s} = \mathfrak{a} + (\mathfrak{g}_\alpha \ominus \mathfrak{v}) + \mathfrak{g}_{2\alpha}$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$. The corresponding subgroup S of $AN \cong \mathbb{C}H^n$ can be viewed as a submanifold of $\mathbb{C}H^n$. If $\mathfrak{v}$ is a k-dimensional complex subspace of $\mathfrak{g}_\alpha$, then S is holomorphically congruent to a totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$. If $\mathfrak{v}$ is 1-dimensional, then S is the ruled real hypersurface in $\mathbb{C}H^n$ which is generated by a horocycle in a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$. This homogeneous ruled real hypersurface was first discovered by Lohnherr [36] in his PhD thesis. Every tube around such a submanifold $S \subset \mathbb{C}H^n$ is a homogeneous hypersurface in $\mathbb{C}H^n$. Finally, every tube a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$ is a homogeneous hypersurface. Berndt and Tamaru proved that every homogeneous hypersurface in $\mathbb{C}H^n$ is holomorphically congruent to one of the above examples.

Consider a hypersurface M in a Hermitian manifold N. The normal bundle νM of M and the complex structure J of N induce a one-dimensional foliation on M through the integral manifolds of $J\nu M$. If these integral manifolds are all totally geodesic in M, then M is called a Hopf hypersurface. The sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ is an elementary example of a Hopf hypersurface. If N is a Kähler manifold, then M is a Hopf hypersurface if and only if the maximal complex subbundle $\mathcal{C}$ of the tangent bundle TM of M is invariant under the shape operator of M, or equivalently, if the nonzero vectors in $J\nu M$ are all principal curvature vectors of M. It is remarkable that every homogeneous hypersurface in $\mathbb{C}P^n$ is a Hopf hypersurface, whereas in $\mathbb{C}H^n$ most of the homogeneous hypersurfaces are not Hopf hypersurfaces.

The classification of hypersurfaces in $\mathbb{C}P^n$ or $\mathbb{C}H^n$ with constant principal curvatures is still an open problem. There are some partial results though. Kimura [28] (for $\mathbb{C}P^n$) and Berndt [2] (for $\mathbb{C}H^n$) established that every Hopf hypersurface with constant principal curvatures in $\mathbb{C}P^n$ or $\mathbb{C}H^n$ is an open part of a homogeneous hypersurface. The number g of distinct principal curvatures for homogeneous hypersurfaces satisfies $g \in \{2, 3, 5\}$ in the projective case (Takagi [45]) and $g \in \{2, 3, 4, 5\}$ in the hyperbolic case (Berndt and Díaz Ramos [6]). Takagi ([46] for $g = 2$, [47] for $g = 3$ and $n \geq 3$) and Wang ([48] for $g = 3$ and $n = 2$) proved that every hypersurface in $\mathbb{C}P^n$ with constant principal curvatures and $g \leq 3$ is an open part of a homogeneous hypersurface. The analogous result for $\mathbb{C}H^n$ was established by Montiel [39] for $g = 2$ and by Berndt and Díaz Ramos for $g = 3$ ([4] for $n \geq 3$ and [5] for $n = 2$). It is an elementary consequence of the Codazzi equation that there are no umbilical hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$, that is, $g = 1$ is impossible. Little is known about the situation for other values of g, although there has been some recent progress by Díaz Ramos and Dominguez Vazquez [22].

It is not known whether or not a hypersurface with constant principal curvatures in $\mathbb{C}P^n$ is necessarily a Hopf hypersurface. A positive answer would provide a complete classification in view of Kimura's result. It is also not known whether or not there exist complete inhomogeneous hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ or $\mathbb{C}H^n$.

3 Hypersurfaces in quaternionic space forms

The homogeneous hypersurfaces in quaternionic projective space $\mathbb{H}P^n$ were classified by Iwata [26] and D'Atri [21]. Any such hypersurface is either a tube around a totally geodesic $\mathbb{H}P^k \subset \mathbb{H}P^n$ for some $k \in \{0, \ldots, n-1\}$ or a tube around a totally geodesic $\mathbb{C}P^n \subset \mathbb{H}P^n$. The number g of distinct principal curvatures for a homogeneous hypersurface in $\mathbb{H}P^n$ satisfies $g \in \{2, 3, 4\}$. Martinez and Pérez proved that every hypersurface in $\mathbb{H}P^n$ with two distinct principal curvatures is an open part of a homogeneous hypersurface. Complete classifications for $g = 3$ or larger g are not known, and $g = 1$ is impossible.

A hypersurface M in a Riemannian manifold N is curvature-adapted if the normal Jacobi operator and the shape operator of M commute, or equivalently, are simultaneously diagonalizable. Let ξ be a local unit normal vector field of M and denote by R^N the Riemannian curvature tensor of N. The normal Jacobi operator K_ξ of M is defined by $K_\xi X = R^N(X, \xi)\xi$ for all $X \in TM$. It is evident that every hypersurface in a real space form is curvature-adapted, as K_ξ is a multiple of the identity transformation at each point. A hypersurface in a non-flat complex space form is curvature-adapted if and only if it is a Hopf hypersurface. The concept of curvature-adapted hypersurfaces therefore provides some kind of generalization of Hopf hypersurfaces in Hermitian manifolds to more general manifolds. However, it appears that there are not many such hypersurfaces in more general manifolds. Berndt [3] proved that a hypersurface in $\mathbb{H}P^n$ is curvature-adapted if and only if it is an open part of a homogeneous hypersurface in $\mathbb{H}P^n$. It is quite surprising that homogeneous hypersurfaces in $\mathbb{H}P^n$ can be characterized by such a simple algebraic condition. An interesting question is whether or not every hypersurface in $\mathbb{H}P^n$

with constant principal curvatures is curvature-adapted. A positive answer would provide a complete classification of hypersurfaces in $\mathbb{H}P^n$ with constant principal curvatures.

Homogeneous hypersurfaces in quaternionic hyperbolic space $\mathbb{H}H^n$ are not yet classified. Berndt and Tamaru [12] reduced the classification problem of such hypersurfaces to the problem of classifying all linear subspaces of a quaternionic vector space with constant quaternionic Kähler angle. Through this reduction many new examples of homogeneous hypersurfaces in $\mathbb{H}H^n$ were found, but the complete classification of such subspaces is still elusive. Berndt [3] showed that every curvature-adapted hypersurface in $\mathbb{H}H^n$ with constant principal curvatures is an open part of a horosphere in $\mathbb{H}H^n$, or of a tube around a totally geodesic $\mathbb{H}H^k \subset \mathbb{H}H^n$ for some $k \in \{0, \ldots, n-1\}$, or of a tube around a totally geodesic $\mathbb{C}H^n \subset \mathbb{H}H^n$. All these hypersurfaces are homogeneous. It is an open problem whether or not the assumption of constant principal curvatures is necessary for this result. The classification of hypersurface in $\mathbb{H}H^n$ ($n \geq 3$) with two distinct constant principal curvatures was obtained by Ortega and Pérez [41]; any such hypersurface is an open part of a homogeneous hypersurface. For $n = 2$ the problem seems to be still open.

4 Hypersurfaces in octonionic space forms

The octonionic space forms are just the Cayley projective plane $\mathbb{O}P^2 = F_4/Spin_9$ and the Cayley hyperbolic plane $\mathbb{O}H^2 = F_4^{-20}/Spin_9$. The problem with these spaces is that there is no reasonable octonionic structure on them, something like a Kähler structure on a complex space form or a quaternionic Kähler structure on a quaternionic space form. This makes it quite difficult to make effective use of the Gauss-Codazzi equations, and presumably just for this reason there are only few results about hypersurfaces in $\mathbb{O}P^2$ and $\mathbb{O}H^2$. There is though a useful explicit expression for the Riemannian curvature tensor on octonionic space forms, which was derived by Brown and Gray [14].

The homogeneous hypersurfaces in $\mathbb{O}P^2$ were classified by Iwata [27]. There are only two families of homogeneous hypersurfaces. One is given by the geodesic hyperspheres in $\mathbb{O}P^2$, and the other one by the tubes around a totally geodesic $\mathbb{H}P^2 \subset \mathbb{O}P^2$. The first family arises as the principal orbits of the action of the isotropy group $Spin_9$, and the second one as the principal orbits of the action of the maximal subgroup Sp_3Sp_1 of F_4.

The homogeneous hypersurfaces in $\mathbb{O}H^2$ were classified by Berndt and Tamaru [12]. Consider an Iwasawa decomposition $F_4^{-20} = Spin_9AN$ and the corresponding decomposition on Lie algebra level, $\mathfrak{f}_4^{-20} = \mathfrak{so}_9 + \mathfrak{a} + \mathfrak{n}$. The 15-dimensional nilpotent Lie algebra $\mathfrak{n}$ decomposes into $\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ with some restricted root spaces in a suitable restricted root space decomposition of $\mathfrak{f}_4^{-20}$. The root space $\mathfrak{g}_{2\alpha}$ is the 7-dimensional center of $\mathfrak{n}$. Let $\mathfrak{v}$ be a k-dimensional linear subspace of $\mathfrak{g}_\alpha \cong \mathbb{R}^8$ and denote by $\mathfrak{g}_\alpha \ominus \mathfrak{v}$ the orthogonal complement of $\mathfrak{v}$ in $\mathfrak{g}_\alpha$. Then $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{g}_\alpha \ominus \mathfrak{v}) + \mathfrak{g}_{2\alpha}$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$. The corresponding subgroup S of AN can be viewed as a submanifold of $\mathbb{O}H^2$ as AN acts simply transitively on $\mathbb{O}H^2$. If $k = 1$, then

S and its equidistant hypersurfaces are homogeneous hypersurfaces in $\mathbb{O}H^2$. If $k \in \{2,3,4,6,7\}$, then every tube around S is a homogeneous hypersurface. If $k = 4$, there exists a one-parameter family of incongruent submanifolds S, in all other cases the submanifold S is unique up to congruence. Every homogeneous hypersurface in $\mathbb{O}H^2$ is isometrically congruent to one of the above homogeneous hypersurfaces, or to a horosphere in $\mathbb{O}H^2$, or to a geodesic hypersphere in $\mathbb{O}H^2$, or to a tube around a totally geodesic $\mathbb{O}H^1 \subset \mathbb{O}H^2$, or to a tube around a totally geodesic $\mathbb{H}H^2 \subset \mathbb{O}H^2$.

The number g of distinct principal curvatures of a homogeneous hypersurface in $\mathbb{O}P^2$ satisfies $g \in \{2,3,4\}$, whereas the possible values for g for homogeneous hypersurfaces in $\mathbb{O}H^2$ have not yet been calculated. No classifications for hypersurfaces with constant principal curvatures in $\mathbb{O}P^2$ or $\mathbb{O}H^2$ in terms of the number g are known. It is also not known what the curvature-adapted hypersurfaces are in these two octonionic space forms.

5 Hypersurfaces in symmetric spaces of higher rank

The symmetric spaces which we discussed above all have rank one. A important geometric feature of these spaces is that any two tangent vectors of the same length can be mapped onto each other by an isometry, that is, all directions are geometrically equivalent. This is no longer true in symmetric spaces of higher rank, and this causes major difficulties, but also interesting new phenomena. Consider for example the symmetric space $SL_4(\mathbb{R})/SO_4$. This space can be identified with the space of all upper triangular matrices $X = (x_{ij})$, $i,j = 1,2,3,4$, with $\det X = 1$. By setting $x_{12} = 0$ or $x_{23} = 0$ we obtain two homogeneous minimal hypersurfaces M_{12} and M_{23} of $SL_4(\mathbb{R})/SO_4$ respectively. These two hypersurfaces have the same principal curvatures with the same multiplicities, but they are not isometrically congruent (see[10]). This means that generically one cannot distinguish homogeneous hypersurfaces in a symmetric space of higher rank by means of their principal curvatures and their multiplicities.

Homogeneous hypersurfaces in irreducible, simply connected, symmetric spaces of compact type have been investigated and classified by Kollross [32]. One of the remarkable consequences of this classification is that there are compact symmetric spaces which do not admit any homogeneous hypersurfaces. This is in sharp contrast to the noncompact case where every symmetric space admits homogeneous hypersurfaces. For example, horospheres in symmetric spaces of noncompact type are homogeneous hypersurfaces. This indicates that the concept of duality between symmetric spaces of compact type and of noncompact type cannot be used for a classification of homogeneous hypersurfaces in symmetric spaces of noncompact type.

Berndt and Tamaru developed in a series of papers (see [10], [11], [12], [13]) a conceptual approach to the classification of cohomogeneity one actions, or equivalently, to the classification of homogeneous hypersurfaces, in symmetric spaces of noncompact type. Every homogeneous hypersurface arises as a principal orbit of a cohomogeneity one action on the space. For topological reasons a cohomogeneity one action on a symmetric space of noncompact type induces either a Riemannian foliation or has exactly one singular orbit. The cohomogeneity one actions whose orbits form a Riemannian foliation were classified in [10]. The moduli space of such actions up to orbit equivalence depends only on the rank of the symmetric space and on possible duality and triality principles on the space.

The homogeneous hypersurfaces with a totally geodesic focal set were classified in [11]. As of today there is no explicit classification of totally geodesic submanifolds in symmetric spaces. Wolf [49] classified totally geodesic submanifolds in symmetric spaces of rank one, and Klein classified in a series of papers ([29], [30], [31]) totally geodesic submanifolds in symmetric spaces of rank two. A particular class of totally geodesic submanifolds are the reflective submanifolds. A submanifold of a Riemannian manifold is reflective if and only if the geodesic reflection in the submanifold is a globally well-defined isometry. Reflective submanifolds in symmetric spaces were classified by Leung ([33], [34]). Reflective submanifolds always come in pairs. For every reflective submanifold F in a symmetric space there exists a reflective submanifold $F^\perp$ in that space such that $F^\perp$ is tangent to the normal space of F at some point. The congruency class of $F^\perp$ does not depend on the point in F. A reflective submanifold F in a symmetric space of noncompact type is the focal set of a homogeneous hypersurface if and only if the rank of $F^\perp$ is one. A complete list of such submanifolds can be obtained from Leung's classification of reflective submanifolds. There are five non-reflective totally geodesic submanifolds which arise as the focal set of a homogeneous hypersurface: $G_2^2/SO_4 \subset SO_{3,4}^o/SO_3SO_4$, $G_2^{\mathbb{C}}/G_2 \subset SO_7^{\mathbb{C}}/SO_7$, $\mathbb{C}H^2 \subset G_2^2/SO_4$, $SL_3(\mathbb{R})/SO_3 \subset G_2^2/SO_4$, $SL_3(\mathbb{C})/SU_3 \subset G_2^{\mathbb{C}}/G_2$. All these exceptions are mysteriously related to the exceptional Lie group G_2.

Homogeneous hypersurfaces with a non-totally geodesic focal set in noncompact symmetric spaces of rank one were discussed above and investigated in [12]. The paper [13] is devoted to higher rank symmetric spaces. Berndt and Tamaru developed two construction methods for cohomogeneity one actions with a non-totally geodesic singular orbit on symmetric spaces of noncompact type. These two methods are based on horospherical decompositions of symmetric spaces of noncompact type and Langlands decompositions of parabolic subalgebras of semisimple real Lie algebras. Horospherical decompositions are in one-to-one correspondence to subsets of a set of simple roots of a restricted root space decomposition of the Lie algebra of the isometry group of the symmetric space. Roughly, every symmetric space of noncompact type is diffeomorphic to the product of a semisimple totally geodesic submanifold, a Euclidean totally geodesic submanifold and a horocyclic submanifold. The first method states that every cohomogeneity one action on the semisimple totally geodesic submanifold can be canonically extended to a cohomogeneity one action on the symmetric space. This leads to many new examples of homogeneous hypersurfaces with a non-totally geodesic focal set. The second method involves the algebraic structure of the horocyclic submanifold and is more delicate. Until now basically only two examples of homogeneous hypersurfaces are known which can be constructed in this way and not by any of the other methods described above, one example in G_2^2/SO_4 and the other one in $G_2^{\mathbb{C}}/G_2$. This conceptual approach has been applied successfully by Berndt and Tamaru to classify explicitly homogeneous hypersurfaces in several noncompact symmetric space of rank 2.

Apart from the homogeneous hypersurfaces, no other examples are known of hypersurfaces with constant principal curvatures in symmetric spaces of higher rank. It would be desirable to find geometric conditions for classifying hypersurfaces in higher rank symmetric spaces. Berndt and Suh ([7], [8], [9]) investigated this in more detail for the rank two symmetric space $SU_{2+m}/S(U_2U_m)$ and its noncompact dual space $SU_{2,m}/S(U_2U_m)$. Both spaces are distinguished by being equipped with both a Kähler structure and a quaternionic Kähler structure. Denote by $\mathcal{C}$ the maximal complex subbundle and by $\mathcal{Q}$ the maximal quaternionic subbundle of the tangent bundle TM of a hypersurface M in one of these two spaces. In the compact case, it was shown in [7] that both $\mathcal{C}$ and $\mathcal{Q}$ are invariant under the shape operator of M if and only if M is congruent to an open part of a tube around a totally geodesic $SU_{2+(m-1)}/S(U_2U_{m-1}) \subset SU_{2+m}/S(U_2U_m)$ or of a tube around a totally geodesic $\mathbb{H}P^n = Sp_{1+n}/Sp_1Sp_n \subset SU_{2+2n}/S(U_2U_{2n})$, where $m = 2n$. The analogous problem for the noncompact space is more complicated, and a partial solution was obtained in [9]. Since $SU_{2+m}/S(U_2U_m)$ is a Kähler manifold, we can construct a unit tangent vector field on it by rotating a unit normal vector field with the Kähler structure. The flow of this vector field is knows as the Reeb flow on the hypersurface. In [8] it was shown that the Reeb flow on a hypersurface M in $SU_{2+m}/S(U_2U_m)$ is isometric if and only if M is congruent to an open part of a tube around the totally geodesic $SU_{2+(m-1)}/S(U_2U_{m-1}) \subset SU_{2+m}/S(U_2U_m)$.

Horospheres play an important role for the geometry of symmetric spaces of noncompact type. Geometrically, horospheres are spheres whose center is at infinity, where infinity refers to the boundary of the symmetric space with regard to its geodesic (or conic) compactification. The geometric concept of singular tangent vectors leads naturally to the concept of singular points at infinity. In [9] a simple characterization of horospheres whose center at infinity is singular was obtained. A horosphere M in $SU_{2,m}/S(U_2U_m)$ has a singular point at infinity if and only if the maximal complex subbundle $\mathcal{C}$ of TM is invariant under shape operator of M, or equivalently, if and only if the maximal quaternionic subbundle $\mathcal{Q}$ of TM is invariant under shape operator of M. An interesting problem is to find a generalisation of this characterization of "singular" horospheres in $SU_{2,m}/S(U_2U_m)$ to symmetric spaces of higher rank in general.

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