Real hypersurfaces with constant principal curvatures in complex space forms

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Abstract. This article contains a survey about the classification problem for real hypersurfaces with constant principal curvatures in complex space forms. It is based on two lectures given by the author at the 10th International Workshop on Differential Geometry at Kyungpook National University in Taegu, Korea, in November 2005.

1. Introduction

A submanifold M of a Riemannian manifold N is called (extrinsically) homogeneous if there exists a closed subgroup G of the isometry group of N such that M is an orbit of the action of G on N. A basic problem in submanifold geometry is to classify the homogeneous submanifolds of a given Riemannian manifold. Since a homogeneous submanifold is an orbit of an isometric action, the second fundamental forms at the points of M are all conjugate to each other. If, in particular, M is a homogeneous hypersurface, then its principal curvatures are constant. A natural problem is to investigate whether in a given Riemannian manifold N the converse holds. More precisely, assume that a hypersurface M in N has constant principal curvatures. Is M an open part of a homogeneous hypersurface? Another basic problem is to classify the hypersurfaces with constant principal curvatures in a given Riemannian manifold.

In real space forms quite a lot is known about these problems, and it is completely solved in Euclidean space $\mathbb{E}^n$ and real hyperbolic space $\mathbb{R}H^n$. A fundamental observation by Élie Cartan is that a hypersurface in a real space form has constant principal curvatures if and only if it is isoparametric. The classification of isoparametric hypersurfaces in $\mathbb{E}^n$ and $\mathbb{R}H^n$ is due to Tullio Levi Civita (for $\mathbb{E}^3$), Benjamino Segre (for $\mathbb{E}^n$) and Élie Cartan (for $\mathbb{R}H^n$). In all these cases the isoparametric hypersurfaces are open parts of homogeneous hypersurfaces. The situation in the sphere S^n is more involved. The homogeneous hypersurfaces are all known due to a classification by Wu-yi Hsiang and Blaine Lawson Jr. Surprisingly, there are inhomogeneous isoparametric hypersurfaces in S^n. The first such examples were discovered by Hideki Ozeki and Masaru Takeuchi, further series of such examples were constructed from Clifford modules by Dirk Ferus, Hermann Karcher and Hans-Friedrich Münzner. The classification of isoparametric hypersurfaces in S^n is still an open problem. For a survey about this topic, relevant references and recent developments we refer to [11] and [22].

In this article we review the classification problems for homogeneous real hypersurfaces and real hypersurfaces with constant principal curvatures in complex space forms, that is, in simply connected and complete Kähler manifolds with constant holomorphic sectional curvature. Since the flat complex space $\mathbb{C}^n$ is isometric to $\mathbb{E}^{2n}$, we restrict to the nonflat case. The relevant complex space forms then are the complex projective space $\mathbb{C}P^n$ and the complex hyperbolic space $\mathbb{C}H^n$. We assume $n \geq 2$ and normalize the Fubini Study metric on $\mathbb{C}P^n$ and $\mathbb{C}H^n$ such that the constant holomorphic sectional curvature is equal to $+4$ and -4, respectively.

2000 *Mathematics Subject Classification.* Primary 53C40.

2. Complex projective spaces

The homogeneous real hypersurfaces in $\mathbb{C}P^n$ were classified by Ryoichi Takagi in 1973:

Theorem 2.1. ([19]) *A real hypersurface in $\mathbb{C}P^n$, $n \geq 2$, is homogeneous if and only if it is congruent to*

(1) *a tube around a k-dimensional totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n$ for some $k \in \{0, \ldots, n-1\}$, or*
(2) *a tube around the complex quadric $Q^{n-1} = \{[z] \in \mathbb{C}P^n \mid z_0^2 + \ldots + z_n^2 = 0\}$ in $\mathbb{C}P^n$, or*
(3) *a tube around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ into $\mathbb{C}P^{2k+1}$ for some $k \geq 2$, or*
(4) *a tube around the Plücker embedding into $\mathbb{C}P^9$ of the complex Grassmann manifold $G_2(\mathbb{C}^5)$ of complex 2-planes in $\mathbb{C}^5$, or*
(5) *a tube around the half spin embedding into $\mathbb{C}P^{15}$ of the Hermitian symmetric space $SO(10)/U(5)$.*

The proof uses the Hopf fibration $S^{2n+1} \to \mathbb{C}P^n$ and the classification of homogeneous hypersurfaces in spheres due to Hsiang and Lawson. A remarkable consequence of this result is that any homogeneous real hypersurface in $\mathbb{C}P^n$ is a Hopf hypersurface. A Hopf hypersurface M in a Kähler manifold N is a real hypersurface with the property that the rank one distribution $J(\nu M)$ on M determined by the normal bundle νM of M and the complex structure J of N is autoparallel. This is equivalent to the property that $J(\nu M)$ is invariant under the shape operator of M. It is not clear why a homogeneous real hypersurface in $\mathbb{C}P^n$ is necessarily a Hopf hypersurface. We will see below that this is not true in complex hyperbolic space $\mathbb{C}H^n$. Makoto Kimura characterized in [13] the homogeneous real hypersurfaces in $\mathbb{C}P^n$ as those real hypersurfaces with constant principal curvatures which are Hopf hypersurfaces:

Theorem 2.2. [13] *Let M be a real hypersurface in $\mathbb{C}P^n$ with constant principal curvatures. Then M is a Hopf hypersurface if and only if it is an open part of a homogeneous real hypersurface.*

All the homogeneous real hypersurfaces in $\mathbb{C}P^n$ have either two, three or five distinct constant principal curvatures. It is known that any real hypersurface in $\mathbb{C}P^n$ with two or three distinct constant principal curvatures is an open part of a homogeneous hypersurface. More precisely, we have

Theorem 2.3. ([20]) *Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 2$, with two distinct constant principal curvatures. Then M is an open part of a geodesic hypersphere in $\mathbb{C}P^n$.*

Note that a geodesic hypersphere in $\mathbb{C}P^n$ is a tube around a totally geodesic $\mathbb{C}P^0$ (a point) in $\mathbb{C}P^n$. Any geodesic hypersphere has two focal sets, a point and a totally geodesic hyperplane $\mathbb{C}P^{n-1}$. Thus a geodesic hypersphere can also be viewed as a tube around a totally geodesic $\mathbb{C}P^{n-1}$. Thomas E. Cecil and Patrick J. Ryan [12] improved the above result for $n \geq 3$ by requiring that M has at most two distinct principal curvatures at each point.

For three distinct constant principal curvatures we have

Theorem 2.4. ([21] for $n \geq 3$ and [23] for $n = 2$) *Let M be a real hypersurface in $\mathbb{C}P^n$, $n \geq 2$, with three distinct constant principal curvatures. Then M is an open part of*

(1) *a tube around a k-dimensional totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n$ for some $1 \leq k \leq n-2$, or*
(2) *a tube around the complex quadric Q^{n-1} in $\mathbb{C}P^n$.*

The focal set of a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n$ is a totally geodesic $\mathbb{C}P^{n-k-1}$, and the focal set of the complex quadric in $\mathbb{C}P^n$ is a totally geodesic real projective space $\mathbb{R}P^n \subset \mathbb{C}P^n$. Thus a tube around $\mathbb{C}P^k$ is a tube around $\mathbb{C}P^{n-k-1}$, and a tube around the complex quadric can be viewed as a tube around $\mathbb{R}P^n$.

It is a simple consequence of the Codazzi equation that there are no totally umbilical real hypersurfaces in $\mathbb{C}P^n$.

We finish this section with two open problems. Let M be a real hypersurface in $\mathbb{C}P^n$ with constant principal curvatures.

1. Is M an open part of a homogeneous real hypersurface? Equivalently, in view of Theorem 2.2, is M a Hopf hypersurface?
2. What is the possible number g of distinct principal curvatures of M?

For a homogeneous real hypersurface in $\mathbb{C}P^n$ we have $g \in \{2,3,5\}$. Zhen Qi Li [14] proved that $g \in \{2,3,5\}$ for all isoparametric real hypersurfaces in $\mathbb{C}P^n$ with constant principal curvatures.

3. Complex hyperbolic spaces

The Codazzi equation implies easily that there are no totally umbilical real hypersurfaces in $\mathbb{C}H^n$. Thus a real hypersurface in $\mathbb{C}H^n$ must have at least two distinct principal curvatures on an open and dense subset. Sebastián Montiel proved in 1985:

Theorem 3.1. ([17]) *Let M be a real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, with at most two distinct principal curvatures at each point. Then M is an open part of*

(1) *a horosphere in $\mathbb{C}H^n$, or*
(2) *a geodesic hypersphere in $\mathbb{C}H^n$, or*
(3) *a tube around a totally geodesic $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$, or*
(4) *a tube of radius $r = \ln(2+\sqrt{3})$ around a totally geodesic real hyperbolic space $\mathbb{R}H^n$ in $\mathbb{C}H^n$.*

For $n = 2$ the problem remains open, even under the additional assumption that M has exactly two distinct constant principal curvatures.

All the hypersurfaces in the previous theorem are Hopf hypersurfaces. The Hopf hypersurfaces in $\mathbb{C}H^n$ with constant principal curvatures have been classified by the author in 1989:

Theorem 3.2. ([2]) *Let M be a Hopf hypersurface in $\mathbb{C}H^n$, $n \geq 2$, with constant principal curvatures. Then M is an open part of*

(1) *a horosphere in $\mathbb{C}H^n$, or*
(2) *a tube around a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n$ for some $0 \leq k \leq n-1$, or*
(3) *a tube around a totally geodesic real hyperbolic space $\mathbb{R}H^n$ in $\mathbb{C}H^n$.*

The proof uses a complex version of Cartan's fundamental formula for isoparametric hypersurfaces in real space forms (see [10]). To derive this formula the restriction for M to be a Hopf hypersurface is essential. In the general case the Gauss-Codazzi equations appear to be too complicated for the deduction of a useful formula.

The question whether there exist non-Hopf hypersurfaces in $\mathbb{C}H^n$ with constant principal curvatures now leads us to the classification problem of homogeneous real hypersurfaces in $\mathbb{C}H^n$. All the real hypersurfaces in the previous theorem are homogeneous. In fact, a horosphere can be realized as an orbit of the action of the nilpotent part in a suitable Iwasawa decomposition of $SU(1,n)$, the connected component of the isometry group of $\mathbb{C}H^n$. A tube around a totally geodesic $\mathbb{C}H^k$ is an orbit of the action of $S(U(1,k) \times U(n-k)) \subset SU(1,n)$, and a tube around a totally geodesic $\mathbb{R}H^n$ is an orbit of the action of $SO^0(1,n) \subset SU(1,n)$.

A natural question is whether any homogeneous real hypersurface in $\mathbb{C}H^n$ is a Hopf hypersurface. It came as a kind of surprise when Michael Lohnherr [15] (see also [16]) discovered a homogeneous ruled real hypersurface in $\mathbb{C}H^n$. The base curve of the ruling is a horocycle in a totally geodesic real hyperbolic plane $\mathbb{R}H^2 \subset \mathbb{C}H^n$. An alternative description of this homogeneous ruled real hypersurface was given by the author in [3]. For this consider an Iwasawa decompositon KAN of $SU(1,n)$ and the corresponding Iwasawa decomposition $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ of the Lie algebra of $SU(1,n)$. The solvable Lie group AN acts simply transitively on $\mathbb{C}H^n$. The nilpotent Lie algebra $\mathfrak{n}$ is isomorphic to the $(2n-1)$-dimensional Heisenberg algebra. Now decompose this algebra orthogonally into $\mathfrak{n} = \mathfrak{v} + \mathfrak{z}$, where $\mathfrak{z}$ is the one-dimensional center of $\mathfrak{n}$, and let $\mathfrak{v}_0$ be a linear hyperplane in $\mathfrak{v}$. Then $\mathfrak{s} = \mathfrak{a} + \mathfrak{v}_0 + \mathfrak{z}$ is a codimension one subalgebra of $\mathfrak{a} + \mathfrak{n}$. The corresponding closed subgroup S of AN acts with cohomogeneity one on $\mathbb{C}H^n$. If o is the point in $\mathbb{C}H^n$ for which K is the stabilizer of $SU(1,n)$ at o, then the orbit $S \cdot o$ is the homogeneous ruled real hypersurface constructed by Lohnherr. The other orbits of the action of S are obviously homogeneous real hypersurfaces of $\mathbb{C}H^n$ as well, but they are neither Hopf nor ruled. The geometry of these orbits has been investigated in detail in [3].

The discovery of these homogeneous real hypersurfaces indicated that the classification of homogeneous real hypersurfaces in $\mathbb{C}H^n$ might be more complicated than expected. I will now describe aspects of some joint work with Hiroshi Tamaru which, among other results, leads to the classification of homogeneous real hypersurfaces in $\mathbb{C}H^n$.

A homogeneous hypersurface M of a Riemannian manifold N is obviously an orbit of a cohomogeneity one action on N. A closed subgroup G of the isometry group of N is said to act with cohomogeneity one if the orbit space N/G is one-dimensional. If N is complete, the orbit space N/G equipped with the quotient topology relative to the canonical projection $N \to N/G$ is homeomorphic to the real line $\mathbb{R}$, the circle S^1, the closed interval $[0,1]$, or the half-open interval $[0,\infty)$, each of them equipped with their standard topology. Geometrically this says that the orbits either form a Riemannian foliation on N (if $N/G \cong \mathbb{R}$ or S^1), or there exists exactly one singular orbit F and the principal orbits are tubes around F (if $N/G \cong [0,\infty)$), or there exist exactly two singular orbits F_1 and F_2, and the principal orbits are tubes around F_1 and F_2 (if $N/G \cong [0,1]$). If F, F_1 or F_2 happen to have codimension one as well, the corresponding tubes are actually equidistant hypersurfaces.

If N is a Hadamard manifold, then the orbit space N/G of a cohomogeneity one action on N cannot be homeomorphic to S^1 or $[0,1]$. From now on we assume that N is an irreducible Riemannian symmetric space of noncompact type. We denote by $\mathfrak{M}$ the set of all cohomogeneity one actions on N up to orbit equivalence. Then $\mathfrak{M} = \mathfrak{M}_F \cup \mathfrak{M}_S$, where $\mathfrak{M}_F$ contains the actions forming a Riemannian foliation and $\mathfrak{M}_S$ contains the actions with exactly one singular orbit. Geometrically, this singular orbit is just the focal set of any of the principal orbits. The homogeneous hypersurfaces arising in a foliation in $\mathfrak{M}_F$ do not have any focal points.

Theorem 3.3. ([7]) *Let N be an irreducible Riemannian symmetric space of noncompact type. Let r be the rank of N and denote by* $\mathrm{Aut}(DD)$ *the automorphism group of the Dynkin diagram associated to N. Then*

$$\mathfrak{M}_F \cong (\mathbb{R}P^{r-1} \cup \{1,\ldots,r\})/\mathrm{Aut}(DD).$$

Note that $\mathrm{Aut}(DD)$ is either trivial, or isomorphic to $\mathbb{Z}_2$ or the symmetric group $\mathfrak{S}_3$ of a set of three elements. If $\mathrm{Aut}(DD) \cong \mathbb{Z}_2$, then there exists a duality principle on N, and if $\mathrm{Aut}(DD) \cong \mathfrak{S}_3$, then there exists a triality principle on N. The symmetric spaces N for which $\mathrm{Aut}(DD) \cong \mathfrak{S}_3$ are $SO^o(4,4)/SO(4)SO(4)$ and $SO(8,\mathbb{C})/SO(8)$.

We have to describe the action of $\mathrm{Aut}(DD)$ on $\mathbb{R}P^{r-1} \cup \{1,\ldots,r\}$. For this we identify $\{1,\ldots,r\}$ with the vertices of the Dynkin diagram associated to N, which gives a natural action of $\mathrm{Aut}(DD)$ on $\{1,\ldots,r\}$. The vertices in the Dynkin diagram correspond to a set of simple roots which span an r-dimensional real vector space V^r. The action of $\mathrm{Aut}(DD)$ on the vertices, and hence on the simple roots, extends to a linear action on V^r which induces an action of $\mathrm{Aut}(DD)$ on the real projective space $\mathbb{R}P^{r-1}$ of V^r.

In the special case $r = 1$, that is, N is a hyperbolic space $\mathbb{F}H^n$ over a normed real division algebra $\mathbb{F} \in \{\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}\}$, we see that $\mathfrak{M}_F$ consists of exactly two elements. In other words, on $\mathbb{F}H^n$ there exist exactly two congruence classes of homogeneous codimension one foliations. For the complex hyperbolic space we already know these two classes, and we thus get

Corollary 3.4. *Let M be a homogeneous real hypersurface in $\mathbb{C}H^n$, $n \geq 2$, without focal points. Then M is congruent to*

(1) *a horosphere in $\mathbb{C}H^n$, or*
(2) *a ruled real hypersurface in $\mathbb{C}H^n$ determined by a horocycle in a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$, or one of its equidistant hypersurfaces.*

We now turn to homogeneous real hypersurfaces in $\mathbb{C}H^n$ with focal points, or equivalently, to cohomogeneity one actions on $\mathbb{C}H^n$ with a singular orbit. We already know from the general situation that the set of focal points of a homogeneous hypersurface in $\mathbb{C}H^n$ must form a smooth submanifold, which arises as the singular orbit of the corresponding cohomogeneity one action. We distinguish the two cases when this focal manifold F is totally geodesic or non-totally geodesic.

The totally geodesic submanifolds in $\mathbb{C}H^n$ are well known, they are just the standard embeddings of complex hyperbolic spaces $\mathbb{C}H^k$, $0 \leq k \leq n-1$, and real hyperbolic spaces $\mathbb{R}H^k$, $1 \leq k \leq n$. We have already seen above that the complex hyperbolic spaces $\mathbb{C}H^k$ and the real hyperbolic space $\mathbb{R}H^n$ arise as a singular orbit of a cohomogeneity one action on $\mathbb{C}H^n$. The real hyperbolic spaces $\mathbb{R}H^k$ for $k < n$ cannot arise as a singular orbit of a cohomogeneity one action on $\mathbb{C}H^n$. This can be seen in the following way. A totally geodesic $\mathbb{R}H^k$ uniquely determines a totally geodesic $\mathbb{C}H^k$. Pick a unit normal vector ξ of $\mathbb{R}H^k$ tangent to $\mathbb{C}H^k$, and a unit normal vector η of $\mathbb{R}H^k$ perpendicular to $\mathbb{C}H^k$. As all isometries of $\mathbb{C}H^n$ are holomorphic or anti-holomorphic, all isometries of $\mathbb{C}H^n$ preserving $\mathbb{R}H^k$ leave $\mathbb{C}H^k$ invariant. For this reason there cannot be an isometry of $\mathbb{C}H^n$ whose differential maps ξ to η. As a cohomogeneity one action is transitive on unit normal vectors of a singular orbit, this shows that $\mathbb{R}H^k$ cannot be a singular orbit of a cohomogeneity one action on $\mathbb{C}H^n$. Altogether this implies

Proposition 3.5. *Let M be a homogeneous real hypersurface in $\mathbb{C}H^n$, $n \geq 2$, with a totally geodesic focal set. Then M is congruent to*

(1) *a tube around a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n$ for some $k \in \{0, \ldots, n-1\}$, or*
(2) *a tube around a totally geodesic $\mathbb{R}H^n$ in $\mathbb{C}H^n$.*

The cohomogeneity one actions with a totally geodesic singular orbit on irreducible Riemannian symmetric spaces of noncompact type have been classified by the author and Hiroshi Tamaru in [8].

We now come to the more difficult situation of a non-totally geodesic singular orbit. The first examples of cohomogeneity one actions on $\mathbb{C}H^n$ with a non-totally geodesic singular orbit have been constructed by the author and Martina Brück in [4]. We denote by $\mathbb{C}H^n(\infty)$ the ideal boundary of $\mathbb{C}H^n$ equipped with the cone topology. A point in $\mathbb{C}H^n(\infty)$ corresponds to an equivalence class of asymptotic geodesics in $\mathbb{C}H^n$. We fix two points $o \in \mathbb{C}H^n$ and $x \in \mathbb{C}H^n(\infty)$. This determines a unique Iwasawa decomposition $SU(1,n) = KAN$. Here, K is the stabilizer of $SU(1,n)$ at o, which is isomorphic to $S(U(1)U(n))$. The solvable Lie group AN acts simply transitively on $\mathbb{C}H^n$, the orbit $A \cdot o$ is the geodesic in $\mathbb{C}H^n$ through o in the equivalence class of asymptotic geodesics determined by x, and the orbit $N \cdot o$ is a horosphere in $\mathbb{C}H^n$ with center at x. Let $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be the corresponding Iwasawa decomposition of the Lie algebra $\mathfrak{su}(1,n)$ of $SU(1,n)$. As above we decompose the nilpotent Lie algebra $\mathfrak{n}$ into $\mathfrak{n} = \mathfrak{v} + \mathfrak{z}$, where $\mathfrak{z}$ is the one-dimensional center of $\mathfrak{n}$. We identify $T_o\mathbb{C}H^n$ with $\mathfrak{a} + \mathfrak{v} + \mathfrak{z}$ in the usual manner. Then $\mathfrak{a} + \mathfrak{z}$ and $\mathfrak{v}$ are invariant under the complex structure of $T_o\mathbb{C}H^n$.

Let $\mathfrak{v}_k$ be a linear subspace of $\mathfrak{v}$ such that the orthogonal complement $\mathfrak{v} \ominus \mathfrak{v}_k$ of $\mathfrak{v}_k$ in $\mathfrak{v}$ is a real subspace of $\mathfrak{v}$ of dimension $2 \leq k \leq n-1$. Then $\mathfrak{s}_k = \mathfrak{a} + \mathfrak{v}_k + \mathfrak{z}$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$. Let S_k be the closed subgroup of AN with Lie algebra $\mathfrak{s}_k$. The orbit $F_k = S_k \cdot o$ of S_k through o is a $(2n-k)$-dimensional submanifold of $\mathbb{C}H^n$ with real normal bundle. Let $\mathbb{C}(\mathfrak{v} \ominus \mathfrak{v}_k)$ be the complex span of $\mathfrak{v} \ominus \mathfrak{v}_k$. Then $\mathfrak{v} \ominus \mathbb{C}(\mathfrak{v} \ominus \mathfrak{v}_k)$ is the maximal complex subspace of $\mathfrak{v}_k$ and $\mathfrak{a} + (\mathfrak{v} \ominus \mathbb{C}(\mathfrak{v} \ominus \mathfrak{v}_k)) + \mathfrak{z}$ is a subalgebra of $\mathfrak{s}_k$. The action of the corresponding closed subgroup of $S_k \subset AN$ induces an autoparallel distribution on F_k whose maximal integral manifolds are totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$. Thus F_k is a ruled submanifold of $\mathbb{C}H^n$. Let $N_K^o(S_k)$ be the connected component of the normalizer of S_k in K. One can show that $N_K^o(S_k)$ is contained in the stabilizer K_x of K at x. The closed subgroup $N_K^o(S_k)S_k \subset KAN$ acts on $\mathbb{C}H^n$ with cohomogeneity one and with singular orbit F_k. As any totally geodesic submanifold of $\mathbb{C}H^n$ is either real or complex, it is clear that F_k cannot be totally geodesic. Different choices of Iwasawa decompositions and subspaces $\mathfrak{v}_k$ lead to congruent actions. Thus for each integer $k \in \{2, \ldots, n-1\}$ there is exactly one congruence class of such cohomogeneity one actions. In particular, for each k the ruled submanifolds F_k are all congruent to each other.

We now fix $\varphi \in \mathbb{R}$ and $k \in \mathbb{Z}$ with $0 < \varphi < \pi/2$ and $0 < 2k < n$. Let $\mathfrak{v}_{k,\varphi}$ be a linear subspace of $\mathfrak{v}$ such that the orthogonal complement $\mathfrak{v} \ominus \mathfrak{v}_{k,\varphi}$ of $\mathfrak{v}_{k,\varphi}$ in $\mathfrak{v}$ is a subspace of $\mathfrak{v}$ of dimension $2k$ and with constant Kähler angle φ. For each choice of k and φ there exists exactly one such subspace up to unitary transformation of $\mathfrak{v}$. Then $\mathfrak{s}_{k,\varphi} = \mathfrak{a} + \mathfrak{v}_{k,\varphi} + \mathfrak{z}$ is a subalgebra of $\mathfrak{a}+\mathfrak{n}$. Let $S_{k,\varphi}$ be the closed subgroup of AN with Lie algebra $\mathfrak{s}_{k,\varphi}$. The orbit $F_{k,\varphi} = S_{k,\varphi} \cdot o$ of $S_{k,\varphi}$ through o is a $2(n-k)$-dimensional submanifold of $\mathbb{C}H^n$ with normal bundle of constant Kähler angle φ. Let $\mathbb{C}(\mathfrak{v} \ominus \mathfrak{v}_{k,\varphi})$ be the complex span of $\mathfrak{v} \ominus \mathfrak{v}_{k,\varphi}$. Then $\mathfrak{v} \ominus \mathbb{C}(\mathfrak{v} \ominus \mathfrak{v}_{k,\varphi})$ is the maximal complex subspace of $\mathfrak{v}_{k,\varphi}$ and $\mathfrak{a} + (\mathfrak{v} \ominus \mathbb{C}(\mathfrak{v} \ominus \mathfrak{v}_{k,\varphi})) + \mathfrak{z}$ is a subalgebra of $\mathfrak{s}_{k,\varphi}$. The action of the corresponding closed subgroup of $S_{k,\varphi} \subset AN$ induces an autoparallel distribution on $F_{k,\varphi}$ whose maximal integral manifolds are totally geodesic $\mathbb{C}H^{n-2k} \subset \mathbb{C}H^n$. Thus $F_{k,\varphi}$ is a ruled submanifold of $\mathbb{C}H^n$. Let $N_K^o(S_{k,\varphi})$ be the connected component of the normalizer of $S_{k,\varphi}$ in K. One can show that $N_K^o(S_{k,\varphi})$ is contained in the stabilizer K_x of K at x. The closed subgroup $N_K^o(S_{k,\varphi})S_{k,\varphi} \subset KAN$ acts on $\mathbb{C}H^n$ with cohomogeneity one and with singular orbit $F_{k,\varphi}$. As any totally geodesic submanifold of $\mathbb{C}H^n$ is either real or complex, it is clear that $F_{k,\varphi}$ cannot be totally geodesic. Different choices of Iwasawa decompositions and subspaces $\mathfrak{v}_{k,\varphi}$ lead to congruent actions. Thus for each $\varphi \in \mathbb{R}$ and $k \in \mathbb{Z}$ with $0 < \varphi < \pi/2$ and $0 < 2k < n$ there is exactly one congruence class of such cohomogeneity one actions. In particular, for each fixed pair φ, k the ruled submanifolds $F_{k,\varphi}$ are all congruent to each other.

These two types of actions provide many examples of cohomogeneity one actions with a non-totally geodesic singular orbit on complex hyperbolic spaces $\mathbb{C}H^n$ for all integers $n \geq 3$. The question whether there are more such actions has been answered by the author and Hiroshi Tamaru in [9].

Let H be a closed subgroup of $SU(1,n)$ acting on $\mathbb{C}H^n$ with cohomogeneity one and assume there exists a non-totally geodesic singular orbit F. A result by Dmitri Alekseevsky and Antonio Di Scala [1] implies that there exists a unique point $x \in \mathbb{C}H^n(\infty)$ which is fixed under the natural extension of the H-action onto $\mathbb{C}H^n \cup \mathbb{C}H^n(\infty)$. Let $o \in F$ and consider the Iwasawa decompositions $SU(1,n) = KAN$ and $\mathfrak{su}(1,n) = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ induced by o and x. As above we decompose $\mathfrak{n}$ into $\mathfrak{n} = \mathfrak{v} + \mathfrak{z} \cong \mathbb{C}^{n-1} + \mathbb{R}$. It was shown in [9] that there exists a linear subspace $\mathfrak{v}_0$ of $\mathfrak{v}$ such that $\mathfrak{s} = \mathfrak{a} + \mathfrak{v}_0 + \mathfrak{z}$ is a subalgebra of $\mathfrak{a}+\mathfrak{n}$, F is the orbit $S \cdot o$ through o of the action of the closed subgroup S of AN with Lie algebra $\mathfrak{s}$, and the actions of $N_K^o(S)S \subset KAN$ and H on $\mathbb{C}H^n$ are orbit equivalent. The strategy for the classification is therefore to find all linear subspaces $\mathfrak{v}_0$ of $\mathfrak{v}$ with codimension greater than one for which there exists a closed subgroup of $K_x \cong U(n-1)$ which acts transitively on the unit sphere in $\mathfrak{v} \ominus \mathfrak{v}_o$. From the resulting cohomogeneity one actions one then has to remove those with a totally geodesic singular orbit, and finally investigate the orbit equivalence. This was carried out in [9] using some results from [4]. The final result is

Proposition 3.6. [9] *Let M be a homogeneous real hypersurface in $\mathbb{C}H^n$, $n \geq 2$, with a non-totally geodesic focal set. Then M is congruent to*

1. *a tube around the ruled submanifold $F_k \subset \mathbb{C}H^n$ for some $k \in \{2, \ldots, n-1\}$, or*
2. *a tube around the ruled submanifold $F_{k,\varphi} \subset \mathbb{C}H^n$ for some $\varphi \in \mathbb{R}$ and $k \in \mathbb{Z}$ with $0 < \varphi < \pi/2$ and $0 < 2k < n$.*

An immediate consequence is

Corollary 3.7. *Any singular orbit of a cohomogeneity one action on $\mathbb{C}H^2$ is totally geodesic.*

Altogether this now gives the classification of homogeneous hypersurfaces in complex hyperbolic spaces.

Theorem 3.8. [9] *Let M be a homogeneous real hypersurface in $\mathbb{C}H^n$, $n \geq 2$. Then M is congruent to*

(1) *a horosphere in $\mathbb{C}H^n$, or*
(2) *a ruled real hypersurface in $\mathbb{C}H^n$ determined by a horocycle in a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$, or one of its equidistant hypersurfaces, or*
(3) *a tube around a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n$ for some $k \in \{0, \dots, n-1\}$, or*
(4) *a tube around a totally geodesic $\mathbb{R}H^n$ in $\mathbb{C}H^n$, or*
(5) *a tube around the ruled submanifold $F_k \subset \mathbb{C}H^n$ for some $k \in \{2, \dots, n-1\}$, or*
(6) *a tube around the ruled submanifold $F_{k,\varphi} \subset \mathbb{C}H^n$ for some $\varphi \in \mathbb{R}$ and $k \in \mathbb{Z}$ with $0 < \varphi < \pi/2$ and $0 < 2k < n$.*

The author and Hiroshi Tamaru used this approach in [9] to classify the cohomogeneity one actions on real hyperbolic spaces and the Cayley hyperbolic plane as well. For the quaternionic hyperbolic space we could reduce the classification problem to a question from quaternionic linear algebra about the existence of subspaces of quaternionic vector spaces with constant quaternionic Kähler angle.

Due to Theorem 3.8 we can now give a negative answer to the question posed after Theorem 3.2, namely there exist non-Hopf hypersurfaces in $\mathbb{C}H^n$ with constant principal curvatures. The next obvious question is whether there exists a real hypersurface with constant principal curvatures in $\mathbb{C}H^n$ which is not an open part of a homogeneous hypersurface.

Let M be a real hypersurface in $\mathbb{C}H^n$ with constant principal curvatures, and denote by g the number of distinct principal curvatures of M. If M is homogeneous, we have $g \in \{2, 3, 4, 5\}$ (see [6]). From Theorem 3.1 we know that if $g \leq 2$ then M is an open part of a homogeneous real hypersurface in $\mathbb{C}H^n$. The author and José Carlos Díaz-Ramos recently proved

Theorem 3.9. ([5]) *Let M be a real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, with three distinct constant principal curvatures. Then M is an open part of a homogeneous real hypersurface on $\mathbb{C}H^n$, that is, M is an open part of*

(1) *a ruled real hypersurface in $\mathbb{C}H^n$ determined by a horocycle in a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$, or one of its equidistant hypersurfaces, or*
(2) *a tube around a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n$ for some $k \in \{1, \dots, n-2\}$, or*
(3) *a tube of radius $r \neq \ln(2+\sqrt{3})$ around a totally geodesic $\mathbb{R}H^n$ in $\mathbb{C}H^n$, or*
(4) *a tube of radius $r = \ln(2+\sqrt{3})$ around the ruled submanifold $F_k \subset \mathbb{C}H^n$ for some $k \in \{2, \dots, n-1\}$.*

Note that Jun-ichi Saito derived in [18] a classification for such hypersurfaces, but the proof is incorrect and leads to an incomplete classification. We do not know whether the analogous result to Theorem 3.9 for $n = 2$ is true or not. Some partial results for this case were obtained by Zhicai Xu [24]. We finish this section with a few open problems. Let M be a real hypersurface in $\mathbb{C}H^n$ with constant principal curvatures.

1. Is M an open part of a homogeneous real hypersurface? (Even an answer for $g \in \{4, 5\}$ would be of interest.)
2. What is the possible number of distinct principal curvatures of M? (The answer is $g \in \{2, 3, 4, 5\}$ if the answer to question 1 is positive.)
3. Do Theorems 3.1 and 3.9 hold for $n = 2$?

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