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Contact hypersurfaces in Kähler manifolds

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Abstract. We present without proofs some basic theory about contact hypersurfaces in Kähler manifolds. We then discuss the classification problem for some Hermitian symmetric spaces.

1 Introduction

A *contact manifold* is a smooth $(2n-1)$-dimensional manifold M together with a one-form η satisfying $\eta \wedge (d\eta)^{n-1} \neq 0$, $n \geq 2$. The one-form η on a contact manifold is called a *contact form*. The kernel of η defines the so-called *contact distribution* $\mathcal{C}$ in the tangent bundle TM of M.

A standard example is a round sphere in an even-dimensional Euclidean space. Consider the sphere $S^{2n-1}(r)$ with radius $r \in \mathbb{R}_+$ in $\mathbb{C}^n$ and denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{C}^n$ given by

$$\langle z, w \rangle = \mathrm{Re} \sum_{\nu=1}^{n} z_\nu \bar{w}_\nu .$$

By defining

$$\xi_z = -\frac{1}{r} iz$$

we obtain a unit tangent vector field ξ on $S^{2n-1}(r)$. We denote by η the dual one-form given by

$$\eta(X) = \langle X, \xi \rangle$$

and by ω the Kähler form on $\mathbb{C}^n$ given by

$$\omega(X, Y) = \langle iX, Y \rangle .$$

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A straightforward calculation gives

$$d\eta(X,Y) = -\frac{2}{r}\omega(X,Y).$$

Since the Kähler form ω has rank $2(n-1)$ on the kernel of η it follows that

$$\eta \wedge (d\eta)^{n-1} \neq 0.$$

Thus $S^{2n-1}(r)$ is a contact manifold with contact form η.

Note that if η is a contact form on a smoth manifold M, then $\rho\eta$ is also a contact form on M for each smooth function ρ on M which is nonzero everywhere.

2 Contact hypersurfaces in Kähler manifolds

Let $\bar{M}$ be a Kähler manifold of complex dimension n and let M be a connected oriented real hypersurface of $\bar{M}$. The hypersurface M can be equipped with what is known as an *almost contact metric structure* (ϕ, ξ, η, g) which consists of

1. a Riemannian metric g on M which is induced canonically from the Kähler metric (also denoted by g) on $\bar{M}$;

2. a tensor field ϕ on M which is induced canonically from the complex structure J on $\bar{M}$: for all vectors fields X on M the vector field ϕX is obtained by projecting orthogonally the vector field JX onto the tangent bundle TM;

3. a unit vector field ξ which is induced canonically from the orientation of M: if N is the unit normal vector field on M which determines the orientation of M then $\xi = -JN$;

4. a one-form η which is defined as the dual of the vector field ξ with respect to the metric g, that is, $\eta(X) = g(X,\xi)$ for all $X \in TM$.

The vector field ξ is also known as the *Reeb vector field* on M. The maximal complex subbundle $\mathcal{C}$ of the tangent bundle TM of M is equal to $\ker(\eta)$.

Let A be the shape operator of M defined by

$$AX = -\bar{\nabla}_X N,$$

where $\bar{\nabla}$ denotes the Levi Civita covariant derivative on $\bar{M}$. Denote by ω the fundamental 2-form on M given by

$$\omega(X,Y) = g(\phi X, Y).$$

Proposition 2.1. *The fundamental* 2*-form* ω *on a real hypersurface in a Kähler manifold is closed, that is,* $d\omega = 0$.

The real hypersurface M is said to be a *contact hypersurface* of $\bar{M}$ if there exists an everywhere nonzero smooth function ρ on M such that

$$d\eta = 2\rho\omega$$

on M (Okumura ([11]). It is clear that if this equation holds then $\eta \wedge (d\eta)^{n-1} \neq 0$, that is, every contact hypersurface in a Kähler manifold is a contact manifold. Note that the equation $d\eta = 2\rho\omega$ means that

$$d\eta(X, Y) = 2\rho g(\phi X, Y)$$

for all tangent vector fields X, Y on M.

Proposition 2.2. *Let M be a connected real hypersurface of an n-dimensional Kähler manifold $\bar{M}^n$ and assume that there exists an everywhere nonzero smooth function ρ on M such that $d\eta = 2\rho\omega$. If $n \geq 3$, then ρ is constant.*

The following proposition gives a useful characterization of contact hypersurfaces in terms of the shape operator and the tensor field ϕ.

Proposition 2.3. *Let M be a connected orientable real hypersurface of a Kähler manifold $\bar{M}$. Then M is a contact hypersurface if and only if there exists an everywhere nonzero smooth function ρ on M such that*

$$A\phi + \phi A = 2\rho\phi.$$

A real hypersurface M of a Kähler manifold is called a *Hopf hypersurface* if the flow of the Reeb vector field is geodesic, that is, if every integral curve of ξ is a geodesic in M. This condition is equivalent to

$$A\xi = \alpha\xi$$

with the smooth function

$$\alpha = g(A\xi, \xi),$$

that is, the Reeb vector field is a principal curvature vector of M at each point. The following result gives an expression for the mean curvature of a contact hypersurface in a Kähler manifold.

Proposition 2.4. (Okumura [11]) *Let M be a contact hypersurface of an n-dimensional Kähler manifold $\bar{M}$. Then M is a Hopf hypersurface and*

$$\mathrm{tr}(A) = \alpha + 2(n-1)\rho.$$

We denote by $\bar{R}$ the Riemannian curvature tensor of $\bar{M}$. For $p \in M$ and $Z \in T_p\bar{M}$ we denote by $Z_{\mathcal{C}}$ the orthogonal projection of Z onto $\mathcal{C}$. The following proposition gives a useful relation between the shape operator of a contact hypersurface and the curvature of the Kähler manifold.

Proposition 2.5. *Let M be a contact hypersurface of a Kähler manifold $\bar{M}$. Then we have*

$$2A^2X - 4\rho AX + 2\alpha\rho X = (\bar{R}(JN, N)JX)_{\mathcal{C}}$$

for all vector fields X on M which are tangent to the contact distribution $\mathcal{C}$.

We know from Proposition 2.2 that ρ is constant for a contact hypersurface M in $\bar{M}^n$ provided that $n \geq 3$. From Proposition 2.4 we see that in this situation M has constant mean curvature if and only if the principal curvature function α corresponding to the Reeb vector field ξ is constant. Using Proposition 2.5 one can prove the following result:

Proposition 2.6. *Let M be a contact hypersurface of a Kähler manifold $\bar{M}^n$, $n \geq 3$. Then we have*

$$d(\mathrm{tr}(A))(X) = g(\bar{R}(JN, N)N, JX)$$

for all vector fields X on M which are tangent to the contact distribution $\mathcal{C}$.

This readily implies the following result:

Proposition 2.7. *Let M be a connected contact hypersurface of a Kähler manifold $\bar{M}$, $n \geq 3$. Then the following statements are equivalent:*

(i) M has constant mean curvature;

(ii) the principal curvature function $\alpha = g(A\xi, \xi)$ corresponding to the Reeb vector field ξ is constant;

(iii) the Reeb vector field $\xi = -JN$ is an eigenvector of the normal Jacobi operator $\bar{R}_N = \bar{R}(\cdot, N)N$ everywhere.

It is a natural problem to determine all contact hypersurfaces in a given Kähler manifold. We are going to discuss this in the next section for the complex space forms.

3 Contact hypersurfaces in complex space forms

The Riemannian curvature tensor $\bar{R}$ of an n-dimensional complex space form $\bar{M}^n(c)$ with constant holomorphic sectional curvature $4c$ is given by

$$\bar{R}(X,Y)Z = c\big(g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ\big).$$

This implies

$$\bar{R}(JN,N)N = 4cJN$$

and hence $\xi = -JN$ is an eigenvector of the normal Jacobi operator $\bar{R}_N$ everywhere. We thus get from Proposition 2.7:

Proposition 3.1. *Let M be a connected contact hypersurface of a complex space form $\bar{M}^n(c)$, $n \geq 3$. Then the functions α and ρ are constant and M has constant mean curvature.*

The contact hypersurfaces in complex Euclidean spaces $\mathbb{C}^n$ were classified by Okumura for $n \geq 3$:

Theorem 3.2. (Okumura [11]) *Let M be a connected orientable real hypersurface of $\mathbb{C}^n$ and $n \geq 3$. Then M is a contact hypersurface if and only if M is congruent an open part of one of the following contact hypersurfaces of $\mathbb{C}^n$:*

(i) the sphere $S^{2n-1}(r)$ of radius $r \in \mathbb{R}_+$ in $\mathbb{C}^n$;

(ii) the tube of radius $r \in \mathbb{R}_+$ around the totally geodesic $\mathbb{R}^n$ in $\mathbb{C}^n$.

Outline of proof: We did already show above that $S^{2n-1}(r)$ is a contact hypersurface of $\mathbb{C}^n$ for all $n \geq 2$. We will now prove that the tube M_r of radius $r \in \mathbb{R}_+$ around $\mathbb{R}^n$ is a contact hypersurface of $\mathbb{C}^n$ for all $n \geq 2$. Let $x \in \mathbb{R}^n$ and let $iy \in i\mathbb{R}^n$ be a unit normal vector of $\mathbb{R}^n$ at x. Denote by p the point on M_r at distance r from $\mathbb{R}^n$ and in direction iy. With the usual identifications we can regard iy also as a unit normal vector of M_r at p. The principal curvatures of M_r with respect to iy are 0 and $-\frac{1}{r}$ and the corresponding principal curvature spaces are $\mathbb{R}^n$ and $i(\mathbb{R}^n \ominus \mathbb{R}y)$, respectively. We can easily see now that the shape operator A_r of M_r satisfies

$$A_r\phi + \phi A_r = -\frac{1}{r}\phi.$$

It then follows from Proposition 2.3 that M_r is a contact hypersurface of $\mathbb{C}^n$.

Conversely, let M be a connected contact hypersurface of $\mathbb{C}^n$. It follows from Proposition 3.1 that ρ and α are constant and then from Proposition 2.5 that M has constant principal curvatures. Thus M is an isoparametric hypersurface of $\mathbb{C}^n$. The isoparametric hypersurfaces in Euclidean spaces were classified by Segre in [12]. Any such hypersurface is either a totally geodesic Euclidean hyperplane, a round sphere, or a tube around a totally geodesic Euclidean subspace. A totally geodesic Euclidean hyperplane satisfies $A\phi + \phi A = 0$ and therefore cannot be contact hypersurface. We therefore have to investigate the tubes around totally geodesic $\mathbb{R}^k \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$ for $k \in \{1, \ldots, 2n-2\}$. Any such tube has exactly two distinct constant principal curvatures 0 and $-\frac{1}{r}$ with multiplicites k and $2n-k-1$ respectively. A necessary condition for a contact hypersurface is that it must be a Hopf hypersurface. A tube around $\mathbb{R}^k$ is a Hopf hypersurface if and only if the complex structure i maps the normal spaces of $\mathbb{R}^k$ either to the normal spaces of $\mathbb{R}^k$ or to the tangent spaces of $\mathbb{R}^k$. In the first case $\mathbb{R}^k$ is embedded in $\mathbb{C}^n$ as a complex submanifold, that is, k is even, say $k = 2m$ and $\mathbb{R}^k \cong \mathbb{C}^m \subset \mathbb{C}^n$. Since the tube around $\mathbb{C}^m$ has an m-dimensional i-invariant principal curvature space with corresponding principal curvature 0 it follows that $A\phi + \phi A = 0$ on that space. Therefore a tube around $\mathbb{C}^m$ in $\mathbb{C}^n$ cannot be a contact hypersurface. In the second case, when i maps normal spaces of $\mathbb{R}^k$ to tangent spaces of $\mathbb{R}^k$, the Euclidean space must be of the form

$$\mathbb{R}^k \cong \mathbb{C}^m \oplus \mathbb{R}^{n-m}.$$

In this case the principal curvatures of the tube are 0 and $-\frac{1}{r}$ with multiplicities $n+m$ and $n-m-1$ respectively. The principal curvature space of 0 is $\mathbb{C}^m \oplus \mathbb{R}^{n-m}$ and we get $A\phi + \phi A = 0$ on the $\mathbb{C}^m$-part of this space. Thus we must have $m = 0$ and therefore we get a tube around $\mathbb{R}^n$ which is a contact hypersurface. This finishes the proof of Theorem 3.2.

Next we consider the n-dimensional complex projective space $\mathbb{C}P^n$ equipped with the Fubini Study metric of constant holomorphic sectional curvature 4.

Theorem 3.3. (Okumura [11]) *Let M be a connected orientable real hypersurface of $\mathbb{C}P^n$ and $n \geq 3$. Then M is a contact hypersurface if and only if M is congruent an open part of one of the following contact hypersurfaces of $\mathbb{C}P^n$:*

(i) the geodesic hypersphere of radius $r \in (0, \frac{\pi}{2})$ in $\mathbb{C}P^n$;

(ii) the tube of radius $r \in (0, \frac{\pi}{4})$ around the totally geodesic $\mathbb{R}P^n$ in $\mathbb{C}P^n$.

Outline of proof: It follows from Proposition 3.1 that α and ρ are constant. From Proposition 2.5 we conclude that A restricted to $\mathcal{C}$ has at most two distinct and constant principal curvatures. Thus M is a Hopf hypersurface with constant principal curvatures. Such hypersurfaces in $\mathbb{C}P^n$ were classified by Kimura in [9]. He proved that any such hypersurface is locally congruent to a homogeneous hypersurface in $\mathbb{C}P^n$. The principal curvatures and principal curvature spaces of the homogeneous hypersurfaces in $\mathbb{C}P^n$ were explicitly calculated by Takagi in [14]. Using this information it is a straightforward calculation to verify that among the homogeneous hypersurfaces in $\mathbb{C}P^n$ with at most three distinct constant principal curvatures the contact hypersurfaces are precisely those listed in Theorem 3.3.

We finally consider the n-dimensional complex hyperbolic space $\mathbb{C}H^n$ equipped with the Bergman metric of constant holomorphic sectional curvature -4.

Theorem 3.4. (Vernon [15]) *Let M be a connected orientable real hypersurface of $\mathbb{C}H^n$ and $n \geq 3$. Then M is a contact hypersurface if and only if M is congruent an open part of one of the following contact hypersurfaces of $\mathbb{C}H^n$:*

(i) the geodesic hypersphere of radius $r \in \mathbb{R}_+$ in $\mathbb{C}H^n$;

(ii) a horosphere in $\mathbb{C}H^n$;

(iii) the tube of radius $r \in \mathbb{R}_+$ around the totally geodesic $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$;

(iv) the tube of radius $r \in \mathbb{R}_+$ around the totally geodesic $\mathbb{R}H^n$ in $\mathbb{C}H^n$.

Outline of proof: The proof is analogous to the one for $\mathbb{C}P^n$, applying the classification of Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$ obtained by the author in [1].

4 Contact hypersurfaces and homogeneous hypersurfaces in Hermitian symmetric spaces

The complex space forms that we discussed in the previous section are the Euclidean Hermitian symmetric spaces and the Hermitian symmetric spaces of rank one. The classification of contact hypersurfaces in other Hermitian symmetric spaces is still an open problem. One complication here is that the structure of the Riemannian curvature tensor of a Hermitian symmetric space of higher rank is more complicated. No analogon of Proposition 3.1 for higher rank Hermitian symmetric spaces is known yet. In particular, it is an open problem whether or not a contact hypersurface in a Hermitian symmetric space has constant mean curvature.

All the examples of contact hypersurfaces we discussed above are open parts of homogeneous hypersurfaces. It is therefore a natural problem to classify first the homogeneous contact hypersurfaces in Hermitian symmetric spaces. Homogeneous hypersurfaces are orbits of cohomogeneity one actions. On irreducible simply connected symmetric spaces such actions were classified by Kollross in [10] for the compact case and investigated thoroughly by the author and Tamaru in [5], [6], [7] and [8] for the noncompact case.

We discuss this now for some irreducible Hermitian symmetric spaces of compact type:

1. The complex Grassmannian $G_p(\mathbb{C}^{p+q}) = SU_{p+q}/S(U_pU_q)$ of p-dimensional complex subspaces in $\mathbb{C}^{p+q}$, $1 \leq p \leq q$ has complex dimension $n = pq$ and its rank is equal to p. For $p = 1$ we get the q-dimensional complex projective space $\mathbb{C}P^q$ which we discussed in the previous section.

For $p = 2$ we have the complex 2-plane Grassmannian $G_2(\mathbb{C}^{q+2})$. There are two types of homogeneous hypersurfaces in $G_2(\mathbb{C}^{q+2})$:
Type (A): Tubes around the totally geodesic $G_2(\mathbb{C}^{q+1}) \subset G_2(\mathbb{C}^{q+2})$.

Type (B): (only if $q = 2m$ is even) Tubes around the totally geodesic $\mathbb{H}P^m \subset G_2(\mathbb{C}^{2m+2})$.

The contact hypersurfaces with constant mean curvature in $G_2(\mathbb{C}^{q+2})$ were classified by Suh:

Theorem 4.1. (Suh [13]) *Let M be a connected real hypersurface with constant mean curvature in $G_2(\mathbb{C}^{q+2})$, $q \geq 3$. Then M is a contact hypersurface if and only if q is even, say $q = 2m$, and M is congruent to an open part of a tube around the totally geodesic $\mathbb{H}P^m \subset G_2(\mathbb{C}^{2m+2})$.*

It is not known if the assumption of constant mean curvature is really necessary here.

For $p \geq 3$ no examples are known of contact hypersurfaces in $G_p(\mathbb{C}^{p+q})$. The only homogeneous hypersurfaces in $G_p(\mathbb{C}^{p+q})$ are the tubes around $G_{p-1}(\mathbb{C}^{p+q})$, or equivalently, around $G_p(\mathbb{C}^{p+q-1})$ since $G_p(\mathbb{C}^{p+q-1})$ is the focal set of $G_{p-1}(\mathbb{C}^{p+q})$.

2. The complex quadric $Q^n = SO_{n+2}/SO_nSO_2$ has complex dimension n and its rank is equal to 2. For $n = 1$ this is a 2-sphere S^2 and for $n = 2$ this is the product of two 2-spheres, so we consider only $n \geq 3$. The complex quadric can be realized as a complex hypersurface in $\mathbb{C}P^{n+1}$ given by the homogeneous equation $z_1^2 + \ldots + z_{n+2}^2 = 0$. Using the Gauss equation for the embedding $Q^n \subset \mathbb{C}P^{n+1}$ one can easily derive an explicit expression for the Riemannian curvature tensor of Q^n in terms of the Riemannian metric, the complex structure and a particular real structure on Q^n. There is in fact an S^1-family of real structures on Q^n coming from the shape operators of Q^n with respect to unit normal vectors in $\mathbb{C}P^{n+1}$ which can be used to describe the curvature of Q^n.

There are five types of homogeneous hypersurfaces in complex quadrics:

Type (A): Tubes around the totally geodesic $Q^{n-1} \subset Q^{n-2}$. The focal set of Q^{n-1} is an n-dimensional sphere S^n which is embedded in Q^n as a real form, that is, as a totally geodesic and totally real submanifold of half dimension. Therefore these hypersurfaces can also be considered as tubes around the real form S^n of Q^n. The author and Suh proved in [3] that every tube around S^n in Q^n is a homogeneous contact hypersurface of Q^n.

Type (B): ($n = 2m$ even) Tubes around the totally geodesic $\mathbb{C}P^m$ in Q^{2m}. These are the principal orbits of the action of $U_{m+1} \subset SO_{2m+2}$ on Q^{2m}.

Type (C): ($n = 4m - 2$) The action of $Sp_mSp_1 \subset SO_{4m}$ on Q^{4m-2} is of cohomogeneity one and the principal orbits are homogeneous hypersurfaces of Q^{4m-2}.

Type (D): ($n = 6$) The 6-dimensional quadric Q^6 can be written as a homogeneous space $Q^6 = SO_8/SO_6SO_2 = SO_8/U_4 = SO_7/U_3$. The exceptional Lie group $G_2 \subset SO_7$ acts on Q^6 with cohomogeneity one and therefore the principal orbits of this action are homogeneous hypersurfaces of Q^6.

Type (E): ($n = 14$) The spin representation of $Spin_9$ on $\mathbb{R}^{16}$ yields an embedding of $Spin_9$ into SO_{16}. The induced action of $Spin_9$ on Q^{14} is of cohomogeneity one and the principal orbits are homogeneous hypersurfaces of Q^{14}.

3. The Hermitian symmetric space SO_{2n}/U_n has complex dimension $n(n-1)/2$ and its rank is equal to $[n/2]$. For $n = 2$ we have $SO_4/U_2 = \mathbb{C}P^1$, for $n = 3$ we have $SO_6/U_3 = \mathbb{C}P^3$ and for $n = 4$ we have $SO_8/U_4 = Q^6$, and therefore we assume $n \geq 5$. The subgroup $SO_{2n-2}SO_2 \subset SO_{2n}$ acts on SO_{2n}/U_n with cohomogeneity one. The principal orbits are tubes around the totally geodesic $SO_{2n-2}/U_{n-1} \subset SO_{2n}/U_n$. These tubes are the only homogeneous hypersurfaces in SO_{2n}/U_n.

4. The Hermitian symmetric space Sp_n/U_n has complex dimension $n(n+1)/2$ and its rank is equal to n. For $n = 2$ we have $Sp_2/U_2 = Q^3$ and therefore we assume $n \geq 3$. The subgroup $Sp_{n-1}Sp_1 \subset Sp_n$ acts on Sp_n/U_n with cohomogeneity one and the principal orbits are tubes around the totally geodesic $Sp_{n-1}/U_{n-1} \times S^2 \subset Sp_n/U_n$. These tubes are the only homogeneous hypersurfaces in Sp_n/U_n.

5. The exceptional Hermitian symmetric space $E_6/Spin_{10}U_1$ has complex dimension 16 and its rank is equal to 2. The subgroup $F_4 \subset E_6$ acts on $E_6/Spin_{10}U_1$ with cohomogeneity one. The principal orbits are the tubes around the real form $\mathbb{O}P^2 = F_4/Spin_9$ of $E_6/Spin_{10}U_1$. There are no other homogeneous hypersurfaces in $E_6/Spin_{10}U_1$.

6. The exceptional Hermitian symmetric space E_7/E_6U_1 has complex dimension 27 and its rank is equal to 3. This symmetric space does not admit any homogeneous hypersurfaces.

There are many examples of cohomogeneity one actions on noncompact Hermitian symmetric spaces, also on the noncompact dual E_7^{-25}/E_6E_1 of E_7/E_6U_1. This makes the problem of classifying homogeneous contact hypersurfaces on these spaces more difficult. For example, one class of homogeneous hypersurfaces in symmetric spaces of noncompact type which have no analogue in the compact case are horospheres. The author and Suh investigated in [4] the geometry of horospheres in the noncompact Grassmannian $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$. This Grassmannian has two distinguished geometric structures, a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$. Horospheres can be seen as spheres with center at infinity with respect to the geodesic compactification, and geodesics correspond to tangent vectors. The Grassmannian $G_2^*(\mathbb{C}^{m+2})$ has two types of singular tangent vectors X which can be characterized by the two geometric properties $JX \in \mathfrak{J}X$ and $JX \perp \mathfrak{J}X$. All other tangent vectors are regular. This gives a corresponding concept of singular and regular points at infinity. From the results in [4] one can easily conclude:

Theorem 4.2. *Let M be a horosphere in $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Then M is a contact hypersurface if and only if M is a horosphere whose center at infinity is a singular point of type $JX \perp \mathfrak{J}X$.*

In recent work with Lee and Suh [2] the author investigated contact hypersurfaces in $G_2^*(\mathbb{C}^{m+2})$, and from their result and Proposition 2.7 one can deduce the following classification result:

Theorem 4.3. *Let M be a connected orientable real hypersurface with constant mean curvature in $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Then M is a contact hypersurface if and only if M is a horosphere whose center at infinity is a singular point of type $JX \perp \mathfrak{J}X$, or m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic quaternionic hyperbolic space $\mathbb{H}H^k \subset G_2^*(\mathbb{C}^{2k+2})$.*

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