Singular Solutions of a Seimilinear Parabolic equation

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We consider singular solutions of the Fujita equation

$$u_t = \Delta u + u^p \quad \text{in } \mathbb{R}^N, \qquad p > 1.$$

Typical superlinear equation

Appears naturally as a scaling limit

Scaling invariance

Simple-looking but rich mathematical structure

Various critical exponents

1. Singular steady states
2. Moving singularity
3. Dynamic singularity
4. Asymptotic behaviour of singular solutions
5. On-going and future works

1. Singular steady states

It has been known that if $N > 2$ and $p > p_{sg} := \dfrac{N}{N-2}$, then the equation

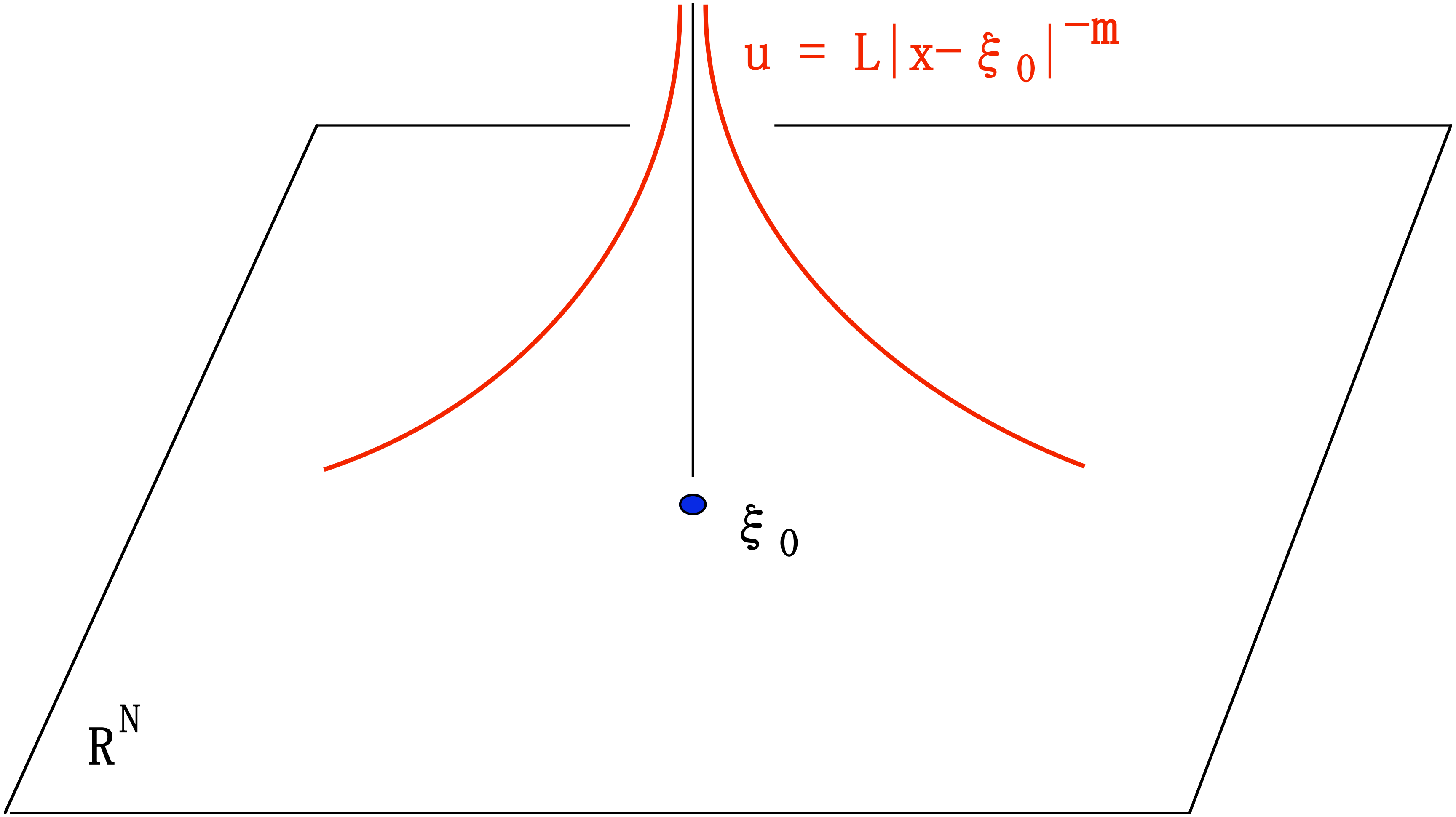
$$u_t = \Delta u + u^p, \qquad x \in \mathrm{R}^N,$$

has a singular steady state

$$u = \varphi_\infty(r) := Lr^{-m}, \quad r := |x - \xi_0|,$$

where $\xi_0 \in \mathrm{R}^N$ is arbitrary and

$$m := \frac{2}{p-1}, \qquad L := \left\{m(N-m-2)\right\}^{\frac{1}{p-1}}.$$



Singular steady state

Concerning other singular solutions, the exponents

$$p_* := \frac{N+2\sqrt{N-1}}{N-4+2\sqrt{N-1}}, \quad N > 2,$$

and

$$p_S := \frac{N+2}{N-2}, \quad N > 2$$

play crucial role.

(i) If $p_{sg} < p < p_S$, then for any $\alpha > 0$, the solution φ_α of

$$\begin{cases} \varphi_{rr} + \dfrac{N-1}{r}\varphi_r + \varphi^p = 0, & r > 0. \\ \lim_{r\to\infty} r^{N-2}\varphi(r) = \alpha. \end{cases}$$

is positive for all $r > 0$ and $\varphi(r) \to \infty$ as $r \to 0$. Then $u = \varphi_\alpha(|x|)$ is a singular steady state.

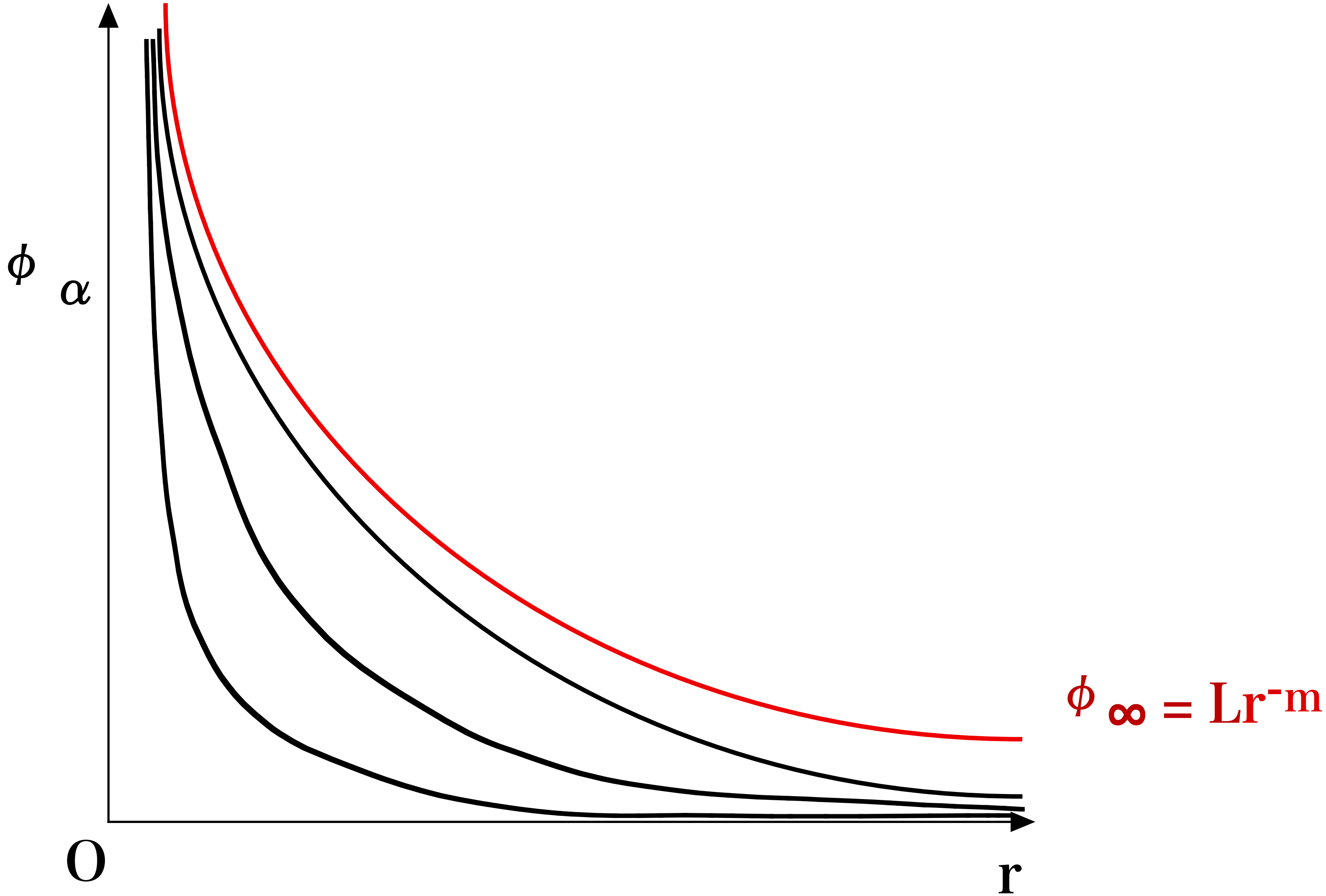
(ii) It was shown by Chen-Lin (1999) that for $p_{sg} < p < p_*$, $\{\varphi_\alpha\}$ the set of singular steady states $\{\varphi_\alpha\}$ has ordered structure (or separation property): $0 < \varphi_{\alpha_1}(r) < \varphi_{\alpha_2}(r) < \varphi_\infty(r)$ for all $0 < \alpha_1 < \alpha_2$ and $r > 0$. Moreover φ_α satisfies

$$\varphi_\alpha(r) = Lr^{-m} - a_\alpha r^{-\lambda_2} + o(r^{-\lambda_2}) \quad \text{as } r \to 0,$$

where

$$\lambda_1 := \frac{N-2-\sqrt{(N-2)^2-4pL^{p-1}}}{2},$$
$$\lambda_2 := \frac{N-2+\sqrt{(N-2)^2-4pL^{p-1}}}{2}.$$

and $0 < \lambda_1 < \lambda_2 < m$. The constant a_α is positive and monotone decreasing in α and satisfies $a_\alpha \to 0$ as $\alpha \to \infty$. We note that $u = \varphi_\infty(|x|)$ and $u = \varphi_\alpha(|x|)$ satisfy the Fujita equation in the distribution sense.



Structure of the singular steady states

2. Time-dependent singular solutions

The singularity of $u = \varphi_\alpha$ and $u = \varphi_\infty$ persists for all $t > 0$, but it does not move in time.

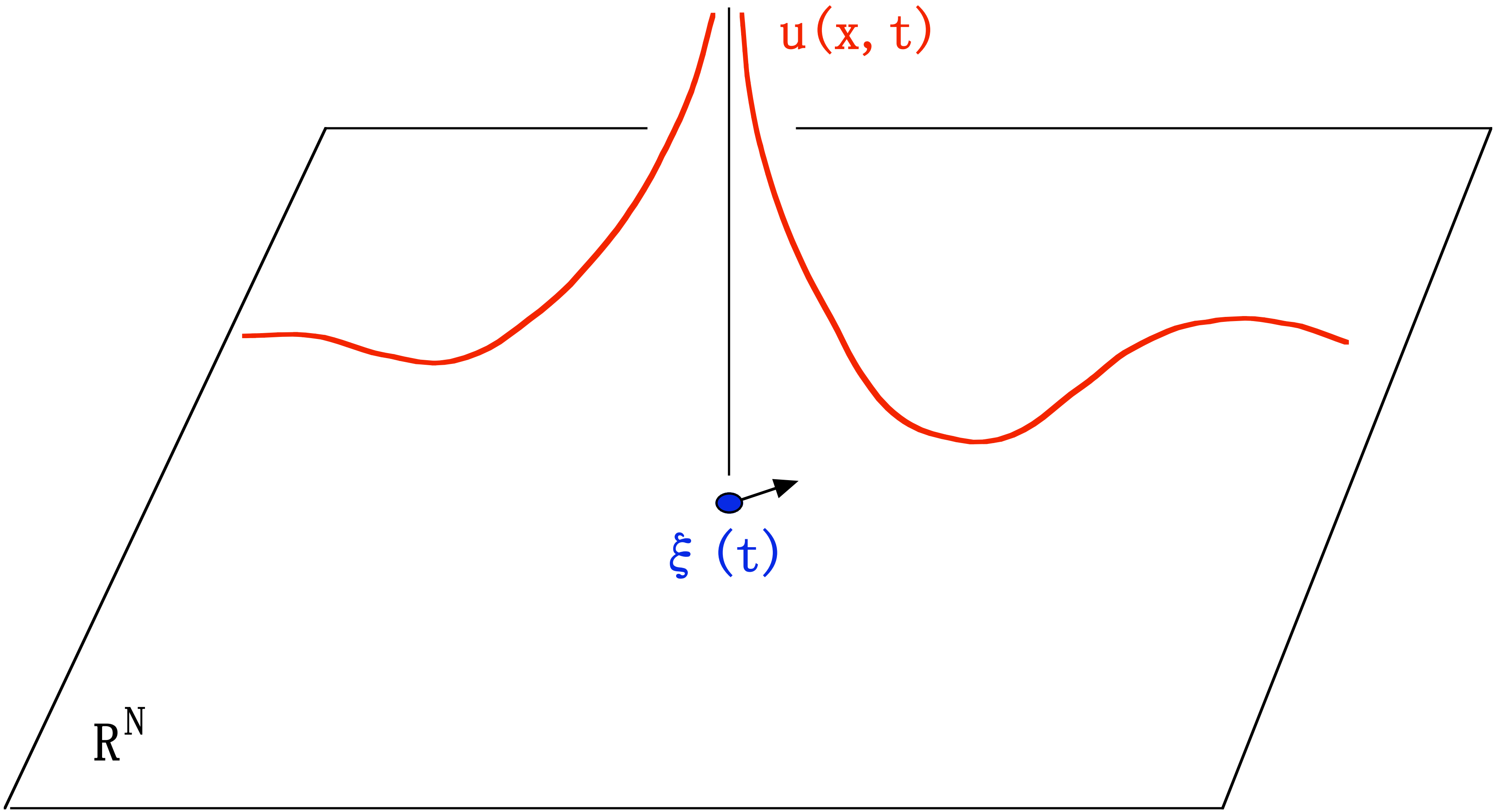
We define a solution with a moving singularity as follows.

Definition 1. $u(x,t)$ is a solution of the Fujita equation with a singularity at $\xi(t) \in \mathrm{R}^N$ if the following conditions are satisfied for some $T \in (0,\infty]$:

(i) $u(x,t)$ satisfies the equation in the distribution sense.

(ii) $u(x,t)$ is defined for $(x,t) \in \mathrm{R}^N \setminus \{\xi(t)\} \times [0,T)$, C^2 with respect to x, and C^1 with respect to t.

(iii) $u(x,t) \to \infty$ as $x \to \xi(t)$ for every $t \in [0,T)$.



Solution with a moving singularity

Consider the initial value problem

$$\text{(P)} \qquad \begin{cases} u_t = \Delta u + u^p, & x \in \mathrm{R}^N \setminus \{\xi(t)\}, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathrm{R}^N \setminus \{\xi(0)\}, \end{cases}$$

where $\xi(t) : [0, \infty) \to \mathrm{R}^N$ is prescribed.

[Assumptions]

(A1) $N \geq 3$ and $\dfrac{N}{N-2} < p < p_* := \dfrac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}}$.

(A2) $\xi(t)$ is sufficiently smooth.

(A3) $u_0(x)$ is nonnegative and continuous in $x \in \mathrm{R}^N \setminus \xi(0)$, and is uniformly bounded for $|x - \xi(0)| \geq 1$.

(A4) $u_0(x) = Lr^{-m} + o(r^{-m})$ as $r = |x - \xi(0)| \to 0$.

Under the assumptions (A1) - (A4), the following results are obtained by Sato-Y (2009, 2010, 2011) and Sato (2011).

(i) (Time-local existence) For some time interval $[0, T)$, there exists a solution u of (P) with a singularity at $\xi(t)$ such that

$$u(x,t) = Lr^{-m} + o(r^{-\lambda_2})$$

as $r = |x - \xi(t)| \to 0$ for all $t \in [0, T)$.

(ii) (Uniqueness) If u_1 and u_2 are two solutions of (P) such that

$$|u_1(x,t) - u_2(x,t)| = o(r^{-\lambda_2})$$

as $r = |x - \xi(t)| \to 0$, then $u_1 \equiv u_2$.

(iii) (Comparison principle) If $u_1 \leq u_2$ at $t = t_0$, then $u_1 \leq u_2$ for $t > t_0$.

(iv) (Time-global existence) For some $\xi(t) \not\equiv Const.$ and $u_0(x)$, the solution exists globally in time and is asymptotically radially symmetric as $t \to \infty$.

(v) (Sudden appearance and dissapaearcne of singularities) Singularities can appear or disappear at any time.

(vi) (Appearance of anomalous singularities) At some $t = T < \infty$, the leading term of u at $\xi(t)$ may become different from Lr^{-m}:

$$u(x,t) \simeq L|x - \xi(t)|^{-m} \quad \text{for } t \in (0,T),$$

$$u(x,t) \not\simeq L|x - \xi(t)|^{-m} \quad \text{at } t = T.$$

(vii) (Blow-up at spatial infinity of singular solutions) Blow-up can occur at spatial infinity, but the possibility of blow-up at a finite point is an open question.

Why $\frac{N}{N-2} < p < p_*$?

Assume that a solution $u(x,t)$ with a singularity at $\xi(t)$ is close to the singular steady state $u = L|x-\xi(t)|^{-m}$, and formally expand the solution $u(x,t)$ at $r=0$ as follows:

$$u(x,t) = Lr^{-m} + \sum_{i=1}^{[m]} b_i(\omega,t) r^{-m+i} + v(y,t),$$

where

$$m = \frac{2}{p-1}, \quad y = x - \xi(t), \quad r = |y|, \quad \omega = \frac{1}{|y|} y \in S^{N-1}.$$

Substitute this expansion into the equation and equate each power of r to obtain a system of equations for $b_i(\omega,t)$.

These equations are solvable and the remainder term $v(y,t)$ must satisfy

$$v_t = \Delta v + \xi_t \cdot \nabla v + \frac{pL^{p-1}}{|y|^2} v + o(|y|^{-2}).$$

This equation is well-posed if and only if

$$0 < pL^{p-1} < \frac{(N-2)^2}{4}.$$

These inequalities hold if

$$N > 2 \quad \text{and} \quad \frac{N}{N-2} < p < p_* = \frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}}.$$

$$L := \left\{ \frac{2}{p-1} \left(N - \frac{2}{p-1} - 2 \right) \right\}^{\frac{1}{p-1}}.$$

3. Existence of a solution with a dynamic singularity

Hereafter, we consider the case where the solution is time-dependent but the singular point is fixed to the origin (i.e., $\xi(t) \equiv 0$).

$$
\text{(P)} \qquad \begin{cases} u_t = \Delta u + u^p, & x \in \mathrm{R}^N \setminus \{0\}, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathrm{R}^N \setminus \{0\}, \\ u(x,t) \to \infty \ \text{ as } x \to 0, & t > 0. \end{cases}
$$

We shall show

- More general results for the existence and uniqueness.
- Convergence to φ_∞ from below.
- Convergence to φ_α.

Theorem 1. Let $N \geq 3$, $p_{sg} < p < p_*$ and $a(t) \in C^1([0,\infty))$ be given. Assume that

$u_0(x)$ is continuous and positive for $x \neq 0$,

$u_0(x)$ is uniformly bounded for $|x| > 1$,

$u_0(x) = L|x|^{-m} + O(|x|^{-\lambda})$ as $|x| \to 0$ for $\exists\lambda < \min\{m, \lambda_2 + 2\}$.

Then there exist $T > 0$ and a positive solution $u(x,t)$ of (P) defined on $\mathrm{R}^N \setminus \{0\} \times (0,T)$ with the following properties:

(i) $u(x,t)$ satisfies the equation in the distribution sense.

(ii) $u(x,t)$ is C^2 with respect to $x \neq 0$ and C^1 with respect to $t > 0$.

(iii) $u(x,t) = L|x|^{-m} - a(t)|x|^{-\lambda_2} + o(|x|^{-\lambda_2})$ as $|x| \to 0$.

Remarks

- **We can also show more general results about the uniqueness and comparison principle.**

- **For solutions with a moving singularity, we mainly considered the case where $a(t) \equiv 0$. When $a(t)$ is not constant, we say that the solution has a dynamic singularity.**

Outline of the proof

Step 1: Construct suitable comparison functions with a singularity at the origin.

Step 2: Construct a sequence of approximate solutions on annular domains

$$D_n := \{x \in \mathrm{R}^N : \frac{1}{n} < |x| < n\}$$

with suitable boundary conditions.

Step 3: Extract a convergent subsequence, and show that the limiting function is indeed a solution of (P) with desired properties.

4. Convergence from below to φ_∞

Theorem 2. Let $N \geq 3$ and $p_{sg} < p < p_*$. Assume that the initial value $u_0(x)$ satisfies

$$u_0(x) \text{ is continuous in } x \neq 0,$$

$$0 \leq u_0(x) \leq \varphi_\infty(|x|) \text{ for } x \in \mathrm{R}^N \setminus \{0\},$$

$$u_0(x) = \varphi_\infty(|x|) + O(|x|^{-\lambda}) \text{ as } |x| \to 0 \text{ for } {}^\exists\lambda < \min\{m, \lambda_2 + 2\}.$$

Then the singular solution $u(x,t)$ of (P) with $a(t) \equiv 0$ exists globally in time and has the following properties:

(i) $0 < u(x,t) < \varphi_\infty(|x|)$ for all $(x,t) \in \mathrm{R}^N \setminus \{0\} \times (0,\infty)$.

(ii) $u(x,t) \to \varphi_\infty(|x|)$ as $t \to \infty$ uniformly on any compact set in $\mathrm{R}^N \setminus \{0\}$.

(iii) If u_0 satisfies

$$0 \le \varphi_\infty(|x|) - u_0(x) \le c_1(1+|x|)^{-l} \quad \text{for } x \in \mathrm{R}^N \setminus \{0\}$$

with some $c_1 > 0$ and $l \in (m, N-\lambda_1)$, then there exists $c_2 > 0$ such that the singular solution satisfies

$$0 < |x|^{\lambda_1}\left|\varphi_\infty(|x|) - u(x,t)\right| \le c_2 t^{-\frac{l-\lambda_1}{2}} \quad \text{for all } t > 1.$$

Here, the range $l \in [m, N-\lambda_1)$ and the rate $\frac{l-\lambda_1}{2}$ are optimal.

Proof. The proof is based on the comparison method. We look for a subsolution of the form

$$u^-(x,t) := \max\{\varphi_\infty(r) - U(|x|,t),\ 0\}.$$

It becomes a subsolution if U is positive and satisfies the linearized equation at φ_∞:

$$U_t = U_{rr} + \frac{N-1}{r}U_r + p\varphi_\infty(r)^{p-1}U,$$

where $p\varphi_\infty(r)^{p-1} = \dfrac{pL^{p-1}}{r^2}$. Here we set $V(r,t) := r^{\lambda_1}U(r,t)$, where $0 < \lambda_1 < \lambda_2$ be the roots of

$$\lambda^2 - (N-2)\lambda + pL^{p-1} = 0.$$

Then the linearized equation is rewritten as a generalized radial heat equation

$$V_t = V_{rr} + \frac{d-1}{r} V_r, \qquad r > 0,\ t > 0,$$

where

$$d := N - 2\lambda_1 = \lambda_2 - \lambda_1 + 2 > 2.$$

The generalized radial heat equation has been extensively studied in 1960's. Among others, we use a result by Bragg (1966) to show that $U \to 0$ as $t \to 0$ with a desired rate.

5. Convergence to the singular steady state φ_α

Theorem 3. Assume the same conditions as in Theorem 2. Then the singular solution $u(x,t)$ of (P) with $a(t) \equiv a_\alpha$ exists globally in time and has the following properties:

(i) $0 < u(x,t) < \varphi_\infty(|x|)$ for all $(x,t) \in \mathrm{R}^N \setminus \{0\} \times (0,\infty)$.

(ii) $u(x,t) \to \varphi_\alpha(|x|)$ as $t \to \infty$ uniformly on any compact set in $\mathrm{R}^N \setminus \{0\}$.

(iii) If u_0 satisfies

$$\big|u_0(x) - \varphi_\alpha(|x|)\big| \le c_1(1+|x|)^{-l} \qquad \text{for } x \in \mathrm{R}^N \setminus \{0\}$$

with some $c_1 > 0$ and $l \in (m, N)$, then there exists $c_2 > 0$ such that

$$|x|^{\lambda_1}\big|u(x,t) - \varphi_\alpha(|x|)\big| \le c_2 t^{-\frac{l-\lambda_1}{2}} \qquad \text{for all } t > 1.$$

Here, the range $l \in [m, N)$ and the rate $\frac{l-\lambda_1}{2}$ are optimal.

[Idea of the proof]

The proof is more delicate than that of Theorem 2. The linearized equation at φ_α is written as

$$U_t = U_{rr} + \frac{N-1}{r} U_r + p\varphi_\alpha(r)^{p-1} U.$$

Setting $V(r,t) := r^{\lambda_1} U(r,t)$, this equation is rewritten as

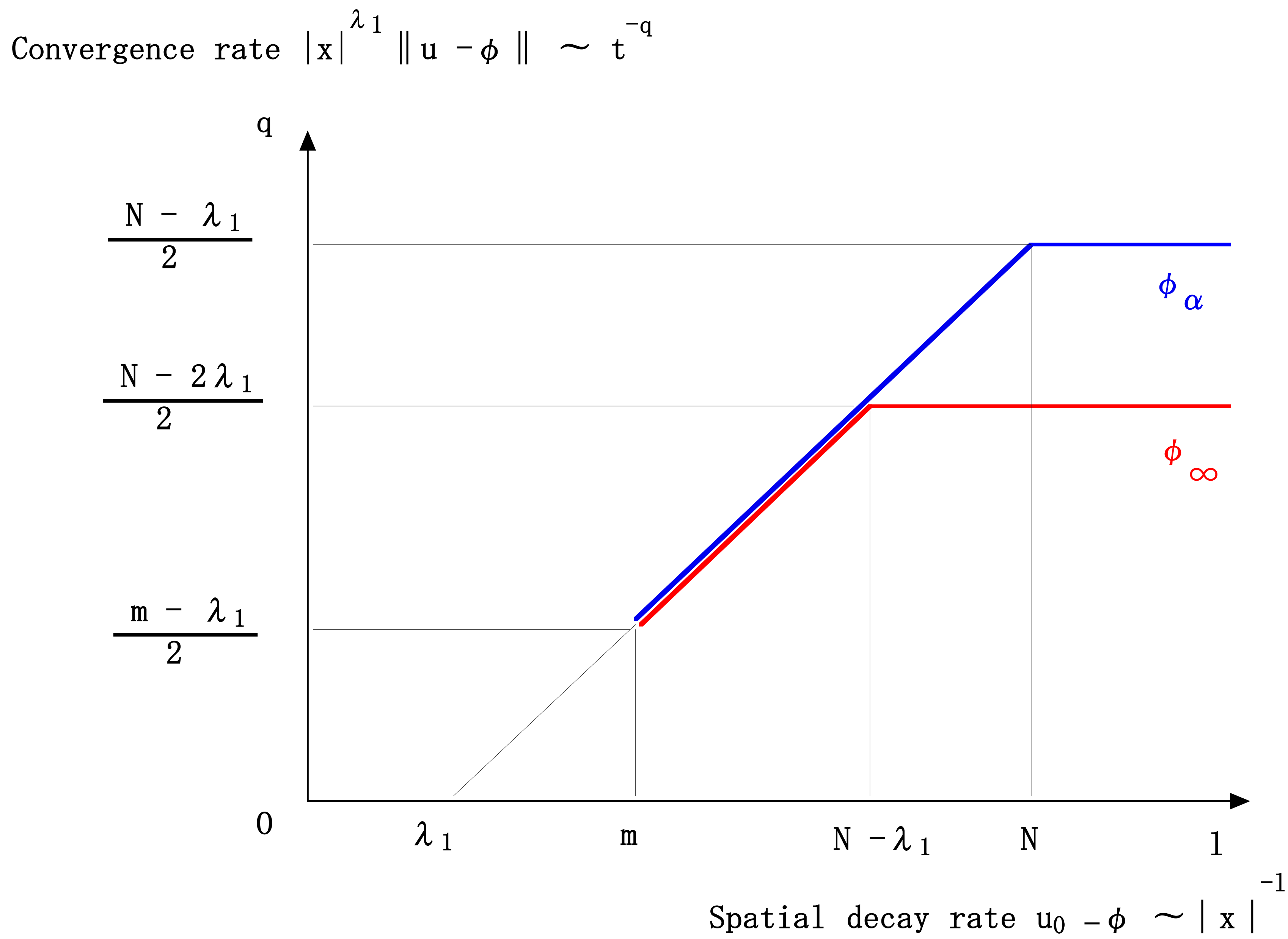
$$V_t = V_{rr} + \frac{d-1}{r} V_r + g(r,t) V,$$

where

$$g(r,t) := p\varphi_\alpha(r)^{p-1} - \frac{pL^{p-1}}{r^2} < 0,$$

$$g(0,t) = 0 \quad \text{and} \quad g(r) \simeq -Cr^{-2} \text{ at } r \simeq \infty.$$

We study the behavior of solutions of this equation by using the matched asymptotics.



Convergence rate $|x|^{\lambda_1} \| u - \phi \| \sim t^{-q}$
q
$\frac{N - \lambda_1}{2}$
$\frac{N - 2\lambda_1}{2}$
$\frac{m - \lambda_1}{2}$
0
λ_1
m
$N - \lambda_1$
N
1
ϕ_α
ϕ_∞
Spatial decay rate $u_0 - \phi \sim |x|^{-1}$

Remark 1.

In Theorems 2 and 3, the rate $\frac{l-\lambda_1}{2}$ is common, but the ranges are different. Namely, φ_α is slightly more stable than φ_∞.

Remark 2.

The convergence is faster in the inner region.

In the inner region ($|x| < Ct^{1/2}$)

$$|x|^{\lambda_1}\left|u(x,t) - \varphi_\alpha(|x|)\right| \le c_2 t^{-\frac{l}{2}} \qquad \text{for all } t > 1.$$

In the whole space,

$$|x|^{\lambda_1}\left|u(x,t) - \varphi_\alpha(|x|)\right| \le c_2 t^{-\frac{l-\lambda_1}{2}} \qquad \text{for all } t > 1.$$

This suggests that the convergence rate may vary depending on the spatial weight.

[Convergence from above to φ_∞]

Theorem 4. Let $p_{sg} < p < \begin{cases} p_* & \text{for } 2 < N \leq 10, \\ \dfrac{N+2}{N-1} & \text{for } N > 10. \end{cases}$

Assume that the initial value $u_0(x)$ satisfies

$$u_0(x) \text{ is continuous in } x \neq 0,$$

$$\varphi_\infty(|x|) \leq u_0(x) \leq (1+\delta)\varphi_\infty(|x|) \; x \in \mathbf{R}^N \setminus \{0\},$$

$$u_0(x) = L|x|^{-m} + O(|x|^{-\lambda}) \text{ as } |x| \to 0 \text{ for } {}^\exists\lambda < \min\{m, \lambda_2 + 2\}.$$

If $\delta > 0$ is sufficiently small and $a(t) \equiv 0$, then the singular solution $u(x,t)$ of (P) exists globally in time and has the following properties:

(i) $\varphi_\infty(|x|) \leq u(x,t) < \infty$ for $(x,t) \in \mathbf{R}^N \setminus \{0\} \times (0,\infty)$,

(ii) $u(x,t) \to \varphi_\infty(|x|)$ as $t \to \infty$ uniformly on any compact set in $\mathbf{R}^N \setminus \{0\}$.

We construct a supersolution by using a forward self-similar solution with a singularity at the origin. We have found that such a solution exists above the singular steady state φ_∞ if and only if

$$p_{sg} < p < \begin{cases} p_* & \text{for } N \le 10, \\ \dfrac{N+2}{N-1} & \text{for } N > 10. \end{cases}$$

Ongoing and future works:

Behaviour of solutions in the case $a(t) \not\equiv Const.$

Asymptotic behaviour in the case $\xi(t) \not\equiv Const.$

Time-periodic solution with a singularity

Multiple and higher dimensional singularities

Bounded domain

Singularities on a boundary

Removability of singularities

Collision and splitting of singularities

Other parameter regions

Other equations