An application of weighted Hardy spaces to the Navier-Stokes equations

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Introduction

$n \geq 2$. L^2-decay estimates of solutions to

$$\text{(N-S)} \quad \begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla\pi = 0, & \text{in} \quad (0,\infty)\times\mathbb{R}^n \\ \operatorname{div} u = 0, \quad \text{in} \quad (0,\infty)\times\mathbb{R}^n \\ \operatorname{div} a = 0, \quad u(0) = a & \text{in} \quad \mathbb{R}^n. \end{cases}$$

- Kato ('84), Giga-Miyakawa ('85); $a \in L^n$ with small $\|a\|_{L^n}$ and $a \in L^p$, $(1 < p < 2)$

$$\Rightarrow \|u(t)\|_{L^2} \lesssim t^{-\gamma} \quad \text{with} \quad \gamma = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right).$$

- Wiegner ('87): $a \in L^2$ with $\|e^{t\Delta}a\|_{L^2} \lesssim t^{-\theta}, (\theta \geq 0), \iff a \in \dot{B}^{-2\theta}_{2,\infty}$

$$\Rightarrow \|u(t)\|_{L^2} \lesssim t^{-\gamma_W} \quad \text{with} \quad \gamma_W = \min\left(\theta, \frac{n+2}{4}\right).$$

$$\gamma < \frac{n+2}{4} \left(\iff \frac{n}{n+1} < p\right).$$

Introduction

u : sol. of (N-S) $\Longleftrightarrow$

$$\text{(I.E.)} \quad u(t) = e^{t\Delta}a - \int_0^t e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla)u(s)ds,$$

where

$$\mathbb{P} = \left(\delta_{i,j} + R_iR_j\right)_{1\le i,j\le n} \text{ : Helmholtz projection.}$$

- Convolution estimates (Theorem 1) $\Rightarrow e^{t\Delta}a$.
- Div-curl estimates (Theorems 2 and 3) $\Rightarrow (u\cdot\nabla)u$.

Contents

- Muckenhoupt classes A_p and Weighted Hardy spaces $H^p(w)$
- Linear Estimate: Convolution estimates on weighted Hardy spaces
- Non Linear Estimate: Div - Curl estimates
- Their application to decay estimates of solutions to the Navier-Stokes equations

Muckenhoupt classes A_p, $(1 \le p \le \infty)$

w: weight $\iff$ $0 \le w \in L^1_{loc}(\mathbb{R}^n)$.
$w(E) = \int_E w dx$, $(E \subset \mathbb{R}^n)$.
B: ball in $\mathbb{R}^n$.

- $w \in A_p \ (1 \le p < \infty) \iff |B|^{-1} \int_B |f| dx \le c \left(w(B)^{-1} \int_B |f|^p w dx \right)^{1/p}$.
- $A_\infty = \bigcup_{1 \le p < \infty} A_p$.
- $p < q \Rightarrow (A_1 \subset) \quad A_p \subset A_q \quad (\subset A_\infty)$.
- $$|x|^\alpha, \ \langle x \rangle^\alpha \in A_p \iff \begin{cases} -n < \alpha < n(p-1) & \text{if } 1 < p \le \infty \\ -n < \alpha \le 0 & \text{if } p = 1 \end{cases}$$

Weighted Hardy spaces

With $w \in A_\infty$ and $0 < p < \infty$,

$$H^p(w) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n); \sup_{t>0} \left| e^{t\Delta} f \right| \in L^p(w) \right\}$$

with

$$\|f\|_{H^p(w)} = \left\| \sup_{t>0} \left| e^{t\Delta} f \right| \right\|_{L^p(w)}, \text{where } \|g\|_{L^p(w)}^p = \int |g|^p w dx,$$

$e^{t\Delta} f = f * G_{\sqrt{t}}$, $G(x) = (4\pi)^{-n/2} \exp(-|x|^2/4)$ and $g_t(x) = t^{-n} g(x/t)$.

- $$p \in (1, \infty) \quad \text{and} \quad w \in A_p \Rightarrow H^p(w) = L^p(w).$$

- $$q > p \geq 1 \Rightarrow^{\exists} w \in A_q \quad \text{s.t.} \quad \delta \in H^p(w)$$

  (from Strömberg-Torchinsky).

Convolution estimates

Let $0 < p \le q < \infty$.

Theorem 1

Let $w, \sigma \in A_\infty$. Assume that there exists $K > 0$ such that

$$[w,\sigma]_{X^K_{p,q}} = \sup_B \min\left(1, |B|^K\right) \frac{\sigma(B)^{1/q}}{w(B)^{1/p}} < \infty. \tag{1}$$

Then, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\|f * \varphi\|_{H^q(\sigma)} \lesssim [w,\sigma]_{X^K_{p,q}} \|f\|_{H^p(w)}. \tag{2}$$

The proof uses atomic decomposition and molecular characterization theories for weighted Hardy spaces.

Convolution estimates

Let $0 < p \le q < \infty$, $-n/q < \beta \le \alpha < \infty$, $w(x) = |x|^{\alpha p}$ and $\sigma(x) = |x|^{\beta q}$.

Lemma

There exists $K > 0$ so that $[w, \sigma]_{X^K_{p,q}} < \infty$.

- Homogeneity:

$$\|f(\lambda\cdot)\|_{H^p(w)} = \lambda^{-(\alpha+n/p)}\|f\|_{H^p(w)}.$$

Corollary

$$\|e^{t\Delta} f\|_{H^q(\sigma)} \lesssim t^{-\gamma}\|f\|_{H^p(w)} \tag{3}$$

with

$$\gamma = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{\alpha - \beta}{2}.$$

Convolution estimates

Remark

If

$$\|e^{t\Delta} f\|_{L^q(\sigma)} \lesssim \|f\|_{L^p(w)},$$

then it have to fulfill that

$$-n/q < \beta \le \alpha \le n(1 - 1/p)$$

which almost implies $w \in A_p$ and $\sigma \in A_q$.

Sketch of the proof of Theorem 1

Let $1 < p \le q < \infty$ and $f \in H^p(w)$. From Strömberg-Torchinsky,

$$f = \sum_j \lambda_j a_j \text{ with supp} a_j \subset B_j,\ \|a_j\|_{L^\infty} \le 1 \text{ and } \partial^\gamma \hat{a}_j(0) = 0, (|\gamma| \le N)$$

with for all $s > 0$

$$\left\| \sum_j \lambda_j^s \chi_{B_j} \right\|_{L^{p/s}(w)}^{1/s} \lesssim \|f\|_{H^p(w)}.$$

$$\|f * \varphi\|_{H^q(\sigma)} = \left\| \sup_{t>0} \left| e^{t\Delta} (f * \varphi) \right| \right\|_{L^q(\sigma)} \lesssim \left\| \sum_j \lambda_j \sup_{t>0} \left| e^{t\Delta}(a_j * \varphi) \right| \chi_{2B_j} \right\|_{L^q(\sigma)}$$

$$+ \left\| \sum_j \lambda_j \sup_{t>0} \left| e^{t\Delta}(a_j * \varphi) \right| \chi_{(2B_j)^c} \right\|_{L^q(\sigma)} =: I + II.$$

Sketch of the proof of Theorem 1; Estimate of I

$$I \lesssim \sum_{j=1}^{\infty} \lambda_j \left\| \sup_{t>0} \left| e^{t\Delta}(a_j * \varphi) \right| \right\|_{L^\infty} \int_{2B_j} |g| \sigma dx,$$

with some $\|g\|_{L^{q'}(\sigma)} \le 1$. For some $r > 1$ and any $s > 0$,

- $$\left\| \sup_{t>0} \left| e^{t\Delta}(a_j * \varphi) \right| \right\|_{L^\infty} \lesssim \min\left(1, |B_j|^{1+(1+N)/n}\right).$$

- $$\int_{2B_j} |g| \sigma dx \lesssim \frac{\sigma(B_j)^{1/q}}{w(B_j)^{1/p}} \left(\int_{2B_j} M_{r;w}\left(|g| w^{\alpha - 1/r} \sigma^{1/q'} \right)^s dx \right)^{1/s},$$

  where $M_{r;w}h(x) = \left(\sup_{B \ni x} w(Q)^{-1} \int_Q |h|^r w dx \right)^{1/r}$.

- $$K < 1 + (1+N)/n \Rightarrow \min\left(1, |B_j|^{1+(1+N)/n}\right) \frac{\sigma(B_j)^{1/q}}{w(B_j)^{1/p}} \lesssim [w, \sigma]_{X^K_{p,q}}.$$

Div-Curl estimates

Coifman-Lions-Meyer-Semmes ('93):
$n/(n+1) < p, q < \infty,\ 1/r = 1/p + 1/q < 1 + 1/n$ and div $u = 0$,

$$\Rightarrow \|(u\cdot\nabla)v\|_{H^r} \lesssim \|u\|_{H^p}\|\nabla v\|_{H^q}.$$

Here $u = (u_1, \cdots, u_n),\ v = (v_1, \cdots, v_n)$ and

$$(u\cdot\nabla)v = \left(\sum_{j=1}^{n} u_j\partial_j v_1, \cdots, \sum_{j=1}^{n} u_j\partial_j v_n\right).$$

Aim:

$$\|(u\cdot\nabla)v\|_{H^r(\mu)} \lesssim \|u\|_{H^p(w)}\|\nabla v\|_{H^q(\sigma)}$$

with $1/r = 1/p + 1/q < 1 + 1/n,\ \mu = w^{r/p}\sigma^{r/q}$ and div $u = 0$.

Div-Curl estimates; Non-endpoint cases $p < \infty$

Theorem 2

Let $n/(n+1) < p, q < \infty$ and $1/r = 1/p + 1/q < 1 + 1/n$. Suppose that there exist $p^* \in (1, p(1 + 1/n))$ and $q^* \in (1, q(1 + 1/n))$ so that

$$w \in A_{p^*},\ \sigma \in A_{q^*} \quad \text{and} \quad p^*/p + q^*/q < 1 + 1/n.$$

Then,

$$\|(u \cdot \nabla)v\|_{H^r(\mu)} \lesssim \|u\|_{H^p(w)} \|\nabla v\|_{H^q(\sigma)} \tag{4}$$

with div $u = 0$ and $\mu = w^{r/p}\sigma^{r/q}$.

$$\|(u \cdot \nabla)v\|_{H^r(\mu)} \lesssim \sum_{j,k=1}^{\infty} \|u_j R_j(|\nabla| v_k) - (R_j u_j)|\nabla| v_k\|_{H^r(\mu)}$$

The proof uses pointwise estimates for a maximal function due to Miyachi ('00) and the boundedness of Riesz transforms R_k on $H^p(w)$.

Div-Curl estimates; Endpoint cases $p = \infty$

Theorem 3

Let $n/(n+1) < q < \infty$ and $\sigma \in A_\infty$. Then

$$\|(u \cdot \nabla)v\|_{H^q(\sigma)} \lesssim \|u\|_{L^\infty} \|\nabla v\|_{H^q(\sigma)} \tag{5}$$

provided that div $u = 0$ and $\sigma \in A_{q(1+1/n)}$ when $q \leq 1$.

Auscher-Russ-Tchamitchian ('05):

$$\sup_{t>0} \left| e^{t\Delta} \left(\sum_{j=1}^{\infty} u_j \partial_j v_k \right)(x) \right| \lesssim \|u\|_{L^\infty} N_m(\nabla v_k)(x),$$

with $N_m(\nabla v_k)(x) = \sup_{\Psi \in \Lambda_m(x)} |\int \nabla v_k \cdot \Psi dy|$ where

$\Psi \in \Lambda_m(x) \iff {}^\exists B \ni x$ s.t. $\Psi \in W^{1,m}(B)$ and $\|\Psi\|_{L^m} + |B|^{1/n} \|\nabla \Psi\|_{L^m} \leq |B|^{-1/m'}$.

Application to N.-S.

Assumptions:

$$1 \le p \le 2,\ 0 \le \alpha < n(1-1/p)+1 \quad \text{and} \quad w(x) = |x|^{\alpha p} \in A_{p(1+1/n)}.$$

Theorem 4

If $a \in L^n(\mathbb{R}^n) \cap H^p(w)$, div $a = 0$ and $\|a\|_{L^n} \ll 1$, then we can find a global solution $u \in C((0,\infty); L^n \cap H^p(w))$ to (N-S) satisfying

$$\|u(t)\|_{H^q(\sigma)} \lesssim t^{-\gamma}\|a\|_{H^p(w)} \text{ with } \gamma = \frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{\alpha-\beta}{2} \tag{6}$$

and $\sigma(x) = |x|^{\beta q} \in A_{q(1+1/n)}$ where $p \le q < \infty$ and $-n/q < \beta \le \alpha$.

Application to N.-S.; Comparison with Wiegner

(6) with $(q,\beta) = (2,0) \Rightarrow$

$$\|u(t)\|_{L^2} \lesssim t^{-\gamma} \quad \text{with} \quad \gamma = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{\alpha}{2}.$$

Now, $1 \le p \le 2$ and $0 \le \alpha < n(1 - 1/p) + 1$ are assumed.

- $$\gamma = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{\alpha}{2} < \frac{n+2}{4} \quad \Longleftrightarrow \alpha < n(1 - 1/p) + 1.$$

- $$\alpha = n(1 - 1/p) + 1 \Longleftrightarrow \frac{n}{2}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{\alpha}{2} = \frac{n+2}{4}.$$