Real-Analytic Operator Equations

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Classical Setting for Bifurcation Theory

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$$F(\lambda, x) = x - \lambda Lx - R(\lambda, x)$$

where L is a compact linear operator R is compact (maps bounded sets into relatively compact sets) with $\|R(\lambda, x)\|/\|x\| \to 0$ as $\|x\| \to 0$.

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Definition: λ_0 is a *bifurcation point* if a sequence $\{(\lambda_k, x_k)\}$ of non-trivial solutions exists with

$$F(\lambda_k, x_k) = 0, \quad \lambda_k \to \lambda_0, \quad x_k \to 0, \quad x_k \neq 0$$

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Since $\lambda_k \to \lambda_0$, L is compact, $\|R(\lambda_k, x_k)\|/\|x_k\| \to 0$ and

$$\frac{x_k}{\|x_k\|} - \lambda_k L\left(\frac{x_k}{\|x_k\|}\right) - \frac{R(\lambda_k, x_k)}{\|x_k\|} = 0$$

and since L is compact, it follows that a subsequence of $\frac{x_k}{\|x_k\|}$ converges strongly to v where $\|v\| = 1$ is a characteristic vector of v with characteristic value λ_0

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All bifurcation points are characteristic values of L

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- the generalised kernel $\mathcal{N}(\lambda_0) = \bigcup_{n\in\mathbb{N}} \ker(\lambda I - L)^n$ is finite dimensional - its dimension equals the codimension of the generalised range $\mathcal{R}(\lambda_0) = \bigcap_{n\in\mathbb{N}} \text{range}(\lambda I - L)^n$ is called the *the multiplicity of λ_0*

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Question: Which characteristic values are bifurcation points.

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If λ_0 is simple with characteristic vector $\xi_0 \neq 0$ there exists a C^{k-1}-function $(\Lambda, \kappa) : (-\epsilon, \epsilon) \to \mathbb{R} \times X$ such that

$$F(\Lambda(s), \kappa(s)) = 0 \text{ for all } s \in (-\epsilon, \epsilon),$$
$$(\Lambda(0), \kappa(0)) = (\lambda_0, 0), \;\; \kappa'(0) = \xi_0$$

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- $\ker(I - \lambda_0 L) \neq \{0\}$ *and* $q \in \mathbb{N}$ *is the codimension of range*$(I - \lambda_0 L)$.

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Then there exist: open sets U *and* V *with*
$(\lambda_0, 0) \in U \subset \mathbb{R} \times X,\ (\lambda_0, 0) \in V \subset \mathbb{R} \times \ker(L),$

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$$F(\lambda, x) = 0,\ (\lambda, x) \in U \Leftrightarrow$$
$$\omega(\lambda, \xi) = x \text{ where } (\lambda, \xi) \in V \text{ and } h(\lambda, \xi) = 0.$$

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When λ_0 is simple, $h : \mathbb{R}^2 \to \mathbb{R}$ and the occurrence of bifurcation for $h(\lambda, \xi) = 0$ is almost trivial from the implicit function theorem applied to

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Note: the kernel being one-dimensional is not enough

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- Characteristic values of even multiplicity may not be bifurcation points, even when operators are polynomials. Here is an example:

  $\lambda z - z - i|z|^2 z = 0$ has no non-trivial solutions $(\lambda, z) \in \mathbb{R} \times \mathbb{C}^2$
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- There are C^∞ examples where non-simple characteristic values are bifurcation points but no continuum bifurcates
- When X is a Hilbert space and $F(\lambda, x) = \nabla_x \Phi(\lambda, x)$, L is self-adjoint and *all characteristic values* are bifurcation points

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- $\mathcal{C}_0$ is unbounded;
- $(\lambda^*, 0) \in \overline{\mathcal{C}}_0$ for some characteristic value $\lambda^* \neq \lambda_0$ with odd multiplicity

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Now let $F(\lambda, x) = h(\lambda, x)(x - \lambda L x)$ for any compact linear L

From MathSciNet:

MR0375019 (51 #11215) Dancer, E. N. *Global structure of the solutions of non-linear real analytic eigenvalue problems.* Proc. London Math. Soc. (3) 27 (1973), 747765.

Let E and G be real Banach spaces. Suppose that $F : E \times \mathbb{R} \to G$ is a real analytic and Fredholm mapping. **The author considers the equation $F(x, \lambda) = 0$ and, proving some results on finite-dimensional real analytic germs, he obtains results on the local and global structure of solutions,** i.e., results on the properties of the set $D = \{(x, \lambda) : E \times (-\infty, \infty) : \ F(x, \lambda) = 0\}$ (e.g., D is locally compact, σ-compact, locally path-connected and closed). Under the assumption that F is real analytic, the set D has a number of rather nice properties **(it is impossible to present briefly here these properties)**; this result complements earlier results. [see, e.g., P. H. Rabinowitz, J. Functional Analysis 7 (1971), 487513]

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$$\Lambda'(s) \neq 0 \text{ for } s \in (0, \epsilon), \quad \kappa'(s) \neq 0 \text{ for } s \in (-\epsilon, \epsilon),$$
$$\mathcal{R}^+ := \{(\Lambda(s), \kappa(s)) : s \in (0, \epsilon)\} \subset \mathcal{T} \cap \mathfrak{N}.$$

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(f) If $(\Lambda(s_1), \kappa(s_1)) = (\Lambda(s_2), \kappa(s_2)) \in \mathfrak{N}$, $\quad s_1 \neq s_2$, then (e)(ii) occurs and $|s_1 - s_2|$ is an integer multiple of T.

In particular, $(\Lambda, \kappa) : [0, \infty) \to \mathcal{S}$ is locally injective.

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- (e)(i) is stronger than saying $\mathfrak{R}$ is unbounded in $\mathbb{R} \times X$.

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    $$(\lambda_{n+1}, x_{n+1}) \in \big(\partial\mathcal{A}_n \cap \partial\mathcal{A}_{n+1}\big) \setminus \{(\lambda_n, x_n)\}$$

    and there exists an injective $\mathbb{R}$-analytic map $\rho : (-1, 1) \to \mathcal{A}_n \cup \mathcal{A}_{n+1} \cup \{(\lambda_{n+1}, x_{n+1})\}$ with $\rho(0) = (\lambda_{n+1}, x_{n+1})$. Hence $\mathcal{A}_{n+1}$ is uniquely determined by $\mathcal{A}_n$ and vice versa.

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$\{\mathcal{A}_0\}$, $\{(\lambda_0, 0)\}$ is a route of length 1 with $(\lambda_0, 0) \in \partial \mathcal{A}_0$

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Once we understand that structure, the global unique continuation result is more-or-less obvious

The Story So Far

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Theorem. F is analytic on U if and only if for each $x_0 \in U$ there exist constants r, C, $R > 0$, depending on x_0, such that

$$\left\| d^k F[x] \right\| \leq \frac{C\, k!}{R^k} \text{ for all } x \in U \text{ with } \|x - x_0\| < r.$$

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Theorem *Suppose that that $U \subset X$ is an open connected set and that $F : U \to Y$ is $\mathbb{F}$-analytic. Suppose also that $F \equiv 0$ on a non-empty open set $W \subset U$. Then F is identically zero on U.*

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(Riemann Extension Theorem) If f is $\mathbb{C}$-analytic on $U \setminus E$ and $\sup\{|f(x)| : x \in U \setminus E\} < \infty$, there exists a $\mathbb{C}$-analytic function $\widetilde{f}$ on U with $f = \widetilde{f}$ on $U \setminus E$.

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X, Y, Z Banach spaces, $(x_0, y_0) \in U$ (open) $\subset X \times Y$, $F : U \to Z$ analytic and $\partial_x F[(x_0, y_0)] \in \mathcal{L}(X, Z)$ bijective.

Then $y_0 \in V$(open) $\subset Y$, $(x_0, y_0) \in W$(open) $\subset U$ and an $\mathbb{F}$-analytic mapping $\phi : V \to X$ such that $\phi(y_0) = x_0$ and

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If $F : \mathbb{R} \times X \to X$ is $\mathbb{R}$-analytic and λ_0 is a simple characteristic value of L with characteristic vector $\xi_0 \neq 0$. Then there exists an $\mathbb{R}$-*analytic function* $(\Lambda, \kappa) : (-\epsilon, \epsilon) \to \mathbb{R} \times X$ such that

$$F(\Lambda(s), \kappa(s)) = 0 \text{ for all } s \in (-\epsilon, \epsilon),$$

$$(\Lambda(0), \kappa(0)) = (\lambda_0, 0), \;\; \kappa'(0) = \xi_0$$

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This means that when $x_0 \in U \cap \mathbb{R}^n$ the coefficients f_p are real.

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For given q, any function which is analytic at 0 is on one of these classes for some choice of r sufficiently small.

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Suppose $0 \in U$ *(open)* $\subset \mathbb{F}^n$, $f : U \to \mathbb{F}$ *is analytic,* $f(0) = 0$ *and, for* $(0, \cdots, 0, x_n) \in U$,

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Now it is not difficult to see that

$$\|(\Gamma - I)u\|_{r,q} =\leq r^{-q} \|u\|_{r,q} \big(C(f) r^{1+q} + r^q \|1 - v\|_{r,q} \big) \to 0 \text{ as } r \to 0.$$

Key Step in Proof of Division theorem

Note that if the result is true for a given f and any g, then formally the coefficients of the functions h and h_k can be obtained by comparing coefficients.

It suffices therefore to show that, for $r > 0$ sufficiently small, a bijection $\Gamma : C_r^q \to C_r^q$ is defined by

$$\Gamma u(x) = f(x)Lu(x) + Au(x), \quad x \in \mathcal{B}_r^q,$$

where for $u \in C_r^q$ and $x \in \mathcal{B}_r^q$,

$$Au(x) = \sum_{\substack{p \in \mathbb{N}_0^n, \\ p_n < q}} u_p\, x^p, \quad Lu(x) = \sum_{\substack{p \in \mathbb{N}_0^n, \\ p_n \geq q}} u_p\, x_1^{p_1} \cdots x_{n-1}^{p_{n-1}} x_n^{p_n - q}$$

Now it is not difficult to see that

$$\|(\Gamma - I)u\|_{r,q} = \leq r^{-q}\|u\|_{r,q}\big(C(f)r^{1+q} + r^q\|1 - v\|_{r,q}\big) \to 0 \text{ as } r \to 0.$$

Hence Γ is a bijection on C_r^q and for $g \in C_r^q$ there is a unique $u \in C_r^q$ with $\Gamma u = g$. The uniqueness of h and h_k follow from the definition of L and A.

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a_k and h are uniquely determined by f.

If $\mathbb{F} = \mathbb{C}^n$ and f is real-on-real, then h and a_k are real-on-real.
Proof. Let $g(x) = x_n^q$ and then let $a_k = -h_k$.

Multiple Roots and the Discriminant

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Let $\xi = (z_1, \cdots, z_m) \in \mathbb{C}^m$ If $A_k = a_k(\xi)$ where the a_k are $\mathbb{C}$-analytic the discriminant

$$D(\xi) := D(a_1(\xi), \cdots, a_{p-1}(\xi))$$

is a $\mathbb{C}$-analytic function of ξ and the A has simple roots when $D(\xi) \neq 0$.

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The solution set will equivalent to a set of the very special form

$$\forall\ k \in \{m+1, \cdots, m\}, \quad h_k(z_1, \cdots, z_n) = 0 \subset \mathbb{C}^n$$

Analytic Varieties Germs

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For Weierstrass polynomials $\{A_{m+1}, \cdots, A_n\}$ on $V \subset \mathbb{C}^m$

$H = \{h_{m+1}, \cdots, h_n\}$ where $h_k(z_1, \cdots, z_n) = A_k(z_k; z_1, \cdots, z_m)$

A Weierstrass analytic variety is a set of the form $\mathrm{var}\left(V \times \mathbb{C}^{n-m}, H\right) \subset \mathbb{C}^n$.

Its discriminant $D(H) : V \to \mathbb{C}$ is the product of the discriminants of the A_ks

Its branches are the connected components of

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Globally, $z_{m+1}, \cdots, z_n$ are not analytic functions on $V \setminus \mathrm{var}\left(V, \{D(H)\}\right)$ if the latter set is multiply connected

Example

Three Weierstrass polynomials

$$Z^2 - z_1; \qquad Z^3 - z_1^2, \qquad Z^4 - z_1^3$$

define an analytic variety in $\mathbb{C}^4$ as follows:

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When $m = 1$ a variety is the union of its branches:

$$\alpha = \gamma_0 \left(\bigcup_{\alpha \cap \gamma_0(B) \neq \{0\}} B \cup \{0\} \right)$$

One-dimensional Branches $m = 1$

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B is a branch of E if and only if $\widehat{B}$ is a branch of $\widehat{E}$, where

$$B = \{(e^z, \xi) : (z, \xi) \in \widehat{B}\}, \quad \xi \in \mathbb{C}^{n-1}.$$

Since $D(H)$ is nowhere zero on $V \setminus \{0\}$, $D(\widehat{H})$ is nowhere zero on $\widehat{V}$ and every point of $\widehat{E}$ is 1-regular and

$$\big(\{z\} \times \mathbb{C}^{n-1}\big) \cap \widehat{E} = \{(z, \xi_q(z)) : 1 \leq q \leq p\},$$

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Recall that, for $z \in \widehat{V}$, each component of $\xi_q(z) \in \mathbb{C}^{n-1}$ is a simple root of a polynomial $A_k(Z; e^z)$, $2 \leq k \leq n$.

Therefore

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Moreover if, for some $\widehat{z} \in \widehat{V}$ and some $m \in \mathbb{Z}$,

$$\xi_{q_1}(\widehat{z}) = \xi_{q_2}(\widehat{z} + 2\pi m i), \quad q_1,\, q_2 \in \{1, \cdots p\},$$

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Hence, for $q \in \{1, \cdots, p\}$, the mapping

$$z \mapsto (e^z, \xi_q(z)) \in E,\ z \in \widehat{V}, \tag{1}$$

is periodic with period $2\pi K_q i$ and is injective on the set $V_q = \{z = \rho + i\theta \in \widehat{V} : 0 < \theta \le 2\pi K_q\}$, $K_q \in \{1, \cdots, p\}$.

This is a branch of the variety E where $m = 1$:

$$B = \left\{(e^z, \xi_q(z)) : z \in V_q\right\}$$

is an injective parameterization of B. Since $z \mapsto \xi_q(K_q z)$ has period (not necessarily minimal) $2\pi i$, we can define an analytic function $\widetilde{\psi} : \{z : 0 < |z| < \delta^{1/K_q}\} \to \mathbb{C}$ by

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The Riemann Extension Theorem means that $\widetilde{\psi}$ has an analytic extension ψ defined on the ball $\left\{z_1 \in \mathbb{C} : |z_1| < \delta^{1/K_q}\right\}$ with $\psi(0) = 0$. Let $K = K_q$ to complete the proof.

Real One-Dimensional Branches

If $\gamma_0(B \cap \mathbb{R}^n) \notin \{\emptyset, \{0\}\}$ there exists $k \in \mathbb{N}_0$ with $0 \leq k \leq 2K - 1$ such that

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Back to Global Bifurcation

Lyapunov-Schmidt Reduction yields an $\mathbb{R}$-analytic function h on a $(q+1)$-dimensional real vector space V into $\mathbb{R}^q$, its $\mathbb{R}$-analytic variety which contains and a 1-dimensional manifold M, namely a $\mathbb{R}$-analytic distinguished arc:

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The q components of $h(\lambda, \xi)$ are real functions defined locally in a neighbourhood of $(\lambda_*, 0) \in V$ by a Taylor series.

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Replacing $(x_1, \cdots, x_{q+1}) \in \mathbb{R}^{q+1}$ with $(z_1, \cdots, z_{q+1}) \in \mathbb{C}^{q+1}$ leads to a real-on-real $\mathbb{C}$-analytic extension h^c of h in a complex neighbourhood V^c of $(\lambda_*, 0)$ and a corresponding $\mathbb{C}$-analytic variety.

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The structure theorem when applied to A^c gives, for each $j \in J^c$, the existence of a real-on-real branch B_j with

$$\gamma_{(\lambda_*,0)}(M_j^c) \subset \gamma_{(\lambda_*,0)}(\overline{B}_j), \quad \dim B_j = 1 \text{ and } B_j \subset A^c$$

with $B_j \setminus \{(\lambda_*, 0)\} \subset M_j^c$. There are finitely many branches and hence finitely many M_j^c and M_j.

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Thus curves in $\mathfrak{N}$ cannot terminate when real-analytic operators are involved.

This leads directly to the advertised properties of maximal routes

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The global result follows easily from this and the local compactness of solution sets.