Variational theory of Bernoulli free-boundary problems

Eugene Shargorodsky

Department of Mathematics
King's College London

Bernoulli Free-Boundary Problem

Free surface:

$$S := \{(u(s), v(s)) |\ s \in \mathbb{R}\},$$

where

- (u, v) is injective and absolutely continuous,
- $u'(s)^2 + v'(s)^2 > 0$ for almost all s,
- $s \mapsto (u(s) - s, v(s))$ is 2π–periodic.

Let Ω denote the open region of $\mathbb{R}^2$ below S.

The boundary value problem:

Find S for which there exists $\psi \in C(\overline{\Omega}) \bigcap C^2(\Omega)$ such that

- $\Delta\psi = 0$ in Ω,
- ψ is 2π–periodic in X,
- $\nabla\psi$ is bounded in Ω and $\nabla\psi(X, Y) \to (0, 1)$ uniformly in X as $Y \to -\infty$,
- $\psi \equiv 0$ on S,
- $|\nabla\psi(X, Y)|^2 + \lambda(Y) = 0$ almost everywhere on S (the Bernoulli boundary condition).

If $\psi \equiv 0$ on S, then the Bernoulli condition

$$|\nabla\psi(X, Y)|^2 + \lambda(Y) = 0 \text{ on } S$$

is equivalent to the Neumann condition

$$\frac{\partial\psi}{\partial\nu}(X, Y) = h(Y) \text{ on } S,$$

where $\lambda = -h^2$ and ν is the outward unit normal to S.

The coefficient:
$\lambda : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and real-analytic on the open set of full measure $\{y \in \mathbb{R} : \; \lambda(y) \neq 0\}$.

Stokes Waves

The case $\lambda(Y) \equiv 2\mu Y - 1$, $\mu = \text{const} > 0$ corresponds to Stokes waves.

A Stokes wave is a steady periodic wave, propagating under gravity with constant speed on the surface of an infinitely deep irrotational flow. The Bernoulli boundary condition is the constant pressure condition for Stokes waves resulting from Bernoulli's theorem in inviscid hydrodynamics.

$\mu^{-1/2}$ is the Froude number, a dimensionless combination of speed, wavelength and gravitational acceleration.

Stokes (1847): nonlinear waves with small amplitudes.

Stokes Conjectures (1880)

First conjecture: There exists a large amplitude wave with a stagnation point and a corner containing an angle of 120° at its highest point. (Stokes wave of extreme form)

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Second conjecture: The Stokes wave of extreme form is convex between successive crests.

A.I. Nekrasov (1921), T. Levi-Civita (1925)
Local existence theory
Bifurcation from a simple eigenvalue

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G. Keady and J. Norbury (1978)
Global bifurcation theory by P.H. Rabinowitz (1971) and its refinement for positive operators by E.N. Dancer (1973) and R.E.L. Turner (1975)

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P.I. Plotnikov and J.F. Toland (2004) proved Stokes second conjecture.

Nekrasov's equation

$$\theta(s) = \frac{1}{6\pi}\int_{-\pi}^{\pi}\left(\log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right|\right)\frac{\sin\theta(t)}{\beta+\int_0^t \sin\theta(\tau)\,d\tau}\,dt, \quad s\in[-\pi,\pi],$$

where β is a constant.

Related 1-d Equations

M.S. Longuet-Higgins, 1978, 1985
K.I. Babenko, 1987
P.I. Plotnikov, 1992
A.I. Dyachenko, E.A. Kuznetsov, M.D. Spector, and V.E. Zakharov, 1996
B. Buffoni, E.N. Dancer, and J.F. Toland, 2000
Bernoulli free-boundary problem case (general λ):

$$\boxed{\lambda(w)\big(1+\mathcal{C}w'\big)+\mathcal{C}\big(\lambda(w)w'\big)+1=0} \tag{1}$$

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where $\mathcal{C}u$ denotes the periodic Hilbert transform of a 2π-periodic function $u:\mathbb{R}\to\mathbb{R}$:

$$\mathcal{C}u(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(y)\cot\frac{x-y}{2}\,dy.$$

Equation (1) has variational structure: it is the Euler-Lagrange equation of the functional

$$\mathcal{J}(w) = \int_{-\pi}^{\pi} \{w + \Lambda(w)(1 + \mathcal{C}w')\} ds,$$

where $\Lambda'(w) = \lambda(w)$.

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B. Buffoni, E.N. Dancer, and J.F. Toland (2000): theory of sub-harmonic (period-multiplying) bifurcations of Stokes waves which, in particular, disproves a conjecture made by Levi-Civita (1925) who had speculated that there might exist an upper bound on the minimal wavelength of a Stokes wave propagating with a given speed.

Function spaces:

- $L^p_{2\pi}$, $p \geq 1$ – the Banach space of locally p^{th}-power summable 2π–periodic functions,

- $W^{1,p}_{2\pi}$ – the Banach space of absolutely continuous, 2π–periodic functions w with weak first derivatives $w' \in L^p_{2\pi}$,

- $\mathcal{H}^{1,1}_{\mathbb{R}}$ – the real Hardy space of absolutely continuous 2π–periodic functions with derivative in the Hardy space $\mathcal{H}^1_{\mathbb{R}} := \{u \in L^1_{2\pi} : \mathcal{C}u \in L^1_{2\pi}\}$.

(Note that $W^{1,p}_{2\pi} \subset \mathcal{H}^{1,1}_{\mathbb{R}}$ for $p > 1$)

Properties of $\mathcal{C}$:

- $\mathcal{C} : L^p_{2\pi} \to L^p_{2\pi}$ is bounded if $1 < p < \infty$ (M. Riesz theorem),
- $\mathcal{C}1 \equiv 0, \quad \mathcal{C}\exp_n \equiv -i\,\mathrm{sign}(n)\exp_n, \forall n \in \mathbb{Z}\backslash\{0\}$, where $\exp_n(t) \equiv e^{int}$,
- $\mathcal{C} \Longleftrightarrow$ a zero order ΨDO on the unit circle with the symbol $-i\,\mathrm{sign}(\xi)$,
- the operator $w \mapsto \mathcal{C}w' \Longleftrightarrow$ a first order ΨDO on the unit circle with the symbol $|\xi|$, i.e. the operator $\sqrt{-\Delta}$.
- the operator $w \mapsto \mathcal{C}w' \Longleftrightarrow$ Dirichlet-to-Neumann operator for the unit disk.

Theorem. (ES & J.F. Toland, 2008)

(a) Let u, v, ψ be a solution of the Bernoulli free-boundary problem. If φ is a harmonic conjugate to $-\psi$, then $\varphi + i\psi$ is a conformal mapping of Ω onto the lower half-plane.
Let Z be the inverse conformal mapping of the lower half-plane onto Ω and $w(t) := \operatorname{Im} Z(-t)$, $t \in \mathbb{R}$. Then $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ is a solution of (1).

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(b) Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ be a solution of (1) such that $\boxed{\lambda(w) \leq 0}$ and

$$\boxed{\mathbb{R} \ni t \mapsto (-(t + \mathcal{C}w(t)), w(t)) \text{ is injective}}.$$

Let $(u(t), v(t)) = (-(t + \mathcal{C}w(t)), w(t))$, $S := \{(-(t + \mathcal{C}w(t)), w(t)) : \ t \in \mathbb{R}\}$, and ψ be the imaginary part of a conformal mapping of the region Ω below S, onto the lower half-plane. Then u, v, ψ is a solution of the Bernoulli free-boundary problem.

Theorem. (ES & J.F. Toland, 2008)

Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ be a solution of (1). Then $\log|\lambda(w)| \in L^1_{2\pi}$*,* $\lambda(w) < 0$ *on a set of positive measure, and w is real-analytic on the open set of full measure where* $\lambda(w) \neq 0$*.*

Theorem. (ES & J.F. Toland, 2008)

Suppose that, for some $k > 0$,

$$|\lambda(y)| \leq \text{const}\,\Big(\text{dist}(y,\,\text{zero set of }\lambda)\Big)^{k}.$$

Let $p(k) = \frac{k+2}{k}$ *and* $r(k) = \frac{k+2}{k+1}$, *and let* $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ *be a solution of (1).*

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(a) The following are equivalent:

- $w \in W^{1,p(k)}_{2\pi}$,
- *w is real-analytic on $\mathbb{R}$,*
- *$\lambda(w) < 0$ on $\mathbb{R}$.*

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(b) If $w \in W^{1,r(k)}_{2\pi}$, then $\lambda(w) \leq 0$.
(The Stokes waves case: $r(1) = 3/2$.)

J.F. Toland, 2000; P.I. Plotnikov & J.F. Toland, 2003:

Let $k \geq 1$. There exists $\hat{\mu} = \hat{\mu}(k) > 0$ and a solution

$$\hat{w} \in \cap_{p<p(k)} W^{1,p}_{2\pi}$$

of (1) with $\lambda(\hat{w}) := -(1 - 2\hat{\mu}\hat{w})^k < 0$ almost everywhere, but which is not Lipschitz continuous at a discrete set of points $t \in \mathbb{R}$ where $1 - 2\hat{\mu}\hat{w}(t) = 0$.

In the case $k = 1$, $\lambda(Y) \equiv 2\hat{\mu}Y - 1$ the above solution gives the profile of the Stokes wave of extreme form.

The main open question:

Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ be a solution of (1).

Can $\mathbf{1 - 2\mu w}$ change sign?

If yes, do such solutions have any physical meaning? (Note that they do not satisfy the Bernoulli condition.)

Heuristic argument

Suppose $2\mu w - 1$ changes sign at t_0 and we have the following in a neighbourhood of t_0:

$$2\mu w(t) - 1 = \begin{cases} c_1|t - t_0|^{\gamma_1} + w_1(t), & t < t_0, \\ c_2|t - t_0|^{\gamma_2} + w_2(t), & t > t_0, \end{cases}$$

where $0 < \gamma_j < 1$, $c_j \in \mathbb{R}$, w_j are "good" functions and $w_j(t_0) = 0$, $j = 1, 2$. Let $\min\{\gamma_1, \gamma_2\} < 1/2$. (This restriction is fulfilled in the case $w' \notin L^{3/2}_{2\pi}$ which we need to deal with.)

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Then one can **prove** that $\boldsymbol{\gamma_1 = \gamma_2 = 1/3}$, $c_1 = -c_2$.
In particular, $\mathbf{w} \notin \mathbf{W^{1,3/2}_{2\pi}}$, but $\mathbf{w} \in \mathbf{W^{1,p}}$, $\forall \mathbf{p} < \mathbf{3/2}$ in a neighbourhood of t_0.

Theorem. (ES & J.F. Toland, 2008)

Suppose that $\lambda'(x) > 0$ *at the points x where* $\lambda(x) \neq 0$*, and that* $\log|\lambda|$ *is concave. (This is fulfilled for the Stokes waves.)*

Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ *be a solution of (1) such that* $\lambda(w) \leq 0$ *and* *the set of zeros of* $\lambda(w)$ *is at most countable*.

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Then $1 + \mathcal{C}w' > 0$ almost everywhere. Hence $\mathbb{R} \ni t \mapsto (-(t + \mathcal{C}w(t)), w(t))$ is injective (is in fact the graph of a function).

Theorem. (ES & J.F. Toland, 2008)

Let $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ be a solution of (1) such that $\lambda(w) \leq 0$. Suppose that, for some $k > 0$,

$$|\lambda(y)| \asymp \Big(\mathrm{dist}(y, \text{ zero set of } \lambda)\Big)^k.$$

Let $p(k)$ be defined as above: $p(k) = (k+2)/k$.

Then $1 - 1/p(k) = 2/(k+2)$ is an upper bound for the lower Minkowski dimension of

$$\mathcal{N} := \{t \in \mathbb{R} : \ \lambda(w(t)) = 0\}$$

(and hence for its Hausdorff dimension).

(Stokes waves: $1 - 1/p(k) = 2/3$,)

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(and hence for its Hausdorff dimension).

If $w \in W^{1,p}_{2\pi}$*,* $1 < p < p(k)$*, then* $1 - p/p(k)$ *is a better upper bound for the lower Minkowski dimension of* $\mathcal{N}$*.*

(Stokes waves: $1 - 1/p(k) = 2/3, \ 1 - p/p(k) = 1 - p/3$*.)*

E. Varvaruca and G.S. Weiss (2010): a Stokes wave can have at most finitely many points with $1 - 2\mu Y = 0$ per period.

Morse index

Let w_0 be a critical point of the functional

$$\mathcal{J}(w) = \int_{-\pi}^{\pi} \Big(w(t) + \Lambda(w(t))(1 + \mathcal{C} w'(t)) \Big)\, dt, \quad w \in W_{2\pi}^{1,2},$$

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where Λ is a primitive of λ.

Consider the quadratic form of the second Fréchet derivative $\mathcal{J}''(w_0)$ (the Hessian):

$$\mathcal{Q}_{w_0}[u] := \int_{-\pi}^{\pi} \Big(2\lambda(w_0(t))u(t)\mathcal{C}u'(t) + \lambda'(w_0(t))(1 + \mathcal{C}w_0'(t))u^2(t) \Big) dt.$$

The Morse index $\mathcal{M}(w_0)$ of w_0 is the number $N_-(\mathcal{Q}_{w_0})$ which is defined below.

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Let $\mathcal{H}$ be a Hilbert space and let $\mathbf{q}$ be a Hermitian form with a domain $\mathrm{Dom}\,(\mathbf{q}) \subseteq \mathcal{H}$. Set

$$N_-(\mathbf{q}) := \sup \{ \dim \mathcal{L} \mid \mathbf{q}[u] < 0, \ u \in \mathcal{L} \setminus \{0\} \},$$

where $\mathcal{L}$ denotes a linear subspace of $\mathrm{Dom}\,(\mathbf{q})$.

Qualitative results

P.I. Plotnikov (for solitary waves, 1991)
B. Buffoni, E.N. Dancer and J.F. Toland (for Stokes waves, 2000)
ES & J.F. Toland (for more general Bernoulli free-boundary problems, 2008)

If the Morse indices of the elements of a set of non-singular Stokes waves are bounded, then none of them is close to a singular one.

A quantitative result

Assume that for some $k > 0$,

$$\text{if } \lambda(y_0) = 0, \quad \text{then} \quad |\lambda(y)| \leq \text{const}\,|y - y_0|^k, \quad \forall y \in \mathbb{R},$$
$$\ln|\lambda| \text{ is concave, and } \lambda' \geq 0 \text{ where } \lambda \neq 0.$$

Let $p(k) = \frac{k+2}{k}$.

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Let $p(k) = \frac{k+2}{k}$.

Every critical point $w_0 \in W^{1,p(k)}_{2\pi}$ of $\mathcal{J}$ is a real analytic function and

$$\min_{t \in \mathbb{R}} \lambda(w_0(t)) > 0.$$

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Let

$$\nu(w_0) := \max_{t\in\mathbb{R}} \frac{|\lambda'(w_0(t))|}{\lambda(w_0(t))},$$

$$\nu_0(w_0) := \max_{t\in\mathbb{R}} \frac{1}{\lambda(w_0(t))} = \frac{1}{\min_{t\in\mathbb{R}} \lambda(w_0(t))}.$$

Morse index

Suppose there exist constants $m_1, m_2 > 0$ such that

$$\frac{m_1}{\lambda(y)^{1/k}} \leq \frac{|\lambda'(y)|}{\lambda(y)} \leq \frac{m_2}{\lambda(y)^{1/k}} \quad \text{for all } \ y \in \mathbb{R} \ \text{ with } \ \lambda(y) \neq 0.$$

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Theorem. (ES, 2012)

There exist constants $M_1, M_2 > 0$ which depend only on k and are such that

$$M_1 \frac{m_1}{m_2} \ln^{\frac{k}{k+2}}(1 + \nu(w_0)) \leq \mathcal{M}(w_0) \leq 1 + M_2\, \nu(w_0) \ln(2 + \nu_0(w_0))$$

holds for every critical point $w_0 \in W^{1,p(\varrho)}_{2\pi}$ of $\mathcal{J}$.

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In the case of Stokes waves, the estimate takes the form

$$M_1 \ln^{1/3}(1 + \nu(w_0)) \leq \mathcal{M}(w_0) \leq 1 + M_2\, \nu(w_0) \ln(2 + \nu_0(w_0)).$$

Morse index

$$\mathbf{q}_V[u] := \int_{-\pi}^{\pi} \Big((\mathcal{C}u'(t))u(t) - V(t)u^2(t) \Big) dt, \quad u \in W^{1,2}_{2\pi}, \quad (V \geq 0).$$

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Theorem. (ES, 2012)

There exist constants $C_1, C_2 > 0$ *such that*

$$C_1 \|V\|_{L^1_{2\pi}} \leq N_-(\mathbf{q}_V) \leq C_2 \|V\|_{\mathcal{B}} + 1, \quad \forall V \in L^1_{2\pi}, \; V \geq 0.$$

Here

$$\mathcal{B}(s) := (1 + |s|) \ln(1 + |s|) - |s|, \quad s \in \mathbb{R}$$

and

$$\|f\|_\Psi := \inf \left\{ \kappa > 0 : \int_{-\pi}^{\pi} \Psi \left(\frac{f(t)}{\kappa} \right) dt \leq 1 \right\}.$$