The Navier-Stokes equations with spatially nondecaying data III

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1. Introduction

The Navier-Stokes initial value problem

(NS) $u_t - \Delta u + (u, \nabla)u + \nabla\pi = 0$ in $R^n \times (0, T)$

$\operatorname{div} u = 0$ in $R^n \times (0, T)$

$u|_{t=0} = u_0 (\operatorname{div} u_0 = 0)$

$u = u(x, t)$: real vector (velocity fields)

$\pi = \pi(x, t)$: scalar (pressure fields)

(kinematic viscosity is normalized to be one)

Regularity criteria

(Extendability) Let u be a smooth solution of (NS) in $(0,T)$. If one assumes extra assumptions, then one can extend the solution beyond T.

(Regularity) Let u be a weak solution of (NS) in $(0,\infty)$. If one assumes extra assumptions, then u is regular.

Typical example (J. Serrin '61…)

If u satisfies

$$\int_0^T \left(\int |u|^p dx \right)^{q/p} dt < \infty$$

with $\frac{n}{p} + \frac{2}{q} \leq 1$, then one can extend the solution beyond T.

If u is a weak solution, u is regular in $R^n \times (0,T]$. Note that the integral is scaling invariant for the equality case of exponents.

Critical exponent and scaling invariance of (NS)

If (u, π) solves (NS) in $R^n \times (0, \infty)$, so does

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t),$$

$$\pi_\lambda(x,t) = \lambda^2 \pi(\lambda x, \lambda^2 t) \ \text{ for } \ \lambda > 0.$$

The norm $\|u_0\|_{L^n}$ is invariant under this scaling. In this sense $p = n$ is critical.

Remark. (i) Since then there is a large literature on regularity criteria. A general principle is that if **a scaling invariant quantity is finite**, then one expect smoothness. In fact, energy inequality is scaling invariant for $n = 2$ while it is not for $n \geq 3$. Energy inequality is too weak to guarantee smoothness for $n \geq 3$.

Remark. (ii) Most of regularity criteria assumes finiteness of some scaling invariant quantity for velocity, vorticity, pressure. New type of criteria called **geometric criteria** is introduced by Constantin-Fefferman '93 on the direction of the vorticity.

2. Geometric Regularity Criteria

Criterion by vorticity direction

$$\xi(x,t) = \omega(x,t)/|\omega(x,t)|$$
$$\omega(x,t) = \operatorname{curl} u$$

(Constantin-Fefferman '93)

If vorticity direction is Lipschitz continuous in space (uniformly in time), then the weak solution is regular (if $u_0 \in H^1$).

[Smooth alignment of vorticity implies regularity for **finite** energy solutions.]

A Key observation of Constantin-Fefferman [CF]

2-D flow: vorticity is scalar and fulfills $\omega_t - \Delta\omega + (u,\nabla)\omega = 0$.

3-D flow: there is a vorticity **stretching** term.

$$\omega_t - \Delta\omega + (u,\nabla)\omega - (\omega,\nabla)u = 0$$

[CF]: $(\partial_t + u\cdot\nabla - \Delta)|\omega|^2 + |\nabla\omega|^2 = \alpha|\omega|^2$

Constantin: $\alpha(x) = \dfrac{3}{4\pi}\ \mathrm{p.v.}\displaystyle\int D\big(\hat{y}, \xi(x+y), \xi(x)\big)\,\omega(x+y)\,\frac{dy}{|y|^3}$

$$\hat{y} = y/|y|\,, D(a,b,c) = (a\cdot c)\mathrm{Det}\,(a,b,c)$$

Several generalizations: Beirao da Veiga - Berselli (2002),
D. Chae (2007), Y. Zhou (2000), etc.

Blow-up argument provides a simple proof

(Solutions are allowed to have an **infinite** energy)

Main Theorem (G-Miura '11 CMP) (simplest form)

u spatially bdd mild sol. for (NS) in $-1 < t < 0$.

Assume that blow-up at zero is **type I**:

$$\|u\|_\infty(t) \leq c(-t)^{-1/2}, -1 < t < 0.$$

If the vorticity direction is uniformly continuous in space, i.e.

(CA) $|\xi(x,t) - \xi(y,t)| \leq \eta(|x-y|)$

for $(x,y),(y,t) \in \Omega_d = \{(x,t) | |\omega(x,t)| > d\}$

for fixed d, then u is bounded up to $t = 0$.

Here η is a modulus of continuity.

3. Blow-up Argument

Assume u blows up at $t = 0$.

Then $\exists (x_k, t_k),\ t_{k+1} \geq t_k,$ s.t.

(i) $|u(x,t)| \leq M_k$ for $t \leq t_k,$

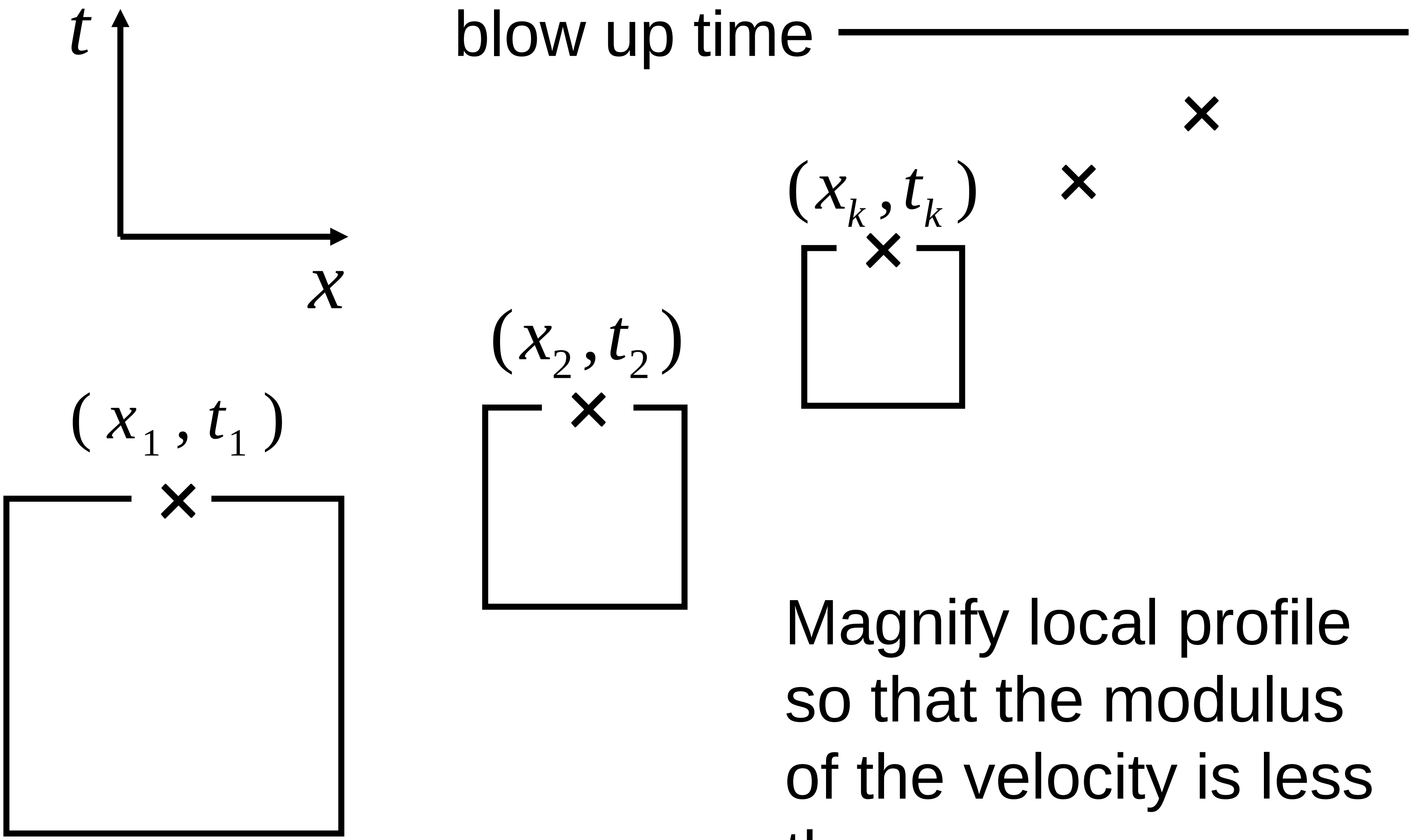
(ii) $M_k = \|u(\cdot, t_k)\|_\infty \to \infty, t_k \to 0$ as $k \to \infty,$

(iii) $|u(x_k, t_k)| \approx M_k.$

Set a rescale function with $\lambda_k = 1/M_k$:

$$\begin{cases} u_k(x,t) = \lambda_k u(x_k + \lambda_k x, t_k + \lambda_k^2 t) \\ \omega_k(x,t) = \lambda_k^2 \omega(x_k + \lambda_k x,\ t_k + \lambda_k^2 t) \end{cases}$$

Rescaling



Magnify local profile so that the modulus of the velocity is less than one.

Convergence of rescaled functions

$|u_k| \leq 1$ in $R^3 \times (-M_k^2, 0]$

⇒G-Sawada (2003): higher derivative estimates

$(u_k, \omega_k) \to (\bar{u}, \bar{\omega})$ Locally unif. in $R^3 \times (-\infty, 0)$

$|\bar{u}(0,0)| = 1,$

$\bar{u}$ is a bounded global mild solution of (NS) in $R^3 \times (-\infty, 0)$

Blow up argument: De Giorgi, minimal surface; parabolic version:
G. (1986) CMP $u_t = \Delta u + u^p$; Polacik-Quittner-Souplet (2007)

Typical regularity criterion

Serrin type $\int_0^T \left(\int |u|^p dx \right)^{q/p} dt < \infty, u$: solution

$q < \infty, \frac{n}{p} + \frac{2}{q} \leq 1 \Rightarrow T$ is not a blow up time

Rescaled function satisfies

$$\left(\int_{-1}^{0} \int_{B(0,1)} |u_k|^p \, dx \right)^{q/p} dt \to 0$$

The limit $\bar{u}$ must be zero so we get contradiction.

The case $p = n$ is not easy: Seregin…

Type one blow-up

$\| u \|_\infty (t) \le C(-t)^{-1/2} \iff$ type one blow up

Lemma. Type I $\Rightarrow \bar{\omega} \not\equiv 0.$

Note $-\Delta \bar{u} = \mathrm{curl}\ \bar{\omega}.$ If $\bar{\omega} \equiv 0,$ then $\bar{u} \equiv \mathrm{const}$ (in x)

by the classical Liouville theorem. The unique existence of local mild
L^∞ solution implies $\bar{u}$ is constant in x and t.
(G-Inui-Matsui, 1999)

Type I implies $|\bar{u}(t)| \le c(-t)^{-1/2}$ for $t < 0$ which is a contradiction.

Lemma. If ξ satisfies the continuous (CA), alignment condition then $\bar{\omega} \equiv 0$.

Sketch:

$$\xi_k(x,t) = \omega_k / |\omega_k|$$

$$\text{(CA)} \Rightarrow |\xi_k(x,t) - \xi_k(y,t)| \leq \eta\left(\frac{|x-y|}{M_k^2}\right) \to 0$$

$\therefore \bar{\xi}$ is independent of x !

Note that $(\overline{u}, \overline{\omega})$ is a mild solution of (NS) in $R^3 \times (-\infty, 0)$.

If ξ is independent of x, it is also independent of time because of the unique existence of local solution with L^∞ initial data (G - Inui - Matsui '99).

Thus $(\overline{u}, \overline{\omega})$ is a **two dimensional** flow.

The next Liouville type theorem implies $\overline{u}$ is constant. $\square$

Two lemmas imply the main theorem.

Liouville type theorem

If u is a bounded mild solution of (NS) in $R^2 \times (-\infty, 0)$, it must be a constant solution.

- Koch-Nadrashvilli-Seregin-Sverak, 2007
  (based on integral estimates)
- G-Miura, Based on strong Max principle of vorticity equation
  (Weak Max principle see: Kato-Fujita 1959 /
  M.-H. Giga, Y. Giga, J. Saal (Book 2010))

Note: mild solution $\Rightarrow \pi = \sum_{i,j} R_i R_j u_i u_j$

Note: any $u = g(t)$ solves (NS) if $\pi = -g'(t) \cdot x$.

Proof of the Liouville type theorem

- We may assume that ω is bounded by unique local-in-time existence theorem for mild solutions.
- We may assume that u and ω are smooth by standard linear regularity theory for parabolic and elliptic equations.
- We may assume that u and ω are smooth up to $t = 0$ by translating time.

Equation for $(\boldsymbol{u}, \boldsymbol{\omega})$

(V) $\omega_t - \Delta\omega + (u, \nabla)\omega = 0, R^2 \times (-\infty, 0]$

$u = (-\Delta)^{-1}\nabla^{\perp}\omega$ in $R^2 \times (-\infty, 0]$

Suppose that $L = \sup \omega > 0$.

Then $^{\exists}(x_k, t_k), t_k < 0$ such that

$\omega(x_k, t_k) \to L$ (as $k \to \infty$).

Shifting

Set $\omega_k(x,t) = \omega(x + x_k, t + t_k)$,

$$u_k(x,t) = u(x + x_k, t + t_k).$$

This solves the vorticity equation (V) in $R^2 \times (-\infty, 0]$. Since $\{(u_k, \omega_k)\}$ is bounded in $L^\infty(R^2 \times (-\infty, 0])$, the linear regularity theory for (V) implies that $\{(u_k, \omega_k)\}$ converges to some $(\bar{u}, \bar{\omega})$ locally uniformly in $R^2 \times (-\infty, 0]$ and $(\bar{u}, \bar{\omega})$ solves (V).

Application of the strong maximum principle

By definition $\omega_k(0,0) \to L (k \to \infty)$ so that $\overline{\omega}(0,0) = L$. Thus $\overline{\omega}$ takes a maximum at $(0,0)$ in $R^2 \times (-\infty, 0]$. By the strong maximum principle for the first equation of (V) implies

$$\overline{\omega} \equiv L.$$

Application of the Liouville theorem for harmonic functions

Since $\bar{u}$ solves the Biot-Savart

$$-\Delta\bar{u} = \nabla^{\perp}\overline{\omega}$$

and $\overline{\omega}$ is a constant, we see that $\bar{u}$ is harmonic. By the Liouville theorem of harmonic functions implies that $\bar{u} \equiv \text{constant}$, which yields $\overline{\omega} = 0$ so we get a contradiction. So $\omega \leq 0$. If we assume $\inf \omega < 0$, the same argument for $-\omega$ implies $\omega \geq 0$. We thus observe that $\omega \equiv 0$ so that $u \equiv \text{const}$.

4. Boundary Effects

What happens when the region fluid occupies in a domain U in R^n not whole space?

We need boundary condition.

Dirichlet BC: $u = 0$ on ∂U

(adherence BC or non-slip BC)

Neumann BC: $u \cdot n = 0,\ \partial u_{\tan} / \partial n = 0$ on ∂U

(slip BC) n: unit normal, $u_{\tan}$: tangential component

Full system with the Dirichlet condition

$$(\mathrm{NS})\quad \begin{cases} u_t - \Delta u + (u,\nabla)u + \nabla\pi = 0 \text{ in } U \times (0,T) \\ \operatorname{div} u = 0 \text{ in } U \times (0,T) \\ u\big|_{t=0} = u_0 \text{ on } U \end{cases}$$

BC: $u = 0$ on $\partial U \times (0,T)$

Note: There is a boundary condition like Robin type called the Navier boundary condition.

$$u_{\tan} + (D(u)n)_{\tan} = 0, u \cdot n = 0$$

Typical domains

The half space $R_+^n = \{(x_1, \ldots, x_n)|\ x_n > 0\}$ a bounded domain, an exterior domain, a bent half space, a layer domain

Solvability is very similar except non-decaying setting.

Geometric regularity criteria with boundary condition

Half space $R^3_+ = \{(x_1, x_2, x_3) \mid x_3 > 0\},\ u = (u^1, u^2, u^3)$

(1) Slip boundary condition:

$$\frac{\partial u^1}{\partial x_3} = \frac{\partial u^2}{\partial x_3} = 0,\ u^3 = 0 \quad \text{on} \quad x_3 = 0$$

$$\bar{\xi}\Big|_{x_3=0} = \left(0, 0, \bar{\xi}_3\right) \Rightarrow \bar{\omega} = (0, 0, \bar{\omega}_3)$$

$\bar{u}$: two dimensional flow $R^2_+ = \{(x_1, x_3) \mid x_3 > 0\}$

$$\begin{cases} \bar{\omega}_{3t} - \Delta\bar{\omega}_3 + (\bar{u}, \nabla)\bar{\omega}_3 = 0 \quad \text{in} \quad R^2_+ \times (-\infty, 0) \\ \bar{\omega}_3 = 0 \ \text{ on } \{x_3 = 0\} \end{cases}$$

Liouville type theorem by strong (Maximum principle)

If $\bar{u}$ is a bdd mild backward global solution with the slip BC, then $\bar{\omega} = 0.$

(2) Dirichlet boundary condition $u^1 = u^2 = u^3 = 0$

$$\bar{\xi}\Big|_{x_3=0} = (*,*,0)$$

By coordinate change of tangential direction

$\bar{u}$: two dimensional flow; $\bar{\omega}$ solves

$$\begin{cases} \omega_t - \Delta\omega + (u,\nabla)\omega = 0 \text{ in } R_+^2 \times (-\infty, 0) \\ u^1 = u^3 = 0 \text{ on } \{x_3 = 0\}, R_+^2 = \{(x_1, x_3) |\, x_3 > 0\}. \end{cases}$$

No similar Liouville Theorem is available.
Vorticity is expected to be created on the boundary. There is even counterexample of Poiseuille type flow (G '11). Nevertheless, we except a similar result. (Hsu, Maekawa, G '12)

Existence of nontrivial entire solutions

(NSD): (NS) in $R^3_+ \times (-\infty, \infty)$ with $u = 0$ on ∂R^3_+

Theorem. (G. '11) $\exists (u, \nabla\pi)$ solution of (NSD) such that

(i) $|u|, |\nabla\pi|$ is bounded;

(ii) $|\nabla u|$ is bounded and $\nabla u \not\equiv 0$;

(iii) u^1 depends only on x_3, t;
π depends only on x_1, t;
$u^2 \equiv u^3 \equiv 0$ (u: parallel to the boundary);

(iv) ${}^\exists C > 0, \sup_{x_3 \geq L} |\nabla u| (x_3, t) \leq C/L$.

(Poiseuille type flow)

Remark (a) This result is obtained by solving the heat equation in x_3 variable.

(b) One may arrange that $\omega > 0$ in $R^3_+ \times (-\infty, \infty)$.

(c) Open problem: If $(u, \nabla\pi)$ is a classical solution of (NS) in $R^3_+ \times (-\infty, 0)$ with non-slip BC, then is all possible solution a Poiseuille type flow provided that $|u|, |\nabla u|$ are bounded?

Idea of the proof
(Construction of a Poiseuille type flow)

We consider

$$u = (u^1(x_3, t), 0, 0), \pi(x, t) = -f(t)x_1$$

with $f \in L^1(-\infty, 0) \cap L^\infty(-\infty, 0)$.

(NSD) is reduced to the heat equation

$$u^1_t - u^1_{x_3 x_3} = f \text{ in } \{x_3 > 0\} \times (-\infty, 0)$$

$$u^1(0, t) = 0 \text{ on } (-\infty, 0)$$

Idea of the proof (continued): Approximation

We construct such a solution by approximation. We set a zero initial condition at $t = -T$ and solve $t > -T$ to get a solution u^1_T. We take $T \to \infty$ to get a desired solution.

Remark. (by Y. Maekawa)

If one requires $\omega \geq 0$ and decay of u^1 itself

$$\bigstar \lim_{R\to\infty} \sup\{|u^1(x, x_3)| x_1 \in R, x_3 \geq R\} = 0$$

then there is no nontrivial solution.

Non existence result

Theorem. Let $u = (u^1, u^2)$ be a C^1 in R^2_+ satisfying $\mathrm{div}\, u = 0$ in R^2_+. Assume that $\omega \geq 0$ and that u, $|\nabla u|$ is bounded. Assume that u is continuous up to the boundary and $u = 0$ on the boundary. If u^1 fulfills the decay condition$\star$, then $\omega \equiv 0$.

New Liouville type result

Lemma. (G-Hsu-Maekawa '12) Let (u,p) be a classical solution of (NSD) in $R_+^2 \times (-\infty, 0)$. Assume that

(C1) $\sup_{-\infty<t<0} \|u\|_{C^{2+\mu}} + \|\partial_t u\|_{C^\mu} < \infty$

(C2) $p = p_F + p_H$

(C3) $\sup_{-\infty<t<0} (-t)^{1/2} \|u\|_\infty(t) < \infty$

(C4) $\omega \geq 0$ in $R_+^2 \times (-\infty, 0)$. Then $u \equiv 0$.

Here

p_F is the sol of $\begin{cases} \Delta q = -\partial_i \partial_j u^i u^j \\ \dfrac{\partial q}{\partial n} = 0 \end{cases}$

s.t.

$\|p_F\|_{BMO} \leq C\|F\|_\infty, \|\nabla p_F\|_{C^\mu} \leq C\|u \otimes u\|_{C^{1+\mu}}$

p_H: harmonic pressure the sol of $\begin{cases} \Delta q = 0 \\ \dfrac{\partial q}{\partial n} = \partial_{\tan}\omega \end{cases}$

s.t. $\sup x_2 \, |\nabla p_M(x)| \leq C\|\omega\|_\infty$

Geometric regularity criterion up to boundary

Applying this lemma one is able to extend Miura-G result for the half space with the Dirichlet B.C.

Theorem. (G-Hsu-Maekawa '12)
Let u be a spatially bounded mild solution for (NSD) in $R^2_+ \times (-1, 0)$. If u is type I near $t = 0$ and u satisfies (CA), then u is bounded up to $t = 0$.

All results so far known needed extra assumptions compared with whole space problem; see e.g. H. Beirao da Veige '07.

More general domain

L^∞-theory is necessary for compactness

L^∞-theory is available for a half space

(V. A. Solonnikov ’03, Bae ’12, Maremonti ’05)

However, it is very recent that one is able to prove that the Stokes semigroup $S(t)$ forms an analytic semigroup when U is a bounded or an exterior domain.

(Ken Abe-Y. G., Acta Math. to appear)