The Navier-Stokes equations with spatially nondecaying data
II

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II. The Stokes semigroup in space of bounded functions

Joint work with Ken Abe (U. Tokyo)

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1. Introduction

Stokes system:

$$v_t - \Delta v + \nabla q = 0, \operatorname{div} v = 0 \text{ in } \Omega \times (0,T)$$

B.C. $\quad v = 0$ on $\partial\Omega$

I.C. $\quad v|_{t=0} = v_0$ in Ω

Here Ω is a uniformly C^3-domain in $R^n (n \geq 2)$

v : unknown velocity field

q : unknown pressure field

v_0 : a given initial velocity

Problem. Is the solution operator (called the Stokes semigroup) $S(t): v_0 \mapsto v(\cdot, t)$ is an analytic semigroup in L^∞-type spaces?

In other words, is there $C > 0$ s.t.

$$\left\| \frac{d}{dt} S(t) f \right\|_X \leq \frac{C}{t} \| f \|_X, \ t \in (0,1), \ f \in X$$

where X is an L^∞-type Banach space.

Analyticity is a notion of regularizing effect appeared in parabolic problems in an abstract level.

Definition of analyticity

Definition 1 (semigroup). Let $S = \{S(t)\}_{t>0}$ be a family of bounded linear operators in a Banach space X. In other words, $\{S(t)\}_{t>0} \in L(X)$. We say that S is a semigroup in X if

(i) (semigroup property) $S(t)S(\tau) = S(t+\tau)$ for $t, \tau > 0$

(ii) (strong continuity) $S(t)f \to S(t_0)f$ in X as $t \to t_0$ for all $t_0 > 0, f \in X$

(iii) (non degeneracy) $S(t)f = 0$ for all $t > 0$ implies $f = 0$.

(iv) (boundedness) $\|S(t)\|_{op} \leq {}^{\exists}C$ for $t \in (0,1)$

Definition 2 (non C_0 analytic semigroup). Let S be a semigroup in X. We say that S is analytic if $^\exists C > 0$ such that

$$\left\| \frac{d}{dt} S(t) \right\|_{op} \leq \frac{C}{t}, t \in (0,1).$$

See a book [ABHN] W. Arendt, Ch. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, Birkhäuser (2011)

Definition 3 (C_0-semigroup). A semigroup S is called C_0-semigroup if $S(t)f \to f$ as $t \downarrow 0$ for all $f \in X$.

Remark. The name of analyticity stems from the fact that $S = \{S(t)\}_{t \geq 0}$ can be extended as a holomorphic function to a sectorial region of t i.e. $|\arg t| < \theta$ with some $\theta \in (0, \pi/2)$.

A simple example

Heat semigroup (Gauss-Weierstrass semigroup)

$$(H(t)f)(x) = e^{t\Delta}f = G_t * f$$

$$= \int_{\mathbf{R}^n} G_t(x-y)f(y)dy$$

$$G_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp(-|x|^2/4t)$$

Proposition 1. The family $H = \{H(t)\}_{t>0}$ is a non C_0-analytic semigroup in $L^\infty(\mathbf{R}^n)$

(and also in $BC(\mathbf{R}^n)$)

but a C_0-analytic semigroup in $BUC(\mathbf{R}^n)$

(and also in $C_0(\mathbf{R}^n)$)

Here

$$BC(\mathbf{R}^n) = C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$$

$$BUC(\mathbf{R}^n) = \{f \in BC(\mathbf{R}^n) | f\text{: uniformly continuous}\}$$

$$C_0(\mathbf{R}^n) = L^\infty\text{-closure of } C_c^\infty(\mathbf{R}^n) = \left\{f \in C(\mathbf{R}^n) \,\middle|\, \lim_{|x|\to\infty} f(x) = 0\right\}$$

Spaces for divergence free vector fields

$C_{c,\sigma}^{\infty}(\Omega) = \{ f \in C_c^{\infty}(\Omega) | \mathrm{div}\, f = 0\}$

$=$ the space of all smooth solenoidal vector fields with compact support

$C_{0,\sigma}(\Omega) = L^{\infty}$-closure of $C_{c,\sigma}^{\infty}(\Omega)$

$= \{ f \in C(\overline{\Omega}) \, | \mathrm{div}\, f = 0$ in $\Omega, f = 0$ on $\partial\Omega\}$

If Ω is bounded. (Maremonti '09)

$L_{\sigma}^{r}(\Omega) = L^{r}$-closure of $C_{c,\sigma}^{\infty}(\Omega)$, $1 \leq r < \infty$

More spaces

Helmholtz decomposition (Ω bounded C^1-domain, …)

$$L^r(\Omega) = L^r_\sigma(\Omega) \oplus G^r(\Omega) \quad (1 < r < \infty)$$
$$G^r(\Omega) = \left\{ \nabla\pi \in L^r(\Omega) \middle| \pi = L^1_{loc}(\Omega) \right\}.$$
$$L^r_\sigma(\Omega) = G^{r'}(\Omega)^\perp$$
$$= \left\{ f \in L^r(\Omega) \middle| \int_\Omega f \cdot \nabla\varphi dx = 0 \text{ for all } \varphi \in G^{r'}(\Omega) \right\}$$

e.g. Fujiwara-Morimoto '79, Galdi's book '11

Here $1/r + 1/r' = 1$

$$L^\infty_\sigma(\Omega) := \left\{ f \in L^r(\Omega) \middle| \int_\Omega f \cdot \nabla\varphi dx = 0 \text{ for all } \varphi \in \widehat{W}^{1,1}(\Omega) \right\}.$$
$$C_{0,\sigma}(\Omega) \subset BUC_\sigma(\Omega) \subset L^\infty_\sigma(\Omega)$$

Typical main results

Theorem 1. (K. Abe – Y. G., Acta Math, to appear) Let Ω be a bounded C^3-domain in $\mathbf{R}^n (n \geq 2)$. Then the Stokes semigroup $S(t)$ is a C_0-analytic semigroup in $C_{0,\sigma}(\Omega)\ (= BUC_\sigma(\Omega))$. It can be regarded as a non C_0 semigroup in $L^\infty_\sigma(\Omega)$.

Remark. Whole space case is reduced to the heat semigroup. This type of analyticity result had been only known for half space where the solution is written explicitly (Desch-Hieber-Prüss '01, Solonnikov '03)

Analyticity of semigroup ···> regularizing effect
Known result for elliptic operators

(i) 2^{nd} order operator on $\mathbf{R}$ (one dim): K. Yosida '66

(ii) 2^{nd} order elliptic operator K. Masuda '71 '72 book in '75

L^r theory, cutoff procedure for resolvent

(iii) higher order, H. B. Stewart '74, '80

Masuda-Stewart method

(iv) degenerate + mixed B. C. K. Taira, '04

See also: P. Acquistapace, B. Terrani (1987)

A. Lunardi (1995) Book.

More recent. nonsmooth coefficient / nonsmooth domain

Heck-Hieber-Stavarakidis (2010) VMO coeff., higher order

Arendt-Schaetzle (2010) 2^{nd} order, Lipschitz domain

Stokes problem in L^p_σ (= L^p-closure of $C^\infty_{c,\sigma}$)

(i) L^2_σ : easy since the Stokes operator is nonnegative self-adljoint.

(ii) L^p_σ : V. A. Solonnikov '77 Y. G. '81 (bdd domain)
(max regularity / resolvent estimate)
$\cdots$ H. Abels-Y. Terasawa '09 (variable coefficient)
bdd, exterior, bent half space.

(iii) $\tilde{L}^r_\sigma$ space $= \begin{cases} L^r \cap L^2_\sigma & r \geq 2 \\ L^r + L^2_\sigma & r < 2 \end{cases}$

W. Farwig, H. Kozono and H. Sohr '05, '07, '09
General uniformity C^2-domain / All except
Solonnikov appeals to the resolvent estimate

Theorem 2. (K. Abe – Y. G. '12)
Let Ω be an C^3-exterior domain in $\mathbf{R}^n$. Then the Stokes semigroup $\{S(t)\}_{t>0}$ is a C_0-analytic semigroup in $C_{0,\sigma}(\Omega)$ and extends to a non C_0-analytic semigroup in $L^\infty_\sigma(\Omega)$. It can be regarded as a C_0-analytic semigroup in $BUC_\sigma(\Omega)$.

Note that for an unbounded domain $C_{0,\sigma}(\Omega)$ is strictly smaller than $BUC_\sigma(\Omega)$ because $f \in C_{0,\sigma}(\Omega)$ implies $|f(x)| \to 0$ as $|x| \to \infty$.

2. A priori estimates and blow-up arguments

A key a priori estimate

$$N(v,q)(x,t)$$
$$= \left|v(x,t)\right| + t^{1/2}\left|\nabla v(x,t)\right| + t\left|\nabla^2 v(x,t)\right|$$
$$+ t\left|\partial_t v(x,t)\right| + t\left|\nabla q(x,t)\right|$$

Theorem 3 (A priori estimate). Let Ω be a bounded domain with C^3-boundary. There exists $T_0 > 0$ and C such that for L^r solution (v,q) we have

$$\sup_{0<t<T_0} \left\|N(v,q)\right\|_\infty(t) \le C\left\|v_0\right\|_\infty, \;\; v_0 \in C^\infty_{c,\sigma}(\Omega).$$

(This estimate implies Theorem 1)

Idea of the proof – a blow-up argument a key observation

(Harmonic) pressure gradient estimate by velocity gradient

$$\sup_{x\in\Omega} d_\Omega(x)\left|\nabla q(x,t)\right| \le C\left\|\nabla v\right\|_{L^\infty(\partial\Omega)}(t)$$

$$d_\Omega(x) = \mathrm{dist}(x, \partial\Omega).$$

A blow-up argument (Argument by contradiction)

Suppose that the priori estimate were false for any choice of T_0 and C. Then there would exist a solution (v_m, q_m) with v_{0m} and a sequence $\tau_m \downarrow 0$ such that

$$\|N(v_m, q_m)\|_\infty(\tau_m) > m\|v_{0m}\|_\infty.$$

There is $t_m \in (0, \tau_m)$ such that

$$\|N(v_m, q_m)\|_\infty(t_m) \geq M_m / 2,\ M_m = \sup_{0<t<\tau_m} \|N(v_m, q_m)\|_\infty(t).$$

We normalize (v_m, q_m) by dividing M_m to observe

$$\sup_{0<t<t_m} \left\| N(\tilde{v}_m, \tilde{q}_m) \right\|_\infty (t) \le 1$$

$$\left\| N(\tilde{v}_m, \tilde{q}_m) \right\|_\infty (t_m) \ge 1/2$$

$$\left\| \tilde{v}_{0m} \right\|_\infty \to 0$$

with $\tilde{v}_m = \dfrac{v_m}{M_m}, \tilde{q}_m = \dfrac{q_m}{M_m}$.

We rescale $(\tilde{v}_m, \tilde{q}_m)$ around a point $x_m \in \Omega$ satisfying

$$N(\tilde{v}_m, \tilde{q}_m)(x_m, t_m) \ge 1/4.$$

Blow-up sequence

$$u_m(x,t) = \tilde{v}_m(x_m + t_m^{1/2}x, t_m t)$$

$$p_m(x,t) = t_m^{1/2}\tilde{q}_m(x_m + t_m^{1/2}x, t_m t)$$

(u_m, p_m) solves the equation in

a rescaled space-time domain

$$\Omega_m \times (0,1]$$

(Ω_m is expanding)

Basic strategy

A. Compactness:

Prove that (u_m, p_m) (subsequently) converges to (u, p) strong enough so that $N(u,p)(0,1) \geq 1/4$.

B. Uniqueness:

The blow-up limit (u, p) solves the Stokes problem with zero initial data so if the solution is unique it must be $u \equiv 0, \nabla p \equiv 0$ which contradicts $N(u,p)(0,1) \geq 1/4$.

M.-H. Giga, Y. Giga, J. Saal, Nonlinear PDEs, 2010

Blow-up argument

E. De Giorgi (1961), regularity of a minimal surface; popular in nonlinear problems

B.Gidas – J. Spruck ’81, a priori bound for semilinear elliptic problems

Y. Giga ’86, First application to a priori bound for parabolic problem (Giga – Kohn ’87)

P. Quittner – Ph. Souplet, ’07, Superlinear parabolic problems

Less for Navier-Stokes equations

- Koch – Nadirashvili, Seregin, Šverák ’09
  (nonexistence of type I axisymmetric singularity)
- Miura – Y. G. ’11
  (nonexistence of type I singularity having continuous vorticity direction)

In our case the problem is linear so we rescale the physical space and velocity in an unrelated way.

Compactness

What estimate is available for (u_m, p_m)?

$$\sup_{0<t<1} \| N(u_m, p_m) \|_\infty (t) \le 1$$

$$N(u_m, p_m)(0,1) \ge 1/4$$

$$\| u_{0m} \|_\infty \to 0.$$

The pressure gradients estimate implies

$$t^{1/2} d_{\Omega_m}(x) \left| \nabla p_m(x,t) \right| \le 1.$$

Set $c_m = d_m / t_m^{1/2}, d_m = d_\Omega(x_m).$

Case 1. $\overline{\lim_{m\to\infty}}\, c_m = \infty \;(\Omega_m \to R^n)$

Case 2. $\overline{\lim_{m\to\infty}}\, c_m < \infty \;(\Omega_m \to$ half space)

Case 2 is more involved.

Even case 1 it is nontrivial because the problem cannot be localized completely.

Ex. Interior regularity of the heat eq

$$u_t - \Delta u = 0 \quad \text{in} \quad B_R(x) \times (-R^2, 0).$$

If u is bdd by $|u| \le M$, then

$$|u|_{C^k} \le {}^{\exists}M_k \quad \text{in} \quad B_{R/2}(x) \times (-R^2/4, 0).$$

This is no longer true for the Stokes equations even if we assume

$$\sup_{0<t<T} \|N(v,q)\|_\infty(t) < \infty - (*).$$

Ex.

$$v(x,t) = g(t), \quad p(x,t) = -g'(t) \cdot x$$

$$g \in C^1[0,\infty)$$

Evidently, $(*)$ is fulfilled.

However v_t is not Hölder if g' is not Hölder.

Lemma 1 (Control of pressure gradient). If the harmonic pressure gradient estimate

$$\|d_\Omega \nabla q\|_{L^\infty(\Omega)} \le C \|\nabla v\|_{L^\infty(\partial\Omega)}$$

holds, then

$$^\exists M = M_\Omega$$

such that

$$[d_\Omega(x)\nabla q]^{(1/2)}_{t,Q_\delta} \le \frac{M}{\delta} \sup_{0<t<T} \|N(v,q)\|_\infty(t)$$

with $Q_\delta = \Omega \times (\delta, T)$. The constant M is invariant under dilation and translation.

Uniqueness

Lemma 2. (Solonnikov '03)

$v \in C^{2,1}\left(R_+^n \times (0,T)\right) \cap C\left(\overline{R}_+^n \times (0,T)\right)$

$\nabla q \in C\left(R_+^n \times (0,T)\right)$ solves the Stokes system in R_+^n with $v = 0$ on $\{x_n = 0\}$.

If $\sup\limits_{0<t<T} \|N(v,q)\|_\infty < \infty$, $v|_{t=0} = 0$

(weak $*$ in L^∞)

and if $\sup\limits_{0<t<T} t^{1/2} |x_n| |\nabla q| < \infty$, then

$v \equiv 0, \nabla q \equiv 0$.

Example of nontrivial solutions

Without decay estimate for ∇q this is not true.

$$\begin{cases} v_t^i - \Delta v^i = g^i(t) \text{ in } R_+^n \times (0,T)\,(1 \le i \le n-1) \\ v^i = 0 \text{ on } \partial R_+^n \times (0,T) \\ v^i \,|_{t=0} = 0,\, v^i = v^i(x_n, t) \text{ (independent of } x'\text{).} \end{cases}$$

Then $v = (v^1, \cdots v^{n-1}, 0)$ and $p(x,t) = -g(t) \cdot x'$ solves the Stokes system in a half space with zero initial data and zero boundary data. Here $x' = (x_1, \ldots, x_{n-1}, 0)$.

3. A priori estimate for harmonic pressure gradient

Equations for the pressure

Consider

$$v_t - \Delta v + \nabla q = 0 \text{ in } \Omega$$

Take divergence to get

$$\Delta q = 0 \text{ in } \Omega$$

since $\operatorname{div} v = 0$. Take inner product with n_Ω : (unit exterior normal) and use $v_t \cdot n_\Omega = 0$ to get

$$\partial q / \partial n_\Omega = n_\Omega \cdot \Delta v \text{ on } \partial\Omega$$

Lemma 3. If $\operatorname{div} v = 0$, then

$$n_\Omega \cdot \Delta v = \operatorname{div}_{\partial\Omega} W(v)$$

with

$$W(v) = -(\nabla v - {}^t(\nabla v)) \cdot n_\Omega$$

In three dimensional case,

$$n_\Omega \cdot \Delta v = -\operatorname{div}_{\partial\Omega}(\omega \times n_\Omega)$$

where $\omega = curl\ v$. In any case W is a tangent vector field.

Neumann problem

The pressure solves

(NP) $\Delta q = 0$ in Ω

$$\partial q / \partial n_\Omega = \mathrm{div}_{\partial\Omega} W$$

Enough to prove that

$$\|d_\Omega \nabla q\|_\infty \leq C \|W\|_\infty$$

for all tangential vector field W.

Strictly admissible domain

Definition 4 (Weak solution of (NP)). (Ken Abe – Y. G., '12) Let Ω be a domain in $\mathbf{R}^n$ $(n \geq 2)$ with C^1 boundary. We call $q \in L^1_{loc}(\overline{\Omega})$ a weak solution of (NP) for $W \in L^\infty(\partial\Omega)$ with $W \cdot n_\Omega = 0$ if q with $d_\Omega \nabla q \in L^\infty(\Omega)$ fulfills

$$\int_\Omega q \Delta \varphi dx = \int_{\partial\Omega} W \cdot \nabla \varphi d\mathcal{H}^{n-1}$$

for all $\varphi \in C_c^2(\overline{\Omega})$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$.

Definition 5 (Strictly admissible domain). Let Ω be a uniformly C^1 domain. We say that Ω is strictly admissible if there is a constant C such that

$$||d_\Omega \nabla q||_\infty \leq C||W||_{L^\infty(\partial\Omega)}$$

holds for all weak solution of (NP) for tangential vector fields. Note that strictly admissibility implies admissibility defined below.

Admissible domain

Let $P:\widetilde{L}^r(\Omega)\to\widetilde{L}^r_\sigma(\Omega)$ be the Helmholtz projection and $Q=I-P$. Applying Q to the Stokes equation to get

$$\nabla q = Q[\Delta v].$$

Here $\widetilde{L}^r = L^r \cap L^2$

$$\widetilde{L}^r_\sigma = L^r_\sigma \cap L^2 \text{ for } r>2.$$

Admissible domain (continued)

Definition 6. (Ken Abe – Y. G., Acta Math to appear) Let Ω be a uniformly C^1-domain. We say that Ω is admissible if there exists $r \geq n$ and a constant $C = C_\Omega$ such that

$$\sup d_\Omega(x) \left| Q[\nabla \cdot f](x) \right| \leq C \| f \|_{L^\infty(\partial\Omega)}$$

hold for all matrix value $f = (f_{ij}) \in C^1(\overline{\Omega})$ satisfy $\nabla \cdot f (= \sum_j \partial_j f_{ij}) \in \tilde{L}^r(\Omega)$,

$$\operatorname{tr} f = 0 \text{ and } \partial_l f_{ij} = \partial_j f_{il}$$

for all $i, j, l = \{1, \dots, n\}$.

Remark. (i) This is a property of the solution of the Neumann problem for the Laplace operator. In fact, $\nabla q = Q[\nabla \cdot f]$ is formally equivalent to

$$-\Delta q = \mathrm{div}\,(\nabla \cdot f) \quad \text{in} \quad \Omega$$

$$\partial q / \partial n_\Omega = n_\Omega \cdot (\nabla \cdot f) \;\; \text{on} \;\; \partial\Omega.$$

Under the above condition for f we see that q is harmonic in Ω since

$$\mathrm{div}\,(\nabla \cdot f) = \sum_{i,j} \partial_i \partial_j f_{ij} = \sum \partial_j \partial_j f_{ii} = 0.$$

(ii) The constant C_Ω depends on Ω but independent of dilation, translation and rotation.

(iii) If Ω is admissible, we easily obtain the pressure gradient estimate by taking $f_{ij} = \partial_j v^i$.

(iv) It turns out that

$$\partial q / \partial n_\Omega = -\mathrm{div}_{\partial\Omega}\left(n_\Omega \cdot (f - {}^t f)\right).$$

Remark. Strictly admissibility implies admissibility.

Example of strictly admissible domains

(a) half space

(b) C^3 bounded domain

(c) C^3 exterior domain

Note that layer domain $\{a < x_n < b\}$ is not strictly admissible.

Consider $q(x_1, \ldots, x_n) = x_1$.

Conjecture: Is Ω strictly admissible if it is NOT quasi-cylindrical $\left(\overline{\lim}_{|x|\to\infty} d_\Omega(x) < \infty\right)$?

A simple example – half space

Proposition 2. A half space R_+^n is strictly admissible for $n \geq 2$.

Sketch of the proof: The solution u of (NP) is of the form

$$u(x', x_n) = \int_{x_n}^{\infty} P_s[\operatorname{div} W]\, ds$$

in $R_+^n = \{(x', x_n) \in R^n | x_n > 0\}$, where P_s is the Poisson semigroup.

Poisson semigroup

$$P_s[f] = \left[\exp\left(-s\,(-\Delta')^{1/2}\right)\right]f$$

Thus

$$q(x', x_n) = \left(-(-\Delta')^{1/2}\left[\exp\left(-(-\Delta')^{1/2}x_n\right)\right]f\right)(x').$$

Clearly,

$$\Delta q = \left(\frac{d^2}{dX_n^2} + \Delta'\right)q = 0.$$

Moreover,

$$\frac{\partial}{\partial n_\Omega}u(x', x_n)|_{x_n=0} = \exp\left(-(-\Delta')^{1/2}x_n\right)\operatorname{div} W|_{x_n=0} = \operatorname{div} W.$$

Basic estimate and completion of the proof

$$\left\|\frac{\partial}{\partial s}P_s\right\|_{\mathrm{op}}(s) \le \frac{C}{s}, s > 0,$$

$$\|\nabla' P_s\|_{\mathrm{op}}(s) \le \frac{C}{s}, s > 0.$$

This is explicitly proved by estimating the Poisson kernel. Thus

$$\left\|\frac{\partial q}{\partial x_n}\right\|_\infty (x_n) \le \left\|\operatorname{div} P_{x_n}[W]\right\|_\infty \le \frac{C}{x_n}\|W\|_\infty,$$

$$\|\nabla' q\|_\infty (x_n) \le \|W\|_\infty \int_{x_n}^\infty \frac{C}{s^2} ds \le \frac{C}{x_n}\|W\|_\infty.$$

This is what we want to prove.

Nontrivial examples

Proposition 3. A C^3 bounded domain is strictly admissible for $n \geq 2$.

Sketch of the proof: We shall prove this estimate by argument by contradiction and blow-up argument. Suppose that the estimate holds there is a sequence of function such that

$$\|d_\Omega u_m\|_\infty > m\|W_m\|_\infty.$$

By normalization we may assume that $\|d_\Omega u_m\|_\infty = 1$ and $\|W_m\|_\infty < 1/m$.

Blow-up argument

We trace maximum point of $d_\Omega |u_m|$. Let x_m be a maximum point. By taking a subsequence we may assume that $x_m \to \hat{x}$ as $m \to \infty$.

Case 1 $\hat{x} \in \Omega$

This contradicts uniqueness of (NP) since the limit of u_m is a nontrivial solution of (NP).

Case 2 $\hat{x} \in \partial\Omega$

We blow up so that distance between x_m and the boundary equals 1. Then we yield a nontrivial solution (NP) as a limit of u_m contradicting the uniqueness of (NP) in a half space.

Summary

- The Stokes semigroup $S(t)$ is analytic in $C_{0,\sigma}(\Omega)$ when uniformly C^3 domain is admissible.
- It can be extended to a C_0 analytic semigroup in $BUC_\sigma(\Omega)$ when Ω is exterior and bounded.
- Blow-up argument is useful to prove establish necessary estimate.

Note: Proof by resolvent estimate is now available. (Ken Abe, Y. G., M. Hieber '12) (Abe's presentation)

It is applicable to other boundary conditions like Navier boundary condition.

Open problems

We have discussed regularizing effect by proving analyticity of the Stokes semigroup $S(t)$. We do not know well about large time behavior.

Problem. (1) Is $S(t)$ bounded in time?

i.e. $\|S(t)\|_{\mathrm{op}} \leq C$ for all $t > 0$.

(2) Is $S(t)$ a bounded analytic semigroup?

i.e. $\|dS(t)/dt\|_{\mathrm{op}} \leq Ct^{-1}$ for all $t > 0$.

[(1), (2) yes for a bounded domain: Abe-Giga, Acta Math]

[(1) yes for an exterior domain: Maremonti '12]

Open problems
(Solvability of the Navier-Stokes equations)

Problem. (3) Do the Navier-Stokes equations admit a local smooth solution even if initial data u_0 is in L^∞_σ or BUC_σ for a domain Ω having a boundary?

[(3) yes for a half space: Solonnikov '03, Bae-Jin '12]

[(3) yes for a three dimensional exterior domain provided that u_0 is Hölder and bounded: Galdi-Maremonti-Zhou '12]

[Ken Abe work in progress]