The Navier-Stokes equations with spatially nondecaying data
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I. The Navier-Stokes equations with L^∞-data

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1.1. The Navier-Stokes initial value problem

(NS) $u_t - \Delta u + (u, \nabla)u + \nabla\pi = 0$ in $R^n \times (0, T)$

$$\operatorname{div} u = 0 \text{ in } R^n \times (0, T)$$

$$u|_{t=0} = u_0 (\operatorname{div} u_0 = 0)$$

$u = u(x, t)$: real vector (velocity fields)

$\pi = \pi(x, t)$: scalar (pressure fields)

(kinematic viscosity is normalized to be one)

One of Clay's Millennium Problems

Does the **three**-dimensional $(n = 3)$ Navier-Stokes initial value problem admit a global-in-time smooth solutions for smooth (compactly supported) initial data even if it is not small?

1.2. Quick overview of known results

(1) Two-dimensional problem

For $n = 2$ there exists a unique global smooth solution for arbitrary u_0 provided that the kinetic energy

$$\frac{1}{2}\int_{R^n} |u_0|^2 dx$$

is finite. (No smallness assumption is necessary) J. Leray '33 ……

(2) Global existence for small data

Even for $n = 3$ if initial data is sufficiently small, say

$$\|u_0\|_{L^n}^n := \int |u_0|^n \, dx$$

is small, then there exists a unique global smooth solution. Smallness depends only on n. J. Leray '34, Kiselev-Ladyzhenskaya ……

G-Miyakawa '85 T. Kato '84

(3) Local existence

Always, there exists a unique smooth locally-in-time solution for arbitrary initial data u_0.

For example, if $\|u_0\|_p$ is finite for $p \geq n$, there is such a solution.

(L^2-theory: Kato-Fujita '62,

L^p-theory: G-Miyakawa '85, Kato '84)

Remark. Local existence and global existence of small data has been established for various function spaces not only L^p but also Besov spaces, BMO space (e.g. Koch-Tataru '01). However, there seems to be 'critical exponent' to guarantee solvability. (Bourgain-Pavlovic, Yoneda, Sawada)

Critical exponent and scaling invariance of (NS)

If (u, π) solves (NS) in $R^n \times (0, \infty)$, so does

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t),$$

$$\pi_\lambda(x,t) = \lambda^2 \pi(\lambda x, \lambda^2 t) \text{ for } \lambda > 0.$$

The norm $\|u_0\|_{L^n}$ is invariant under this scaling. In this sense $p = n$ is critical.

(4) Weak solutions

There is a global weak solution (may not be differentiable, may not be unique) for arbitrary initial data with finite energy.

J. Leray '34, ……

Energy inequality –
a key for construction a weak solution

$$u \times \mathrm{eq}$$

$$\frac{1}{2}\ \frac{d}{dt}\int |u|^2 dx + \int |\nabla u|^2 = 0$$

$\cdots\cdots\cdots\cdots\blacktriangleright$

$$\|u\|_2^2\ (t) + 2\int_0^t \|\nabla u\|_2^2\ ds\ \underset{=}{\leq} \|u_0\|_2^2$$

(5) Regularity of weak solutions

(a) Estimate of possible singularities

J. Leray '34, Scheafer, Cafferelli-Kohn-Nirenberg '82. A suitable weak solution with $P^1(s) = 0$ is constructed for $n = 3$. Here s is a singular set and P^1 is the Hausdorff measure (parabolic). Note that s can be empty.

(b) Regularity criteria

(Extendability) Let u be a smooth solution of (NS) in $(0,T)$. If one assumes extra assumptions, then one can extend the solution beyond T.

(Regularity) Let u be a weak solution of (NS) in $(0,\infty)$. If one assumes extra assumptions, then u is regular.

Typical example (J. Serrin '61…)

If u satisfies

$$\int_0^T \left(\int |u|^p dx \right)^{q/p} dt < \infty$$

with $\frac{n}{p} + \frac{2}{q} \leq 1$, then one can extend the solution beyond T.

If u is a weak solution, u is regular in $R^n \times (0, T]$. Note that the integral is scaling invariant for the equality case of exponents.

Remark. (i) Since then there is a large literature on regularity criteria. A general principle is that if **a scaling invariant quantity is finite**, then one expect smoothness. In fact, energy inequality is scaling invariant for $n = 2$ while it is not for $n \geq 3$. Energy inequality is too weak to guarantee smoothness for $n \geq 3$.

Remark. (ii) Most of regularity criteria assumes finiteness of some scaling invariant quantity for velocity, vorticity, pressure. New type of criteria called **geometric criteria** is introduced by Constantin-Fefferman '93 on the direction of the vorticity.

2.1. Nondecaying initial data

If u_0 does not decay at spatial infinity, does the solution blow-up in finite time?

There is more chance to have a blow-up solution. However, if u_0 is periodic the situation is essentially similar and even easier than whole space problem.

Warning for nondecaying solutions

$$u(x,t) = g(t)$$

$$\pi(x,t) = -g'(t)x$$

always solves the Navier-Stokes initial value problems.

Note: g is arbitrary.
Any spatially constant vector field u is a solution!

(Seregin-Sverak called a Parasitic Solution)

Mild solutions (approximable by decaying initial data)

We asked a special relation between u and π which is automatic for spatially decaying solutions.

Take div of the first equation to get.

$$\mathrm{div}\,(u,\nabla)u + \Delta\pi = 0$$

Mild sol: Solution (u,π) satisfying

$$\boxed{\pi = (-\Delta)^{-1}(\mathrm{div}\,(u,\nabla)u)}$$ (G-Inui-Matsui ’99)

2.2. Local well-posedness

There exists a unique local-in-time **mild** solution for the Navier-Stokes initial value problem for u_0 belonging to a function space X which includes nondecaying functions. The solution is classical for $t > 0$.

(1) Kightly '72 $X = L^{\infty}$ mild sol.

(without explicit proof)

(2) G-Inui-Matsui '99 $X = BUC,\ L^{\infty}$ mild sol. regularity

(3) Koch-Tataru '01 $X = \partial(BMO)$

$\|u_0\|_{\partial(BMO)}$ small ⇒ global existence

(4) Lemarie-Riesset $X = L^2_{ul}$ local weak sol.

(5) Maekawa-Terasawa $X = L^p_{ul},\ p \geq n$ local strong sol.

$$\|u\|_{L^p_{ul}} = \sup_{x \in R^n} \left(\int_{B(x,1)} |u|^p \, dx \right)^{1/p}$$

Regularity for local solution

(6) G-Sawada '03

spatial estimate $\| \partial_x^m u \|_\infty (t) \le C t^{-m/2}$

spatial analyticity

(7) G-Jo-Mahalov-Yoneda '08

time analyticity

Note: Solution depends on initial data uniformly continuously in L^∞.

G-Mahalov-Nicolaenko '08

u_0: almost periodic $\Rightarrow$ u: spatially almost periodic

2.3. Construction of a local solution

Convert eq. to integral eq.

P : Leray-Helmoltz projection $P = \left(P_{ij}\right)$

$P_{ij} = \delta_{ij} + \partial_{x_i}\partial_{x_j}(-\Delta)^{-1}, 1 \le i, j \le n$.

- Apply P to the first eq. of (NS)

$$u_t - \Delta u = -P(u, \nabla)u$$

- Integral eq. (Duhamel's principle)

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta} P(u, \nabla)u \, ds$$

- Picard like successive approximation

Heat semigroup $e^{t\Delta}$

$$\left(e^{t\Delta}f\right)(x) = \int_{R^n} G_t(x-y)f(y)dy = G_t * f$$

$$G_t(x) := (4\pi t)^{-n/2}\exp(-|x|^2/4t)$$

Gauss kernel

Regularizing estimates

Young's inequality for convolution implies

$$\left\|\partial^m e^{t\Delta}f\right\|_p \le Ct^{-m/2}\|f\|_p \ (1 \le p \le \infty).$$

Operator P

Note that P is bounded in L^p $(1 < p < \infty)$ but not in L^∞.

We use regularizing estimate

$$\left\| \partial e^{t\Delta} P f \right\|_p \leq \frac{C}{t^{1/2}} \| f \|_p$$

for $p = \infty$ to prove the convergence of approximate solutions.

For $1 < p < \infty$ it follows from

$$\left\|\partial e^{t\Delta} f\right\|_p \leq \frac{C}{t^{1/2}} \|f\|_p .$$

For $p = \infty$ special device is necessary.

〈Short proof by G-Jo-Mahalov-Yoneda '08〉

Use $(-\Delta)^{-1} = \int_0^\infty e^{s\Delta} ds$ to get

$$\partial_k e^{t\Delta} \partial_i \partial_j (-\Delta)^{-1} = \partial_k \partial_i \partial_j \int_t^\infty e^{s\Delta} ds$$

$$\|\cdots\|_{L^\infty \to L^\infty} \leq C \int_t^\infty \frac{ds}{s^{3/2}} = C' t^{-1/2} .$$

Picard like successive approximation

$$u_{m+1}(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}P(u_m,\nabla)u_m ds,$$

$$u_1(t) = e^{t\Delta}u_0,\ \ m = 1, 2, \dots .$$

Boundedness of the sequence:

Regularizing estimates imply

$$\|u_{m+1}\|_\infty \le \|u_0\|_\infty + \int_0^t \frac{C}{(t-s)^{1/2}}\|u_m\|_\infty^2 ds$$

since $(u,\nabla)u = \nabla\cdot u \otimes u$ by $\mathrm{div}\, u = 0$.

A priori estimate

We set $K_m(T) = \sup\{\|u_m\|_\infty(t) | 0 < t < T\}$. The estimate implies

$$K_{m+1}(T) \le \|u_0\|_\infty + C' K_m(T)^2 T^{1/2}.$$

This implies a bound for $\{K_m\}$ provided that $\|u_0\|_\infty T^{1/2}$ is smaller than a fixed computable constant (depending on C'). One can prove that $\{u_m\}$ is a Cauchy sequence in $L^\infty([0,T), L^\infty)$.

Unique local existence

Theorem (Knightly '72, G-Inui-Matsui '99).
There is a constant C_0 such that there exists a local-in-time mild solution u of (NS) with $u_0 \in L^\infty$ in a time interval $(0, T)$ with $T \geq C_0 / \|u_0\|_\infty^2$.

Corollary (Lower bound of blow-up).
If u blows up at time T_*, then

$$\|u\|_\infty(t) \geq C_0^{1/2} / (T_* - t)^{1/2}.$$

Remark: If $u_0 \in BUC$, then
$u \in C([0, T), BUC)$.

3. Two dimensional problem

3.1. Global well-posedness

If the space dimension $n = 2$,
the solution can be extended globally-in-time.

(G-Matsui-Sawada ’01)

$$\|u\|_\infty(t) \le C\|u_0\|_\infty \exp(C\|\omega_0\|_\infty t)$$

(Sawada-Taniuchi ’07)

$$\omega_0 = \operatorname{curl} u_0$$

key: $2-D$ **vorticity equation**

$$\omega_t - \Delta\omega + (u,\nabla)\omega = 0$$

Maximum principle (Kato-Fujita '59)

$$\|\omega\|_\infty (t) \le \|\omega_0\|_\infty$$

$$\omega = \mathrm{curl}\ u$$

The proof is based on voritcity eq.
〈No stretching term $(\omega,\nabla)u$ unlike $n=3$.〉

3.2. Idea of Proof

A priori global estimate for $\|u\|_\infty(t)$.
(G-Matsui-Sawada '01, double exponential)
(Sawada-Taniuchi '07, single exponential type)
We shall give a sketch of the proof for

$$\|u\|_\infty(t) \le C\|u_0\|_\infty \exp(C\|\omega_0\|_\infty t)$$

following the idea of Sawada-Taniuchi.
Here $\omega_0 = \mathrm{curl}\, u_0$.

Littlewood-Payley decomposition

$\{\varphi_j\}_{j=-\infty}^{\infty} \subset C^{\infty}(R^n)$ such that

(i) $\hat{\varphi}_j(\xi) = \hat{\varphi}_0\left(2^{-j}\xi\right)$

(ii) $\Sigma\hat{\varphi}_j(\xi) = 1 \ (\xi \neq 0)$

(iii) $\mathrm{supp}\,\hat{\varphi}_0 \subset \{1/2 \leq |\xi| \leq 2\}$

Such φ_0 always exists! Here $\hat{\varphi}$ denotes the Fourier transform of φ.

Basic estimates

Set $\hat{\psi}_k = 1 - \sum_{j=k}^{\infty} \hat{\varphi}_j$

Lemma. (a) $\left\|\varphi_j\right\|_1 = \|\varphi_0\|_1 < \infty, j \in Z$

$$\left\|\psi_j\right\|_1 = \|\psi_0\|_1 \; (= \sigma_0) \;\; (j \leq 0\,)$$

(b) $\left\|\nabla(-\Delta)^{-1}\varphi_j\right\|_1 = 2^{-j}\lambda,\; j \in Z$

$$(\lambda = \|\nabla(-\Delta)^{-1}\varphi_0\|_1 < \infty)$$

(c) $\left\|P\nabla\varphi_j\right\|_1 \leq 2^j\sigma \;\; (j \leq 0)$

$$(\sigma = \|P\nabla\varphi_0; \mathcal{H}^1\| < \infty)$$

Decomposition of low and high frequency part

$$u = \psi_{-N} * u + \sum_{j=-N}^{\infty} \varphi_j * u$$

$$\|u\| \le \|\psi_{-N} * u\| + \sum_{j=-N}^{\infty} \left\|\varphi_j * u\right\|$$

$$= \mathrm{I} + \mathrm{II}$$

I: low frequency part

II: high frequency part

($\|\cdot\| = \|\cdot\|_\infty$)

Estimate for low frequency part

$$
\begin{aligned}
\mathrm{I} &\leq \|\psi_0\|_1 \left\|e^{t\Delta}u_0\right\| \\
&\quad + \int_0^t \left\|P\nabla\psi_{-N} * e^{(t-s)\Delta} u \otimes u\right\| ds \\
&\leq \sigma_0 \|u_0\| + 2^{-N}\sigma \int_0^t \|u\|^2 ds
\end{aligned}
$$

Estimate for high frequency part

Use Biot-Savart: $u = \nabla^{\perp}(-\Delta)^{-1}\omega$ to get

$$\left\|\varphi_j * u\right\| \leq 2^{-j}\lambda\|\omega\| \leq 2^{-j}\lambda\|\omega_0\|.$$

The last inequality follows from the maximum principle. We thus obtain

$$\mathrm{II} \leq \lambda \sum_{j=-N}^{\infty} 2^{-j}\|\omega_0\| = \lambda 2^{N}\|\omega_0\|.$$

Choice of cutting number N

$\mathrm{I} + \mathrm{II}$

$$\leq \sigma_0 \|u_0\| + 2^{-N}\sigma \int_0^t \|u\|^2 ds + 2^N \lambda \|\omega_0\|.$$

Take N large such that

$$2^N \leq \left(\sigma \int_0^t \|u\|^2 ds / \|\omega_0\| \lambda\right)^{1/2} \leq 2^{N+1}$$

to get

$$\|u\|^2 \leq \left(\sigma_0 \|u_0\| + 3\left[\sigma\lambda \|\omega_0\| \int_0^t \|u\|^2 ds\right]^{1/2}\right)^2.$$

Application of the Gronwall inequality

Use $(\mathrm{a}+\mathrm{b})^2 \le 2(a^2+b^2)$ to get

$$||u||^2 \le 2\sigma_0||u_0|| + 2\cdot 3^2\sigma\lambda||\omega_0|| \int_0^t ||u||^2\, ds.$$

Gronwall implies

$$||u||^2(t) \le 2\sigma_0||u_0||^2 \exp(18\sigma\lambda||\omega_0||t).$$

Open problems

ⓐ Are there global-in-time weak solutions for $n = 3$?

(cf. J. Leray '34 if u_0 has finite energy i.e., $\|u_0\|_{L^2} < \infty$, then $\exists$ global weak solution.)

(b) Even if $n = 2$ does the problem admit a global solution for $u_0 \in L^\infty$ when we impose the Dirichlet boundary condition.

⟨No maximum principle for ω is expected.⟩