Yang-Baxter maps and integrability

Alexander Veselov, Loughborough University, UK

Complement to the lectures at UK-Japan Winter School, Manchester 2010

History

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Yang-Baxter equation in quantum theory and statistical mechanics

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Dynamical point of view:

A.P. Veselov *Yang-Baxter maps and integrable dynamics.* Physics Letters A, **314** (2003), 214-221.

Quantum Yang-Baxter equation

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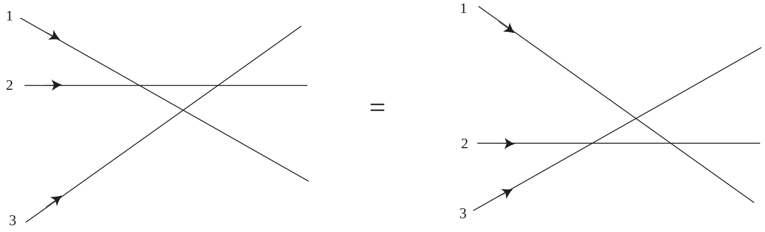

Figure: Yang-Baxter relation

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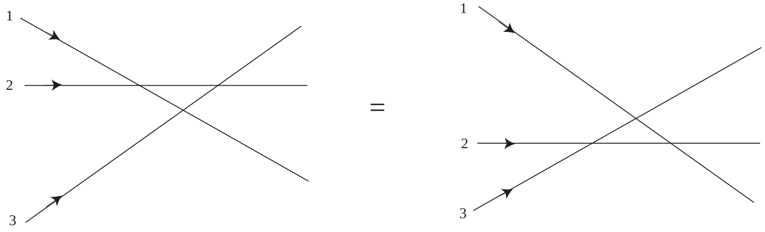

Figure: Yang-Baxter relation

Important consequence: **Transfer-matrices** $T(\lambda) = tr_0 R_{0n} \dots R_{01}$ **commute:**

$$T(\lambda)T(\mu) = T(\mu)T(\lambda).$$

Yang-Baxter maps (= Set-theoretical solutions of YBE)

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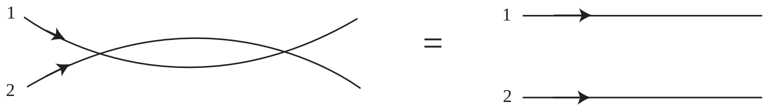

Figure: Reversibility

One can consider also the **parameter-dependent Yang-Baxter maps** $R(\lambda, \mu)$ with λ, μ from some parameter set Λ, satisfying

$$R_{12}(\lambda_1, \lambda_2) R_{13}(\lambda_1, \lambda_3) R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3) R_{13}(\lambda_1, \lambda_3) R_{12}(\lambda_1, \lambda_2)$$

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Although this case can be considered as a particular case of the previous one by introducing $\tilde{X} = X \times \Lambda$ and $\tilde{R}(x, \lambda; y, \mu) = R(\lambda, \mu)(x, y)$ it is often convenient to keep the parameter separately.

Matrix KdV equation

$$U_t + 3UU_x + 3U_xU + U_{xxx} = 0$$

has the soliton solution of the form

$$U = 2\lambda^2 P \; sech^2(\lambda x - 4\lambda^3 t),$$

where "polarisation" P must be a projector: $P^2 = P$.

Example 1: Interaction of matrix solitons

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The change of polarisations P after the soliton interaction is **non-trivial**:

$$\tilde{L}_1 = (I + \frac{2\lambda_2}{\lambda_1 - \lambda_2} P_2) L_1,$$

$$\tilde{L}_2 = (I + \frac{2\lambda_1}{\lambda_2 - \lambda_1} P_1) L_2,$$

where L is the image of P (**Goncharenko, AV** (2003)).

Tsuchida (2004), **Ablowitz, Prinari, Trubatch** (2004): vector NLS equation

Example 2: KdV and Adler map

Darboux transformation

$$L = -\frac{d^2}{dx^2} + u(x) = A^* A \to L_1 = AA^*.$$

$$A = \frac{d}{dx} - f(x), \quad A = -\frac{d}{dx} - f(x).$$

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A.B.Shabat, A.V. (1993): periodic dressing chain

$$(f_i + f_{i+1})' = f_i^2 - f_{i+1}^2 + \alpha_i,\ i = 1, \ldots, 2m+1.$$

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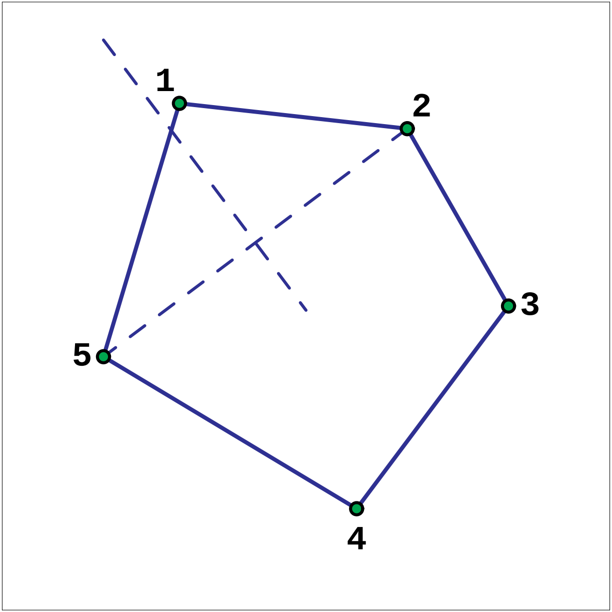
$$(f_i + f_{i+1})' = f_i^2 - f_{i+1}^2 + \alpha_i, \ i = 1, \ldots, 2m+1.$$

V. Adler (1993): symmetry of dressing chain

$$\tilde{f}_1 = f_2 - \frac{\beta_1 - \beta_2}{f_1 + f_2}$$

$$\tilde{f}_2 = f_1 - \frac{\beta_2 - \beta_1}{f_1 + f_2}$$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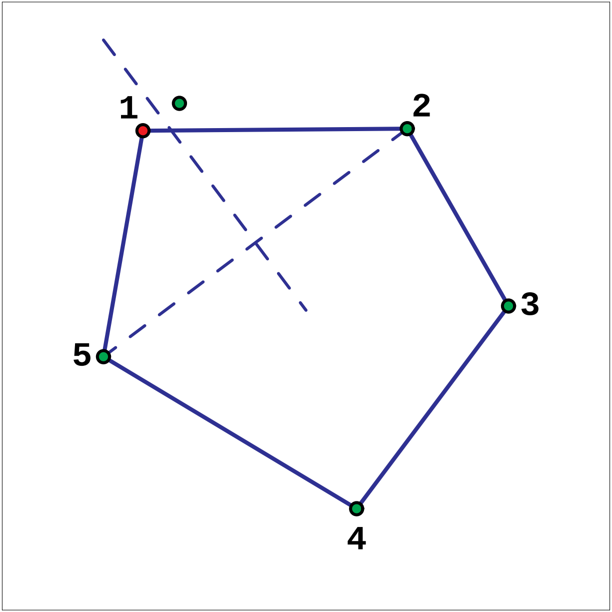
Geometric realisation: Recuttings of polygon



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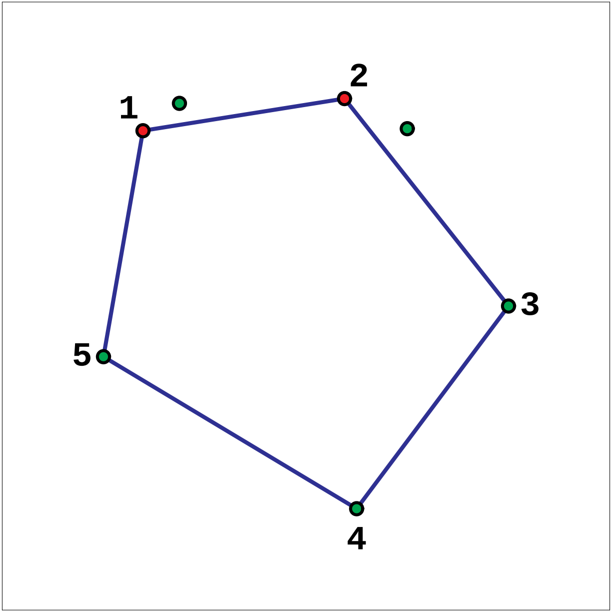

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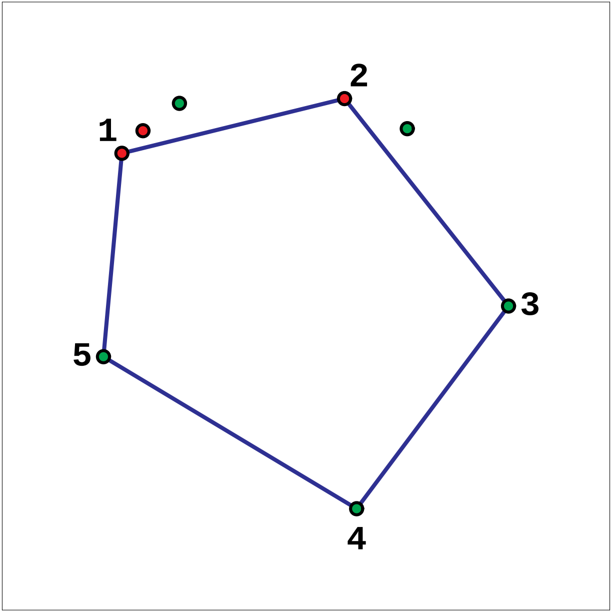

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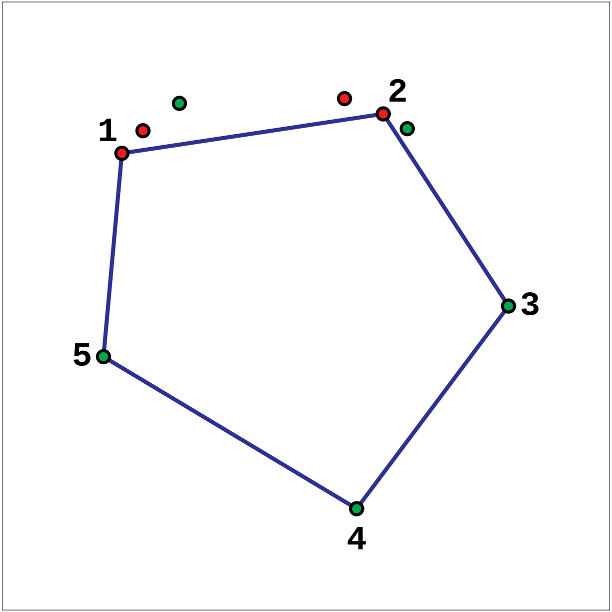

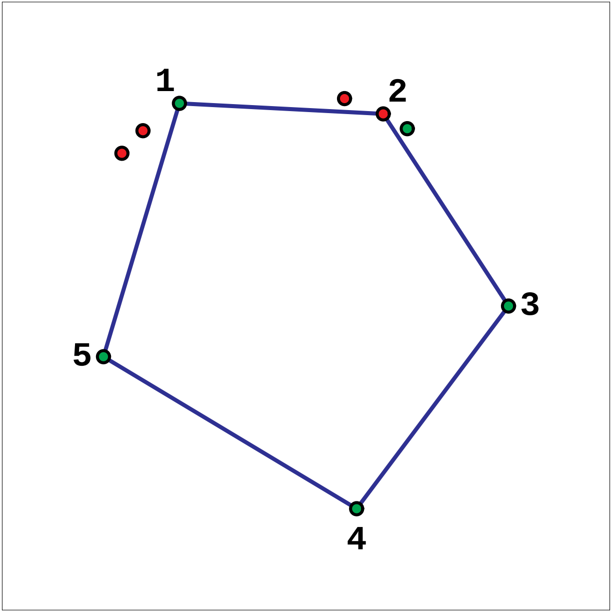
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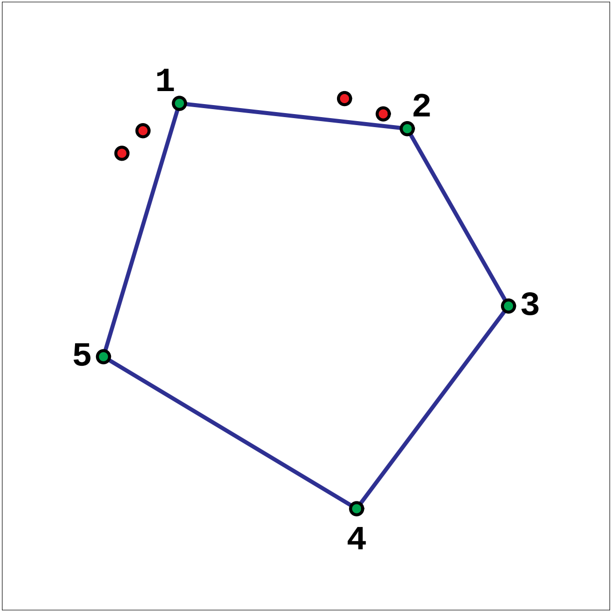

Transfer dynamics

Define the **transfer maps**

$$T_i^{(n)} : X^n \to X^n,\ i = 1, \dots, n$$

by

$$T_i^{(n)} = R_{ii+n-1} R_{ii+n-2} \dots R_{ii+1},$$

where the indices are considered modulo n. In particular $T_1^{(n)} = R_{1n} R_{1n-1} \dots R_{12}$.

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For any reversible Yang-Baxter map R the transfer maps $T_i^{(n)}$ **commute with each other**:

$$T_i^{(n)} T_j^{(n)} = T_j^{(n)} T_i^{(n)}$$

and satisfy the property

$$T_1^{(n)} T_2^{(n)} \dots T_n^{(n)} = Id.$$

Conversely, if $T_i^{(n)}$ satisfy these properties then R is a reversible YB map.

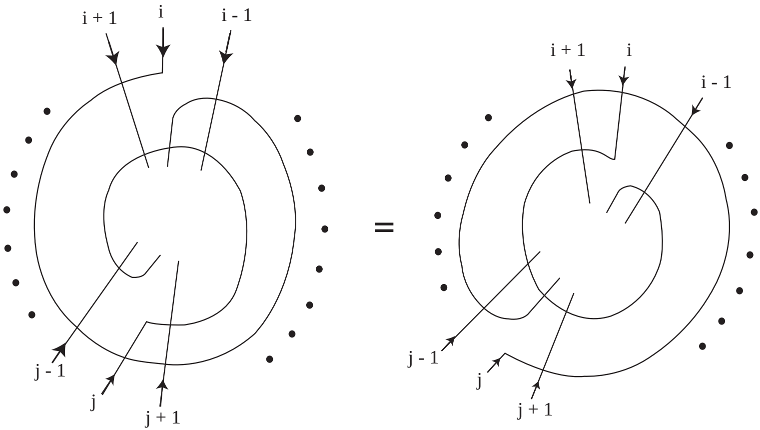
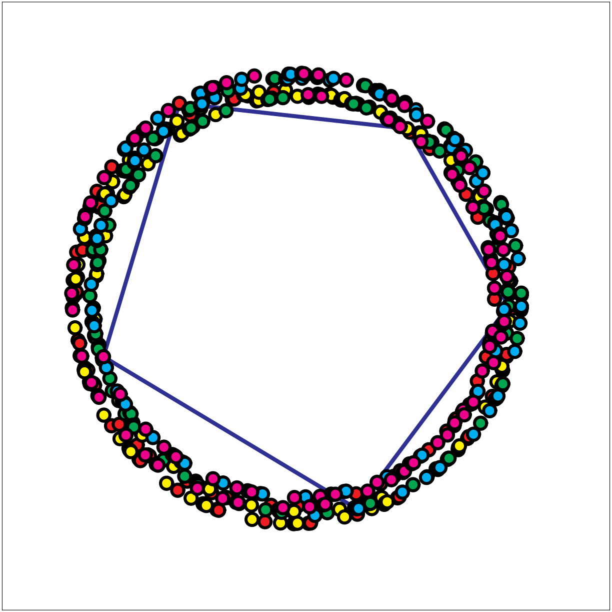
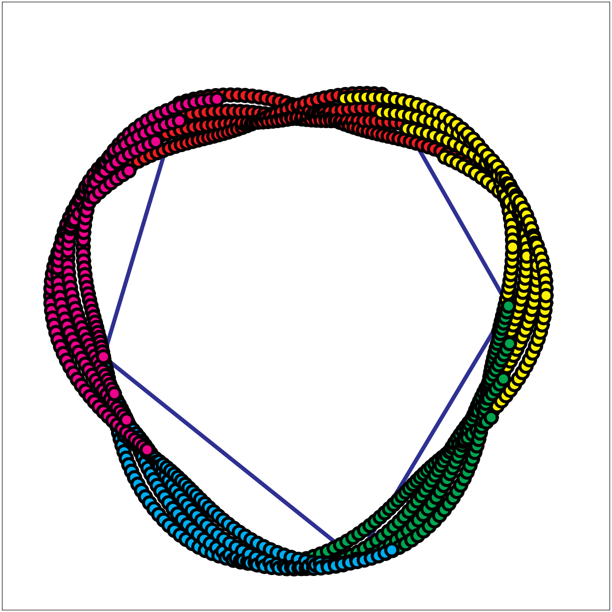


Figure: Commutativity of the transfer maps

Recutting of polygons: dynamics



Some other initial data



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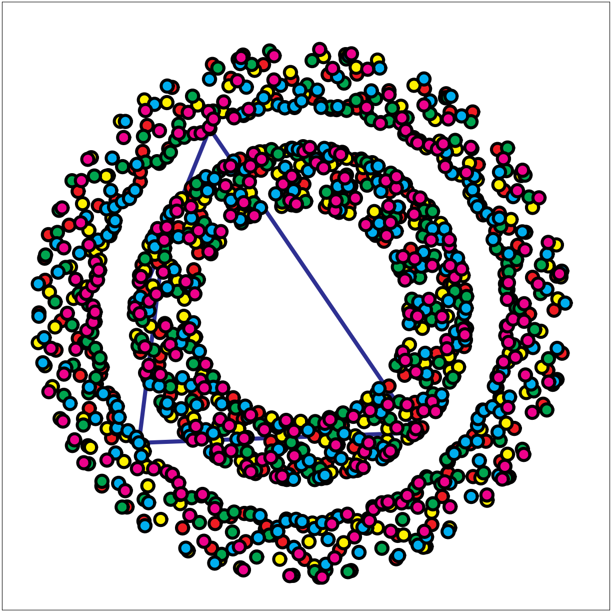

Matrix $A(x, \lambda, \zeta)$ with spectral parameter $\zeta \in \mathbb{C}$ is called **Lax matrix** of the map R if it satisfies the relation

$$A(x, \lambda; \zeta)A(y, \mu; \zeta) = A(\tilde{y}, \mu; \zeta)A(\tilde{x}, \lambda; \zeta),$$

whenever $(\tilde{x}, \tilde{y}) = R(\lambda, \mu)(x, y)$.

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Define **monodromy matrix**

$$M = A(x_n,\lambda_n,\zeta)A(x_{n-1},\lambda_{n-1},\zeta)\dots A(x_1,\lambda_1,\zeta).$$

The transfer maps $T_i^{(n)}$ preserve the spectrum of M for all ζ. The coefficients of the characteristic polynomial

$$\chi = \det(M(x,\lambda,\zeta) - \mu I)$$

are the **integrals of the transfer-dynamics.**

Suris, AV (2003):
Suppose that the Yang-Baxter map $R(\lambda, \mu)$ has the following special form:

$$\tilde{x} = B(y, \mu, \lambda)[x], \quad \tilde{y} = A(x, \lambda, \mu)[y]$$

for some action of $GL(N)$ on X. Then both $A(x, \lambda, \zeta)$ and $B^{\mathrm{T}}(x, \lambda, \zeta)$ are Lax matrices for R.

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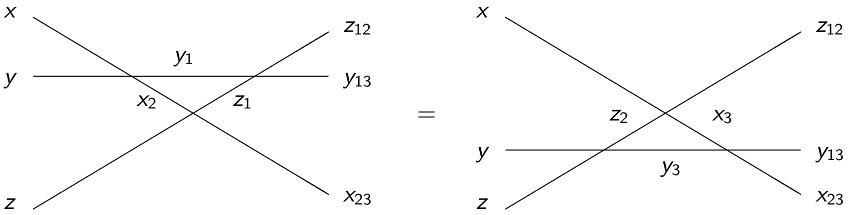

Indeed, LHS gives $z_{12} = A(y_1, \mu, \nu)A(x_2, \lambda, \nu)[z]$, while the RHS gives $z_{12} = A(x, \lambda, \nu)A(y, \mu, \nu)[z]$.

Example: Lax matrix for Adler map

For Adler map

$$\tilde{x} = y - \frac{\lambda - \mu}{x + y}$$
$$\tilde{y} = x - \frac{\mu - \lambda}{x + y}$$

we can write

$$\tilde{y} = x - \frac{\mu - \lambda}{x + y} = \frac{x^2 + xy - (\mu - \lambda)}{x + y} = A(x, \lambda, \mu)[y],$$

so we come to the Lax matrix

$$A = \begin{pmatrix} x & x^2 + \lambda - \zeta \\ 1 & x \end{pmatrix},$$

(which was actually known from the theory of the dressing chain).

Close relative: integrable discrete equations

Bianchi (1880s):

Superposition of Bäcklund transformations:

$$\begin{array}{ccc} v & \longrightarrow & v_1 \\ \downarrow & & \downarrow \\ v_2 & \longrightarrow & v_{12} \end{array}$$

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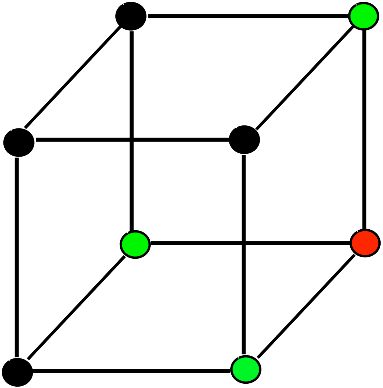
Bianchi's important observation was the results of these commuting transformations satisfy an **algebraic relation**.

In KdV case the Darboux transformations satisfy

$$(v_{12} - v)(v_1 - v_2) = \beta_1 - \beta_2,$$

which is the discrete KdV equation.



Bianchi (1880s), **Tsarev** (1990s), **Doliwa and Santini** (1997), **Bobenko and Suris, Nijhoff** (2001): $3D$ consistency as the definition of integrability.

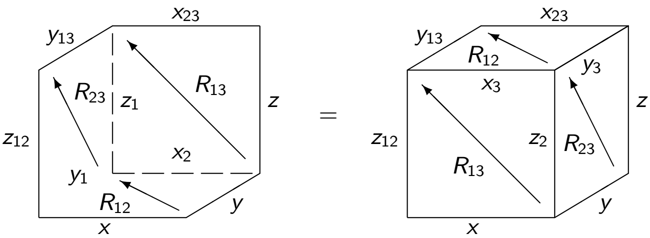


Figure: "Cubic" representation of the Yang–Baxter relation

From IDE to YBM

Papageorgiou, Tongas, AV (2006): symmetry approach

Discrete KdV equation

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is invariant under the translation $v \to v + \mathrm{const.}$

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The invariants

$$x_1 = v_1 - v, \quad x_2 = v_{1,2} - v_1, \quad y_1 = v_{1,2} - v_2, \quad y_2 = v_2 - v,$$

satisfy the relation

$$x_1 + x_2 = y_1 + y_2$$

and the equation itself:

$$(x_1 + x_2)(x_1 - y_2) = \beta_1 - \beta_2.$$

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and the equation itself:

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This leads to the following YB map

$$y_1 = x_2 + \frac{\beta_1 - \beta_2}{x_1 + x_2}, \qquad y_2 = x_1 - \frac{\alpha_1 - \beta_2}{x_1 + x_2},$$

which is nothing else but the Adler map.

Weinstein and Xu (1992), **Reshetikhin, AV** (2005)

Suppose that X can be embedded as a symplectic leaf in a Poisson Lie group G: $\phi_\lambda : X \to G$ and define the correspondence $R(\lambda, \mu) : X \times X \to X \times X$ by the relation

$$\phi_\lambda(x)\phi_\mu(y) = \phi_\mu(\tilde{y})\phi_\lambda(\tilde{x}).$$

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Define the symplectic structure $\Omega^{(N)}$ on $X^{(N)}$ as

$$\Omega^{(N)} = \omega_{\lambda_1} \oplus \omega_{\lambda_2} \oplus \ldots \oplus \omega_{\lambda_N}.$$

Then $R(\lambda,\mu)$ is a reversible Yang-Baxter Poisson correspondence and **transfer dynamics is Poisson** with respect to $\Omega^{(N)}$.

Other relations: "box-ball" systems, geometric crystals

Hatayama, Hikami, Inoue, Kuniba, Noumi, Okado, Takagi, Tokihiro, Yamada (2000-): Takahashi-Satsuma "box-ball" systems and Kashiwara's crystal theory

Berenstein, Kazhdan (2000), **Etingof** (2001): geometric crystals

Yang-Baxter map:

$$R : X \times X \to X \times X, \quad X = \mathbf{C}^n$$

$$\tilde{x}_j = x_j \frac{P_j}{P_{j-1}}, \qquad \tilde{y}_j = y_j \frac{P_{j-1}}{P_j}, \qquad j = 1, \ldots, n,$$

where

$$P_j = \sum_{a=1}^{n} \left(\prod_{k=1}^{a-1} x_{j+k} \prod_{k=a+1}^{n} y_{j+k} \right).$$

with the subscripts $j + k$ taken modulo n.

Adler, Bobenko, Suris (2004):

Quadrirational case, $X = \mathbb{C}P^1$

$$u = \alpha y P, \qquad v = \beta x P, \qquad P = \frac{(1-\beta)x + \beta - \alpha + (\alpha - 1)y}{\beta(1-\alpha)x + (\alpha-\beta)yx + \alpha(\beta-1)y}, \tag{1}$$

$$u = \frac{y}{\alpha} P, \qquad v = \frac{x}{\beta} P, \qquad P = \frac{\alpha x - \beta y + \beta - \alpha}{x - y}, \tag{2}$$

$$u = \frac{y}{\alpha} P, \qquad v = \frac{x}{\beta} P, \qquad P = \frac{\alpha x - \beta y}{x - y}, \tag{3}$$

$$u = yP, \qquad v = xP, \qquad P = 1 + \frac{\beta - \alpha}{x - y}, \tag{4}$$

$$u = y + P, \quad v = x + P, \quad P = \frac{\alpha - \beta}{x - y}, \tag{5}$$

Geometric interpretation

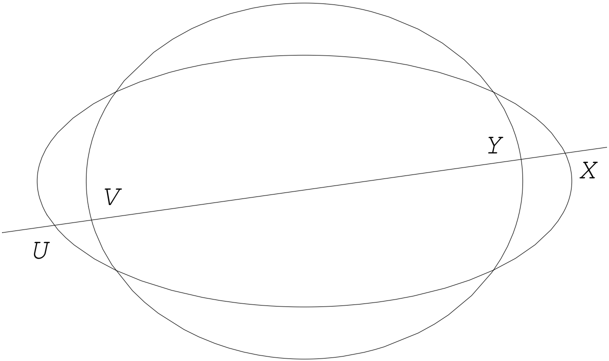
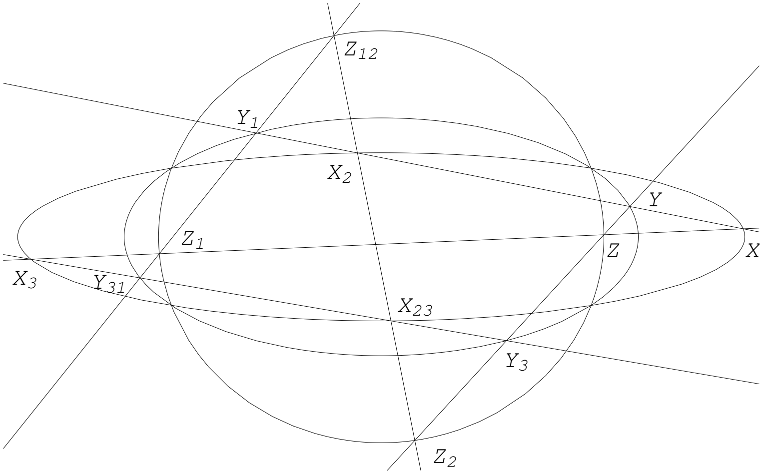
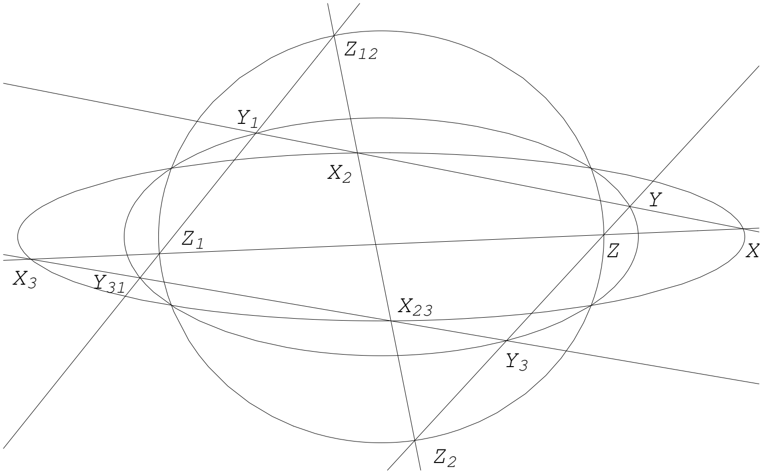


Figure: A quadrirational map on a pair of conics

Yang-Baxter property = Geometric theorem





Konopelchenko, Schief (2001): Menelaus' theorem, Clifford configurations and discrete KP hierarchy.

Papageorgiou, Suris, Tongas, V (2009):

$$u = yQ^{-1}, \quad v = xQ, \qquad Q = \frac{(1-\beta)xy + (\beta-\alpha)y + \beta(\alpha-1)}{(1-\alpha)xy + (\alpha-\beta)x + \alpha(\beta-1)}, \tag{6}$$

$$u = yQ^{-1}, \quad v = xQ, \qquad Q = \frac{\alpha + (\beta-\alpha)y - \beta xy}{\beta + (\alpha-\beta)x - \alpha xy}, \tag{7}$$

$$u = \frac{y}{\alpha}Q, \quad v = \frac{x}{\beta}Q, \qquad Q = \frac{\alpha x + \beta y}{x+y}, \tag{8}$$

$$u = yQ^{-1}, \quad v = xQ, \qquad Q = \frac{\alpha xy + 1}{\beta xy + 1}, \tag{9}$$

$$u = y - P, \quad v = x + P, \qquad P = \frac{\alpha - \beta}{x+y}. \tag{10}$$

The last map is the Adler map.

Some open questions

- Classification

  Adler, Bobenko, Suris (2004), (2009), **PSTV** (2009)

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  Adler, Bobenko, Suris (2004), (2009), **PSTV** (2009)

- Soliton interaction $\Rightarrow$ Integrable hierarchy

  Adler map $\Rightarrow$ KdV hierarchy: **S.P. Novikov** (1974), **Shabat, AV** (1993)

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- Discrete hierarchies and tropicalization

  Kakei, Nimmo, Willox (2008)
  Inoue, Takenawa (2008): tropical algebraic geometry