11th UK-Japan Winter School

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Lecture by A. Hone – Darboux transformations, Bäcklund transformations & symplectic correspondences

Darboux-Bäcklund transformations

Integrable PDEs ⟷ Integrable P∆Es [Veselov, lecture 1]

Symmetry reductions [Mikhailov, Lecture 1]

BTs [Kuznetsov-Sklyanin 1998]

Integrable finite-dimensional Hamiltonian systems ⟷ Integrable symplectic maps/correspondences [Veselov 1991]

Liouville-Arnold definition of integrability: A finite-dimensional Hamiltonian system with N degrees of freedom is completely integrable if it has N independent first integrals $H_1, \ldots, H_N$ in involution, i.e. $\{H_j, H_k\} = 0 \ \forall j, k$. The same definition applies to symplectic maps/correspondences [Veselov 1991].

Aim: Describe the left hand side, and the bottom part, of the above diagram, for the explicit example of the stationary flow of KdV.

KdV: $u_t = u_{xxx} + 6uu_x$

$$\Big\downarrow \quad \text{stationary flow} \quad \left(\frac{\partial u}{\partial t} = 0, \ \frac{\partial}{\partial x} \rightsquigarrow \frac{d}{dx} \right)$$

$$\frac{d}{dx}(u_{xx} + 3u^2) = 0 \quad : \text{3rd order ODE}$$

Let $q \equiv u$, $p \equiv u_x$, $\dot{} \equiv \frac{d}{dx}$.

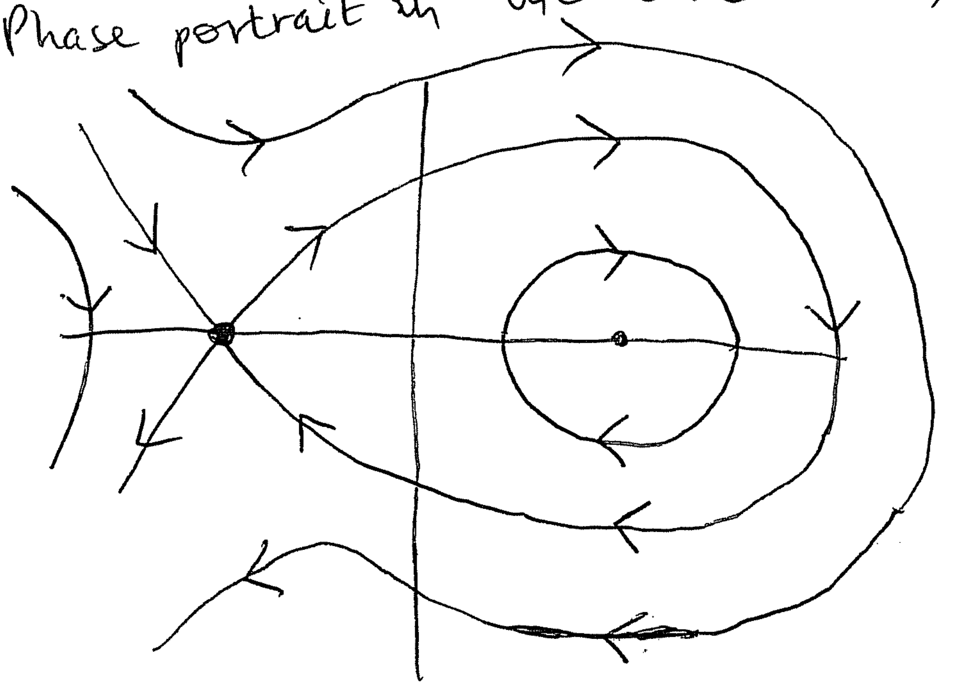
$u_{xx} + 3u^2 = 3k^2 = \text{constant} \Rightarrow \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ -3q^2 + 3k^2 \end{pmatrix}$

ODE system for q, p is in Hamiltonian form:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial q} \\ \frac{\partial h}{\partial p} \end{pmatrix}, \quad h = \tfrac{1}{2}p^2 + q^3 - 3k^2 q.$$

Phase space has canonical symplectic form $\omega = dp \wedge dq$

Phase portrait in the case $k \in \mathbb{R}$, $k^2 > 0$:



Fixed points at $(\pm k, 0)$. Generic level curves $h = \text{const.}$ have genus one (elliptic curves in (q,p) plane, hence system can be integrated with elliptic functions).

Lax pair for KdV, and its reduction

$$\left.\begin{array}{l} L\psi = \lambda\psi \\ \psi_t = P\psi \end{array}\right\} \text{(Peter Lax)} \Rightarrow L_t + [L,P] = 0 \quad \text{(Lax equation)}$$

if $\lambda_t = 0$ (isospectral).

If $L = D_x^2 + u$ and $P = 4D_x^3 + 6uD_x + 3u_x = 4(L^{3/2})_+$
(cf. Sato theory)

then Lax equation $\Rightarrow$ KdV equation $u_t = u_{xxx} + 6uu_x$.

With this choice of L, P, if ψ satisfies the two linear equations $L\psi = \lambda\psi$, $\psi_t = P\psi$ then $\psi_t = (4\lambda + 2u)\psi_x - u_x\psi$

$$\Rightarrow \frac{\partial}{\partial x}\begin{pmatrix}\psi \\ \psi_x\end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}}_{U}\begin{pmatrix}\psi \\ \psi_x\end{pmatrix},$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi \\ \psi_x\end{pmatrix} = \underbrace{\begin{pmatrix} -A_x & 2A \\ -A_{xx} + 2(\lambda - u)A & A_x \end{pmatrix}}_{V}\begin{pmatrix}\psi \\ \psi_x\end{pmatrix}$$

Compatibility of matrix linear system
$\Rightarrow$ zero curvature equation $[\partial_x - U, \partial_t - V] = 0$
i.e. $U_t - V_x + [U, V] = 0$.

For KdV, take $A = 2\lambda + u$, to specify entries of V.
(For local symmetries of KdV, weight $2n+1$, take A = nth degree polynomial in λ; nonlocal symmetries arise for A = rational function of λ.)

Reduction to stationary flows: set $\psi(x,t;\lambda) = e^{\mu t}\tilde{\psi}(x;\lambda)$ and let $\Psi = \begin{pmatrix} \tilde{\psi} \\ \dot{\tilde{\psi}} \end{pmatrix}$ with $\bullet \equiv \frac{d}{dx}$ as above.

Matrix linear system reduces to $\begin{cases} \mathcal{L}\Psi = \mu\Psi, \\ \dot{\Psi} = \mathcal{P}\Psi, \end{cases}$

where $\mathcal{L} = \mathcal{L}(\lambda) \equiv \mathcal{L}(q,p;\lambda) = \begin{pmatrix} -p & 4\lambda+2q \\ 4\lambda^2-2q\lambda+q^2-3k^2 & p \end{pmatrix}$

and $\mathcal{P} = \begin{pmatrix} 0 & 1 \\ \lambda-q & 0 \end{pmatrix}$.

Lax equation $\dot{\mathcal{L}} + [\mathcal{L}, \mathcal{P}] = 0 \Rightarrow$ Hamilton's equations for p, q.

Isospectral evolution of $\mathcal{L} \Rightarrow$ Spectral curve $\det(\mathcal{L}(\lambda) - \mu\mathbb{1}) = 0$ is invariant.

Here $\det(\mathcal{L}(\lambda)-\mu\mathbb{1}) = \mu^2 - 16\lambda^3 + 12k^2\lambda - 2h = 0$ defines an elliptic curve in (λ,μ) plane for fixed level $h = $ const.

(Stationary flows of nth KdV symmetry give hyperelliptic curves $\mu^2 = $ (Poly. in λ, degree $2n+1$) of genus n, cf. [Mumford 1984].)

Darboux transformation, and reduction to BT

$$\text{DT} \quad \begin{cases} \psi_{xx} + u\psi = \lambda\psi \\ \downarrow \\ \tilde{\psi}_{xx} + \tilde{u}\tilde{\psi} = \lambda\tilde{\psi} \end{cases}$$

[Darboux 1882]*

Covariance of 2nd order linear equation for ψ achieved by picking solution ϕ of same equation,

$$\phi_{xx} + u\phi = \alpha\phi, \quad \text{for } \alpha \text{ fixed level,}$$

and take
$$\begin{cases} \tilde{u} = u + 2(\log\phi)_{xx} \\ \tilde{\psi} = K\dfrac{(\psi_x\phi - \psi\phi_x)}{\phi}, \quad K \text{ arbitrary const.} \end{cases}$$

Can write transformation for $\begin{pmatrix}\psi \\ \psi_x\end{pmatrix}$ as

$$\begin{pmatrix}\tilde{\psi} \\ \tilde{\psi}_x\end{pmatrix} = K \underbrace{\begin{pmatrix} -y & 1 \\ y^2 + \lambda - \alpha & -y \end{pmatrix}}_{\text{Darboux matrix [Veselov, Lecture 2]}} \begin{pmatrix}\psi \\ \psi_x\end{pmatrix}, \qquad y = (\log\phi)_x .$$

$$\underline{M} = \underline{M}(\lambda)$$

Restrict to stationary flow $\Rightarrow$ $\tilde{\underline{\Psi}} = \underline{M}\,\underline{\Psi}$

$\Rightarrow$ Discrete Lax equation $\quad \tilde{\mathcal{L}}(\lambda)\,\underline{M}(\lambda) = \underline{M}(\lambda)\,\mathcal{L}(\lambda)$ — (*)

N.B. $\mathcal{L}(\lambda) \equiv \mathcal{L}(q, p; \lambda)$, $\tilde{\mathcal{L}}(\lambda) \equiv \mathcal{L}(\tilde{q}, \tilde{p}; \lambda)$.

(*) For original French, and translations into English, Romanian and Spanish, see arXiv: physics/9908003

$$DT \Rightarrow \begin{cases} u - \alpha = -y_x - y^2 \\ \tilde{u} - \alpha = y_x - y^2 \end{cases}$$ (cf. [Veselov, Lecture 1]).

At the level of the stationary flow, this implies

$$\tilde{q} + q = 2\alpha - 2y^2, \quad \text{— } ①$$

$$\tilde{p} + p = -2y(\tilde{q} - q). \quad \text{— } ②$$

What is y? From d Lax equation $(*)$, reading off coefficients, also recover ① & ②.

Key observation: $\begin{pmatrix} 1 \\ y \end{pmatrix} \in \ker M(\alpha)$

$$\Rightarrow \mathcal{L}(\alpha)\begin{pmatrix} 1 \\ y \end{pmatrix} = \beta \begin{pmatrix} 1 \\ y \end{pmatrix}, \quad \text{eigenvalue } \beta$$

and $\det(\mathcal{L}(\alpha) - \beta \mathbb{1}) = 0$

$$\Rightarrow y = y(q,p) = \frac{\mathring{A}(\alpha) + \beta}{2A(\alpha)} = \frac{p + \beta}{4\alpha + 2q}$$

where (α, β) is a pt. on spectral curve.

$$\Rightarrow \beta = \beta(q,p) = \pm\sqrt{16\alpha^3 - 12k^2\alpha + 2h(q,p)}$$

is uniquely determined up to $\pm$.

$\therefore$ Equations ①, ② define a 2-valued correspondence $\begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix}$.

This is symplectic with generating function

$$F(q, \tilde{q}) = \tfrac{6}{5} y^5 - 4\alpha y^3 + 6(\alpha^2 - k^2) y + \tfrac{1}{2}(\tilde{q} - q)^2 y.$$

N.B. $dF = p\,dq - \tilde{p}\,d\tilde{q}$

and $d^2F = 0 \Rightarrow dp \wedge dq = d\tilde{p} \wedge d\tilde{q}$ as required.

In the formula for F, interpret y as $y(q,\tilde{q})$ determined by (1) : $y = \pm\sqrt{\alpha - \tfrac{1}{2}(q+\tilde{q})}$.

Remarks: After fixing h from the initial conditions (q_0, p_0), and fixing the choice of sign for $\beta = \beta(q_0, p_0)$, the 2-valued symplectic correspondence (1), (2) becomes a rational map restricted to each level $h = h(q_0, p_0) = \text{const}$. Identifying $(\tilde{q}, \tilde{p})$ with (q, p) shifted by time step ε, the map discretizes the Hamiltonian system $\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \cdots$, which arises in the continuum limit $\varepsilon \to 0$ with $\alpha \sim \frac{1}{\varepsilon^2}$.

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