The shape of water: metamorphosis and infinite-dimensional geometric mechanics

Darryl D. Holm

Imperial College London

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Abstract

Whenever we say the words "fluid flows" or "shape changes" we enter the realm of infinite-dimensional geometric mechanics. Water, for example, flows. In fact, Euler's equations tell us that water flows a particular way. Namely, it flows to get out of its own way as adroitly as possible.

The shape of water changes by smooth invertible maps called diffeos (short for diffeomorphisms). The flow responsible for this optimal change of shape follows the path of shortest length, the geodesic, defined by the metric of kinetic energy.

Not just the flow of water, but the optimal morphing of any shape into another follows one of these optimal paths. These lectures will be about the commonalities between fluid dynamics and shape changes and will be discussed in the language most suited to fundamental understanding – the language of geometric mechanics.

A common theme will be the use of momentum maps and geometric control for steering along the optimal paths using emergent singular solutions of the initial value problem for a nonlinear partial differential equation called EPDiff, that governs metamorphosis along the geodesic flow of the diffeos.

Background and motivation: Emergent Singularities

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Many continuum models in physics possess singular delta-like solutions:

- *Euler/Navier-Stokes vortices*,
  (e.g., vortex sheets and filaments in 3D; point vortices in 2D)

- *Magnetic vortex lines in MHD plasmas*,
  (e.g., in sunspots)

- *Collisionless Vlasov-Poisson kinetic theory*,
  (e.g., in self-gravitating clusters of stars and galaxies in nebulae)

- *Certain integrable Hamiltonian continuum systems (shallow water)*

We are interested in singular solutions

that emerge from smooth initial conditions.

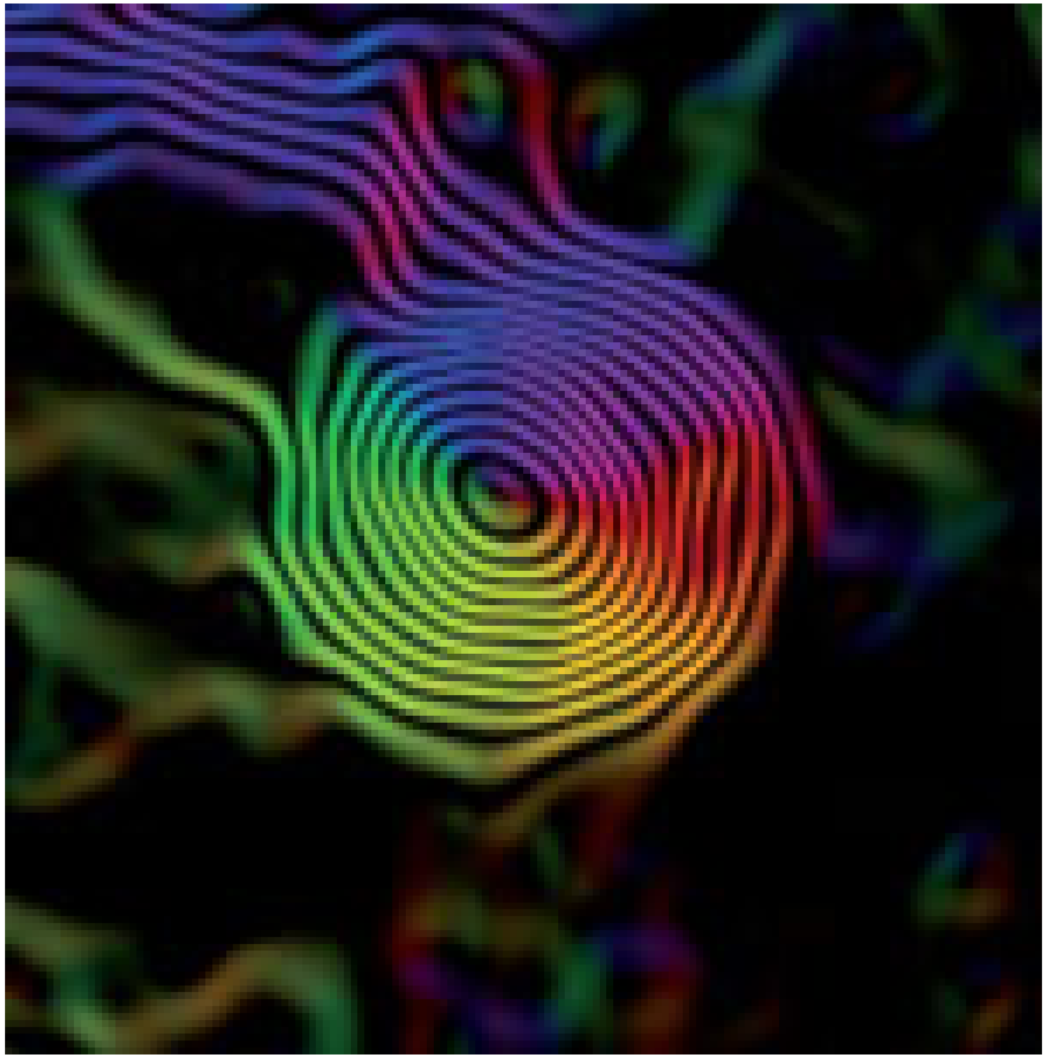

Figure 1: This image shows the magnetic field lines in Fe-Ni (iron-nickel) nanoparticles using electronic holography. The concentric circles of field lines correspond to the magnetic vortex formed in the central particle of 70 nm (1) in diameter. The colours indicate the local orientation of the field. The total width of the image is 140 nm. (Centre d'Etudes de Chimie Métallurgique and University of Cambridge).

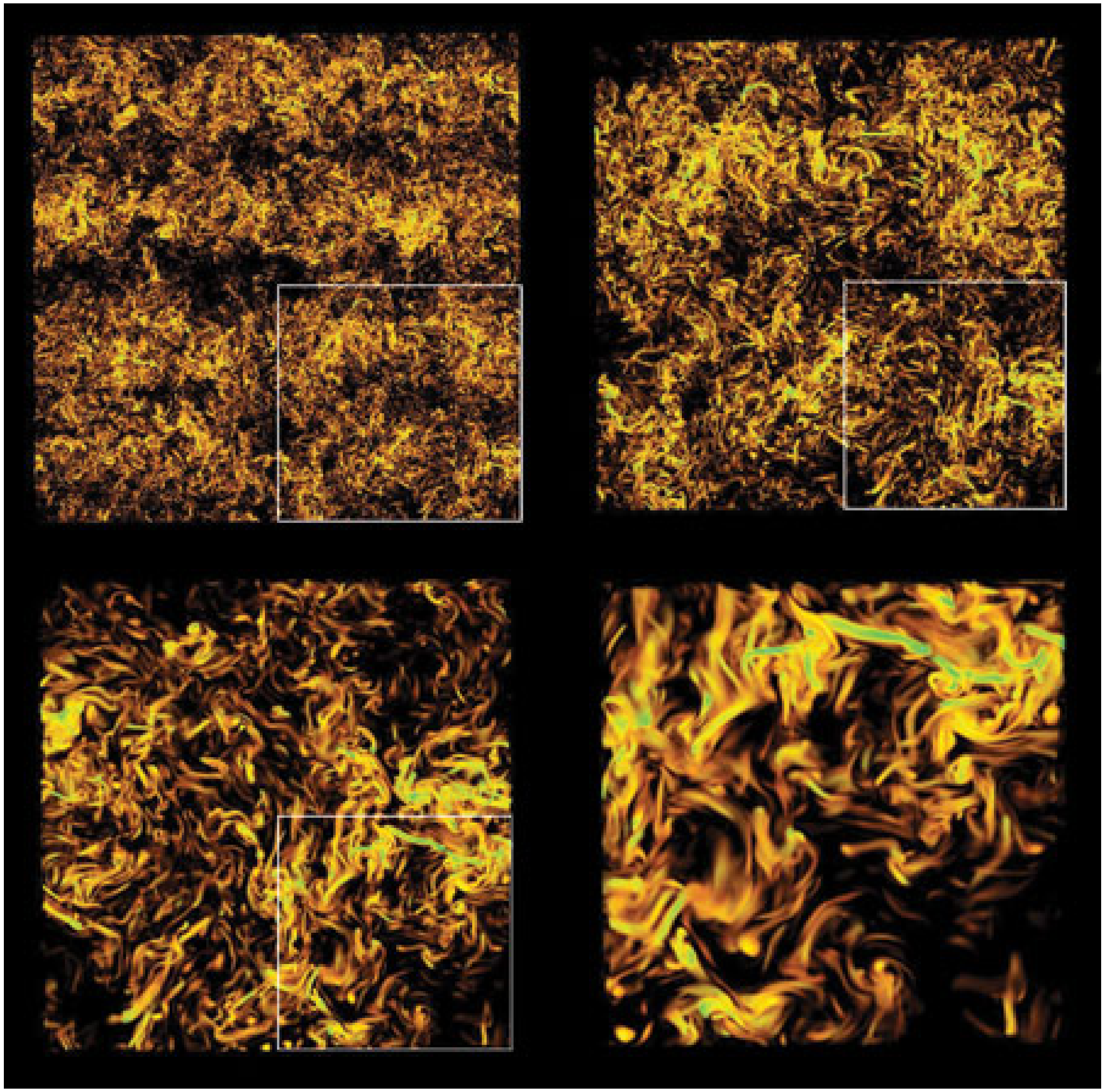

Figure 2: This image zooms 64X to display vortices in numerical simulations of turbulence. – P. Minnini, NCAR

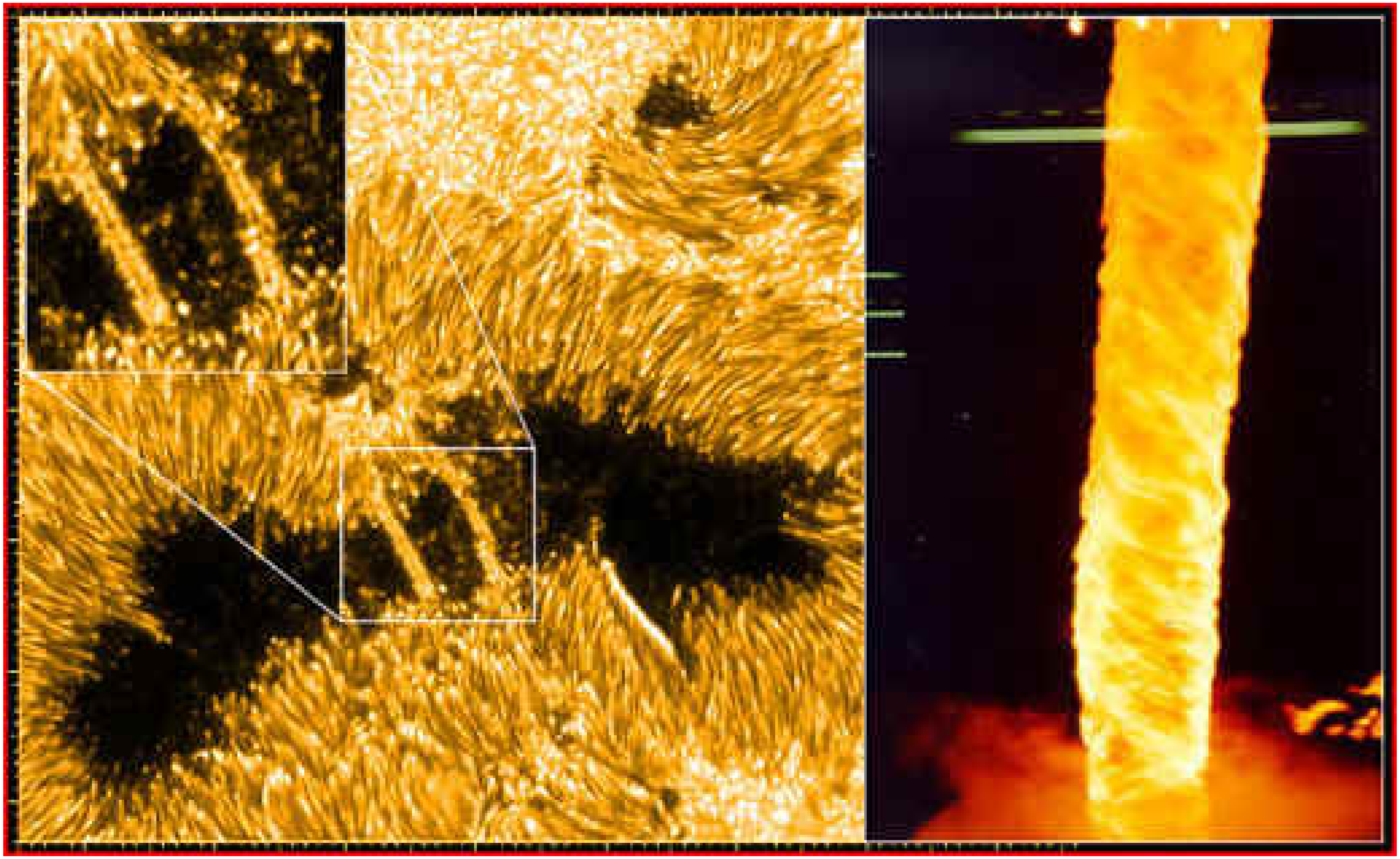

Figure 3: This image shows light emission from particles in the sun following magnetic field lines in the vicinity of a sunspot.

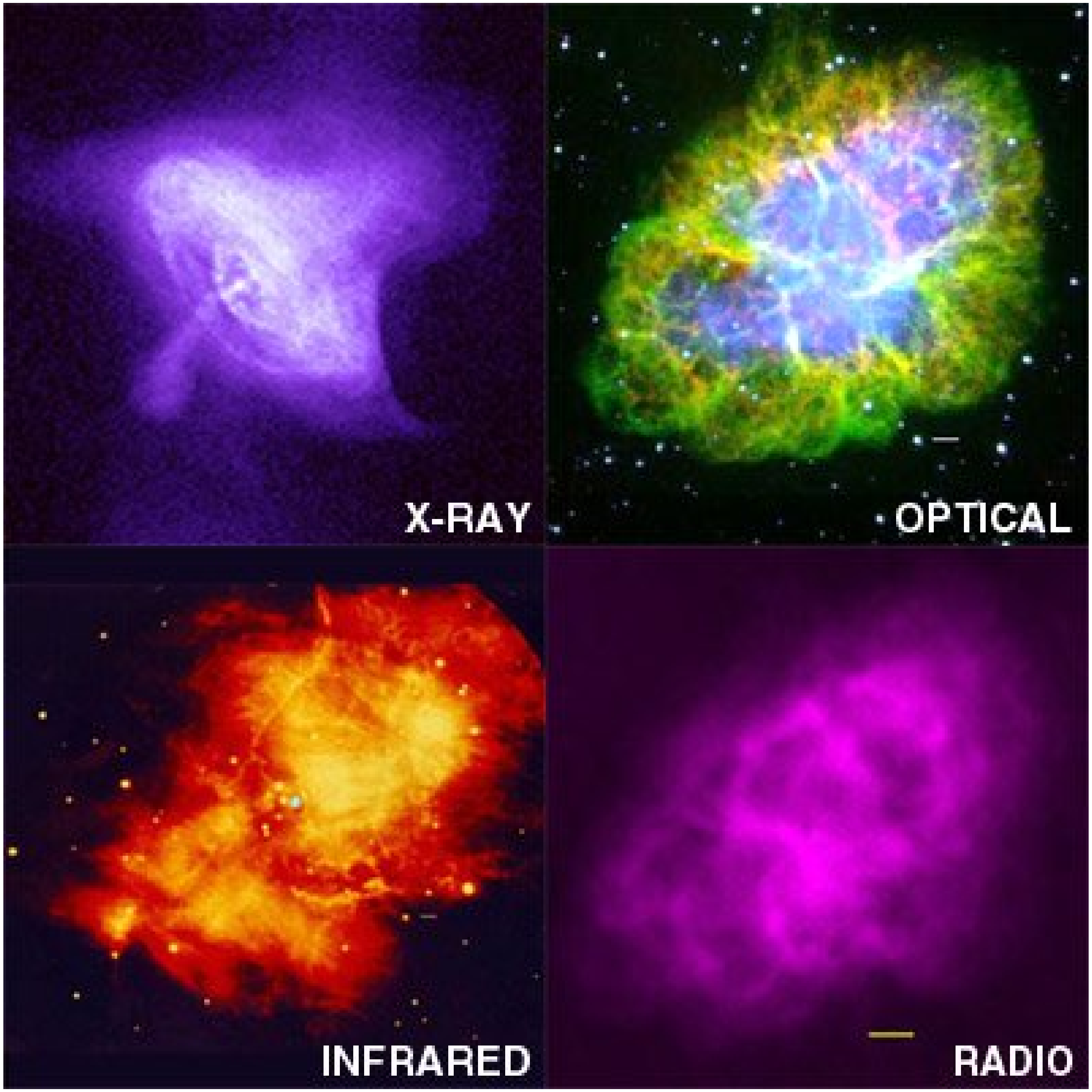

Figure 4: Four different images of the crab Nebula.

Figure 5: This image shows a trailing vortex from an aircraft wing, visualised by red smoke.

Emergent singularities: Camassa-Holm and EPDiff equations

- The CH equation is a 1+1 **integrable PDE** for shallow water dynamics
- Its soliton solutions (*peakons*) **emerge spontaneously** from any initially confined velocity distribution.

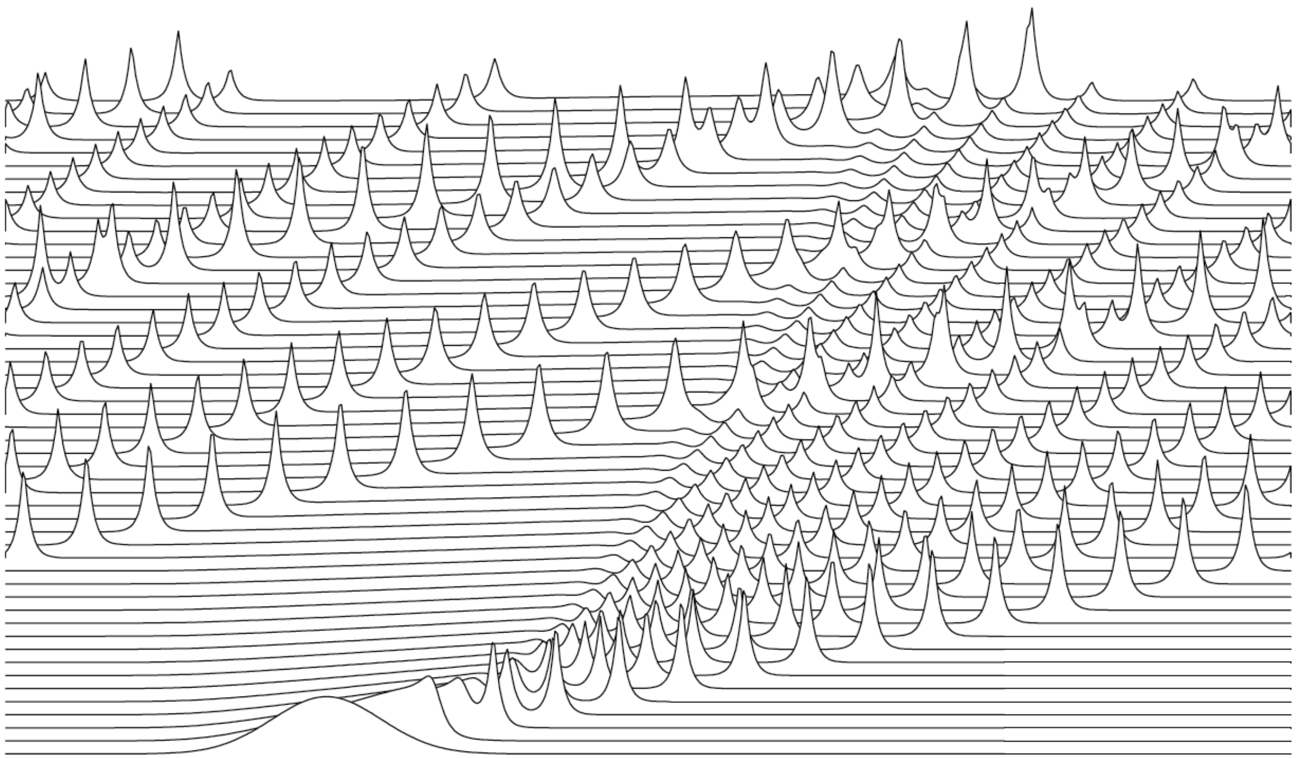

Figure 6: CH equation: singular solutions emerge from an initial Gaussian velocity.

- Extending to any dimension yields the **EPDiff** equation, a general **geodesic flow** on the diffeomorphism group
- EPDiff (Euler-Poincaré equation on the diffeomorphisms) also exhibits **emergent singular solutions** in 2D and 3D

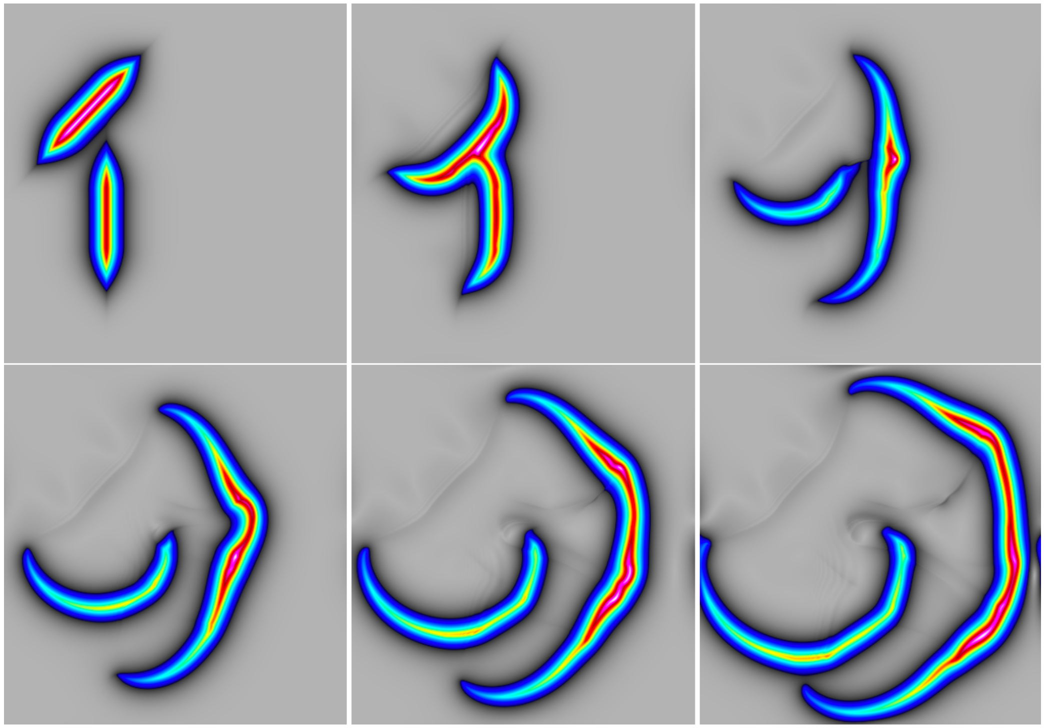

Figure 7: CH equation: singular solutions collide and exchange momentum in 2D.

- EPDiff has a range of applications, from turbulence to imaging science

Metamorphosis

Metamorphosis is an optimal-control matching problem that seeks geodesics on semidirect-product groups – this means that it fits perfectly into the Euler-Poincaré framework!

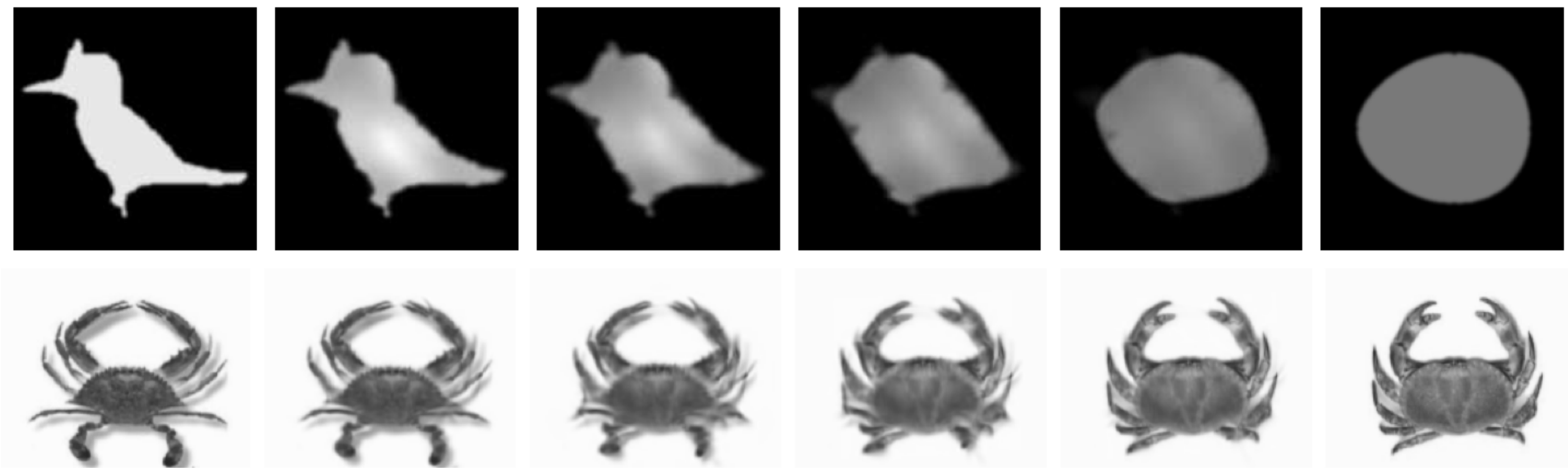

Figure 8: Geodesic flow on Diff$\circledS\mathcal{F}$ governs image morphing. The figure plots the morphing process in the (conjugate) $\mathcal{F}$-variable as a function of time. (Figure from DD Holm, A Trouvé and L.Younes, *Quart Appl Math*, 2009.)

D'Arcy Thompson's transformative approach (1917)

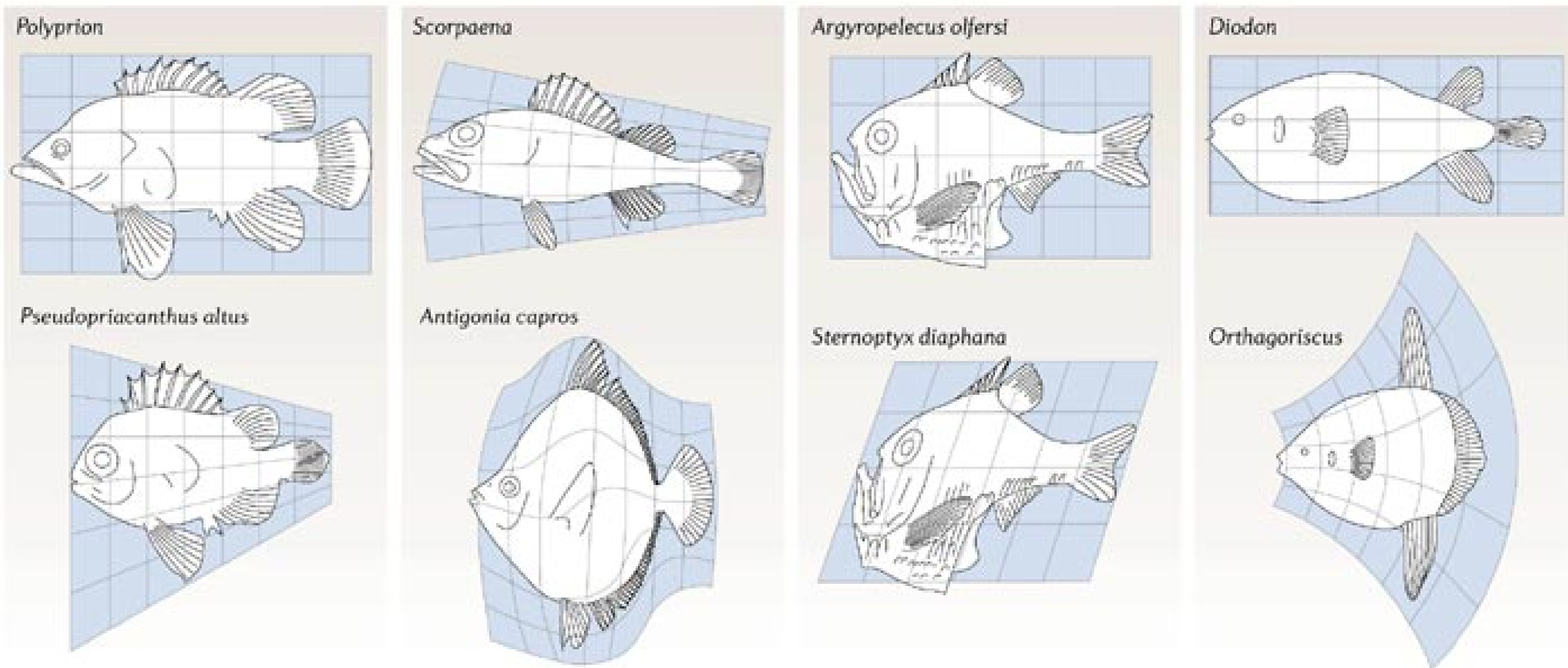

Figure 9: Thompson's illustration of the transformations of shapes of various species of fish (top row) into others (bottom row). From Wallace Arthur, D'Arcy Thompson and the theory of transformations. *Nature Reviews Genetics* **7**, 401-406 (May 2006).

Contents

Lecture #1, EPDiff

1 Brief history of ideal continuum motion

This lecture explains how the Euler equations of ideal incompressible fluid motion may be recognized as Euler–Poincaré equations EPDiff_{Vol} defined on the dual of the tangent space at the identity $T_eG = T_e\,\text{Diff}_{Vol}(\mathcal{D})$ of the **divergence-free right invariant vector fields** $\mathfrak{X}(\mathcal{D})$ over the domain $\mathcal{D}$. Then it develops these ideas to apply to EPDiff, the Euler–Poincaré equations on the full diffeomorphism group.

Arnold [Arn66] applied the Lagrangian and Hamiltonian theories of geometric mechanics to rederive the Euler fluid equations for incompressible motion of an ideal fluid, with:

- **configuration space** $Q = G = \text{Diff}_{Vol}(\mathcal{D})$, the volume preserving **diffeomorphisms** (smooth invertible maps with smooth inverses) of the region $\mathcal{D}$ occupied by the fluid.

- The **tangent vectors** for the maps, $TG = T\,\text{Diff}_{Vol}(\mathcal{D})$, represent the space of **fluid velocities**, which must satisfy appropriate physical conditions at the boundary of the region $\mathcal{D}$.

- **Group multiplication** in $G = \text{Diff}_{Vol}(\mathcal{D})$ is composition of the smooth invertible volume-preserving maps.

Arnold's geometric approach for incompressible ideal fluid motion has been extended and applied to many other cases of **ideal continuum motion**. See, e.g., [HMRW85, HMR98] for references, discussion and progress in these more general applications.

Definition 1 (EPDiff)
EPDiff *is the family of equations governing* **geodesic motion** *on the full diffeomorphism group with respect to whatever right invariant metric is chosen on the tangent space of the diffeomorphisms.*

Remark 1
Arnold's result [Ar1966] that the Euler fluid equations describe geodesic motion on the volume-preserving diffeomorphisms may be understood by interpreting these equations as EPDiff$_{Vol}$ *with respect to the right invariant* L^2 *metric of the Eulerian fluid velocity supplied by the fluid's kinetic energy.*

If **potential energy** *due to advected quantities is present in an ideal continuum flow, then reduction by right invariance of Hamilton's principle under the diffeomorphisms produces the* **EP theorem with advected quantities** *[HMR98].*

2 Geometric setting of ideal continuum motion

Definition 2 *Points in a domain $\mathcal{D}$ represent the positions of material particles of the system in its* **reference configuration**. *These points are denoted by $X \in \mathbb{R}^n$ and called the* **particle labels**.

- *A* **configuration**, *which we typically denote by g, is an element of* Diff$(\mathcal{D})$, *the space of diffeomorphisms from $\mathcal{D}$ to itself.*

- *A* **fluid motion**, *denoted as g_t or alternatively as $g(t)$, is a time-dependent curve in* Diff$(\mathcal{D})$, *providing an evolutionary sequence of diffeomorphism from the reference configuration to the current configuration in $\mathcal{D}$.*

The configuration space Diff$(\mathcal{D})$ is a group, with the group operation being composition and the group identity being the identity map. This group acts on $\mathcal{D}$ in the obvious way: $g \cdot X := g(X)$, where we are using the 'dot' notation $(\,\cdot\,)$ for the group action.

Definition 3 *During a motion* g_t *or* $g(t)$*, the particle labelled by* X *describes a path in* $\mathcal{D}$ *along a locus of points*

$$x(X,t) := g_t(X) = g(t)\cdot X\,, \tag{1}$$

which are called the **Eulerian** *or* **spatial points** *of the path. This locus of points in* $\mathbb{R}^n$ *is also called the* **Lagrangian***, or* **material***,* **trajectory***, because a Lagrangian fluid parcel follows this path in space.*

Definition 4 *The* **Lagrangian***, or* **material***,* **velocity** U *of the system along the motion* g_t *or* $g(t)$ *is defined by taking the time derivative of the Lagrangian trajectory (1) keeping the particle labels* X *fixed:*

$$U(X,t) := \frac{\partial}{\partial t} g_t \cdot X = \frac{\partial}{\partial t} x(X,t)\,. \tag{2}$$

Thus $U(X,t)$ *is the velocity of the particle with label* X *at time* t*.*

The **Eulerian***, or* **spatial***,* **velocity** u *of the system is velocity expressed as a function of spatial position and time, meaning that if* $x = x(X,t) = g_t(X)$

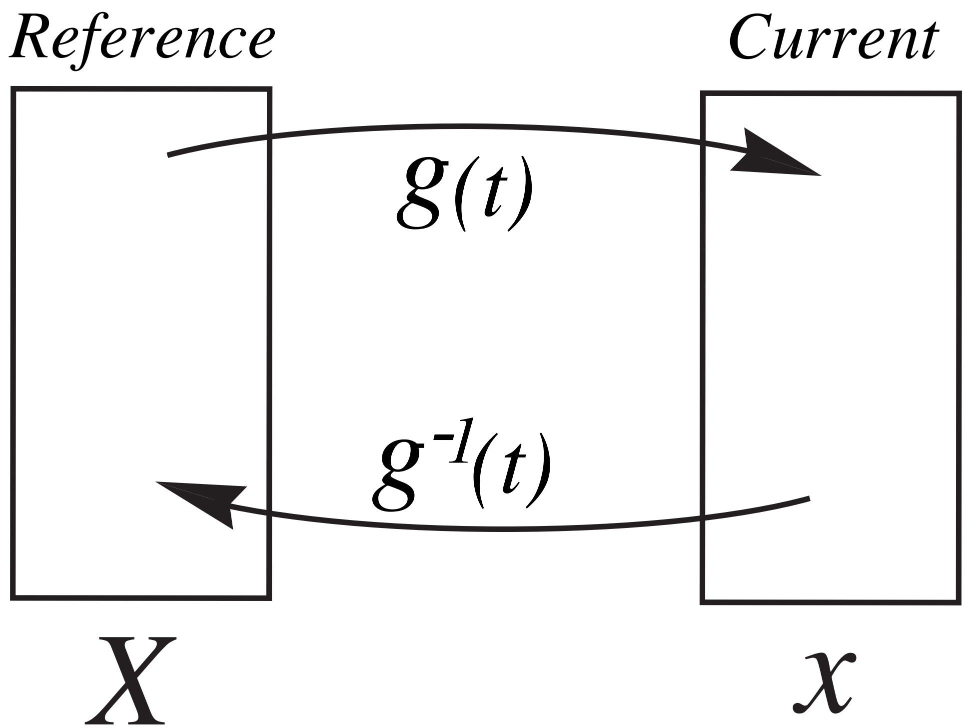

Figure 10: The map from Lagrange reference coordinates X in the fluid to the current Eulerian spatial position x is performed by the time-dependent diffeomorphism $g(t)$, so that $x(t, X) = g(t) \cdot X$.

then

$$u(x, t) := U(X, t) = U(g_t^{-1}(x), t)\,. \tag{3}$$

Thus, $u(x, t)$ *is velocity at time* t *of the particle currently in position* x*.*

Remark 2 (Time-dependent vector fields) *The Eulerian velocity* u *can also be regarded as a time-dependent vector field* $u_t \in \mathfrak{X}(\mathcal{D})$*, where* $u_t(x) := u(x,t)$*. Similarly, we write* $U_t(X) := U(X,t)$*, though this is not really a vector field since* $U_t(X)$ *is a vector based at* $x = x(X,t)$ *rather than* X*. It follows from eqn (3) that*

$$U_t = u_t \circ g_t \,. \tag{4}$$

In this sense, the Lagrangian velocity field at a particular time is a **right translation** *of the Eulerian velocity field. This observation leads to consideration of the Lie-group structure of* Diff$(\mathcal{D})$*.*

Definition 5 *Given a path* $g(t)$ *in* Diff$(\mathcal{D})$*, the corresponding* **Lagrangian velocity fields** U_t *are also denoted* $\dot{g}(t)$ *or* $\frac{\partial}{\partial t} g(t)$*. We use the 'dot' notation,*

$$\dot{g}(t)\cdot X := \dot{g}(t)(X) = U_t(X) = \frac{\partial}{\partial t} g_t \cdot X.$$

For a given t*, the velocity field* $\dot{g}(t)$ *is called a* **tangent vector** *to* Diff$(\mathcal{D})$ *at* $g(t)$*.*

Definition 6 *The* **tangent space** *of* $\mathrm{Diff}(\mathcal{D})$ *at* g*, denoted* $T_g\,\mathrm{Diff}(\mathcal{D})$*, is the set of all tangent vectors to* $\mathrm{Diff}(\mathcal{D})$ *at* g*, i.e. all possible Lagrangian velocity fields* U_t *(for a fixed* t*) such that* $U_t(X) \in T_{g(X)}\mathcal{D}$ *for all* X*. The union of all of these tangent spaces is the* **tangent bundle** $T\,\mathrm{Diff}(\mathcal{D})$*.*

Remark 3 *Any smooth vector field on* $\mathcal{D}$ *can be expressed as* $\dot{g}(0)$ *for* $g(t)$ *equal to the flow of the vector field, so*

$$T_e\,\mathrm{Diff}(\mathcal{D}) = \mathfrak{X}(\mathcal{D}),$$

where $\mathfrak{X}(\mathcal{D})$ *is the set of smooth vector fields on* $\mathcal{D}$*. In general, we have*

$$U_t = u_t \circ g_t\,,$$

and u_t *(for a fixed* t*) is a vector field, so general* **tangent vectors** *are right translations of vector fields, and*

$$\begin{aligned} T_g\,\mathrm{Diff}(\mathcal{D}) &= \{u \circ g : u \in \mathfrak{X}(\mathcal{D})\} \\ &= \{\text{smooth } U : \mathcal{D} \to T\mathcal{D} \mid U(X) \in T_{g(X)}\mathcal{D} \text{ for all } X\}. \end{aligned}$$

Remark 4 (Tangent lift of right translation is Eulerian velocity)
Let $\varphi \in \mathrm{Diff}(\mathcal{D})$ *and let* R_φ *be the right translation map* $g \mapsto g \circ \varphi$*. The* **tangent lift** *of* R_φ *is the map* $TR_\varphi : T\,\mathrm{Diff}(\mathcal{D}) \to T\,\mathrm{Diff}(\mathcal{D})$ *defined as follows: Let* $U = \dot{g}(t_0)$*. Then*

$$TR_\varphi(U) = TR_\varphi\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0} g_t\right) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}(g_t \circ \varphi) = U \circ \varphi\,,$$

since for all $X \in \mathcal{D}$*,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}(g_t \circ \varphi)(X) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}(g_t \circ \varphi(X)) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0} g_t\right) \cdot \varphi(X)) = U \circ \varphi(X).$$

We use the notation $U\varphi = TR_\varphi(U)$*. With this notation, the* **Eulerian velocity** *corresponding to a flow* $g(t)$ *is*

$$u_t = \dot{g}(t)g^{-1}(t)\,.$$

The **Lie algebra** of $\mathrm{Diff}(\mathcal{D})$ is $\mathfrak{X}(\mathcal{D})$ with the **Lie bracket** defined by

$$[u, v]_L := [X_u^L, X_v^L](e) = \mathrm{ad}_u\, v, \text{ for all } u, v \in \mathfrak{X}(\mathcal{D}).$$

Remark 5 (A matter of signs) *Let $\Phi_u(t)$ and $\Phi_v(t)$ be the flows of vector fields u and v, respectively. The adjoint action of* $\mathrm{Diff}(\mathcal{D})$ *on* $\mathfrak{X}(\mathcal{D})$ *is*

$$\mathrm{Ad}_g v = \frac{d}{dt}\Big|_{t=0} g \circ \Phi_v(t) \circ g^{-1} = TL_g \circ v \circ g^{-1} = g_* v \,,$$

the **push-forward** *of v by g. It follows that the adjoint action of $\mathfrak{X}(\mathcal{D})$ on itself is*

$$\mathrm{ad}_u v = \frac{d}{dt}\Big|_{t=0} (\Phi_u(t))_* v = -\frac{d}{dt}\Big|_{t=0} (\Phi_u(t))^* v = -\mathcal{L}_u\, v \;= -[u, v], \tag{5}$$

where the bracket on the right is the standard **Jacobi–Lie bracket** *of the vector fields. In components (summing on repeated indices),*

$$\begin{aligned} -(\mathrm{ad}_u v)^i &= [u, v]^i = u^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial u^i}{\partial x^j}\,, \\ \text{or} \quad -\mathrm{ad}_{\mathbf{u}} \mathbf{v} &= [\mathbf{u}, \mathbf{v}] = \mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{u}\,. \end{aligned} \tag{6}$$

Thus, the Lie bracket on $\mathfrak{X}(\mathcal{D})$, considered as the Lie algebra of $\mathrm{Diff}(\mathcal{D})$*, is* **minus** *the standard Jacobi-Lie bracket.*

3 Euler–Poincaré reduction for continua

Euler–Poincaré reduction starts with a G-invariant Lagrangian $L : TG \to \mathbb{R}$ defined on the tangent bundle of a Lie group G.

Definition 7 *A Lagrangian $L : TG \to \mathbb{R}$ is said to be right G-invariant if $L(TR_h(v)) = L(v)$, for all $v \in T_gG$ and for all $g, h \in G$. In shorter notation,* **right invariance** *of the Lagrangian may be written as*

$$L(g(t)h, \dot{g}(t)h) = L(g(t), \dot{g}(t)) \,,$$

for all $h \in G$.

Remark 6 *For a G-invariant Lagrangian defined on TG, reduction by symmetry takes Hamilton's principle from TG to $TG/G \simeq \mathfrak{g}$. Stationarity of the symmetry-reduced Hamilton's principle yields the Euler–Poincaré equations on $\mathfrak{g}^*$. As we shall discuss later, the corresponding reduced Legendre transformation yields the now-standard* **Lie–Poisson bracket** *for the Hamiltonian formulation of these equations.*

Theorem 1 (Euler–Poincaré reduction)
Let G be a Lie group and $L : TG \to \mathbb{R}$ be a **right invariant Lagrangian**. *Let $\ell := L|_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{R}$ be its restriction to $\mathfrak{g}$. For a curve $g(t) \in G$, let*

$$u(t) = \dot{g}(t) \cdot g(t)^{-1} := T_{g(t)} R_{g(t)^{-1}} \dot{g}(t) \in \mathfrak{g}\,,$$

so $L(g(t), \dot{g}(t)) = L(e, \dot{g}(t) g(t)^{-1}) =: \ell(u)$.

Then the following four statements are equivalent:

(i) *$g(t)$ satisfies the* **Euler–Lagrange equations** *for Lagrangian L defined on G.*

(ii) *The variational principle*

$$\delta \int_a^b L(g(t), \dot{g}(t)) \mathrm{d}t = 0\,, \tag{7}$$

holds, for variations with fixed endpoints.

(iii) *The (right invariant)* **Euler–Poincaré equations** *hold:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \ell}{\delta u} = -\,\mathrm{ad}^*_u \frac{\delta \ell}{\delta u}\,. \tag{8}$$

(iv) *The variational principle*

$$\delta \int_a^b \ell(u(t))\,\mathrm{d}t = 0\,, \tag{9}$$

holds on $\mathfrak{g}$*, using variations of the form*

$$\delta u = \dot{v} + [u, v]\,, \tag{10}$$

where $u(t)$ *is an arbitrary path in* $\mathfrak{g}$ *that vanishes at the endpoints, i.e.* $u(a) = u(b) = 0$*.*

Remark 7 (Avoiding analytical technicalities) *We identify the Lie group* G *with the smooth invertible maps whose inverses are also smooth; that is, we identify* G *with Diff*$(\mathcal{D})$ *the group of* **diffeomorphisms** *acting on the domain* $\mathcal{D}$. *The corresponding Lie algebra will be the algebra of smooth vector fields* $\mathfrak{X}(\mathcal{D})$ *endowed with the ad-operation given by (minus) the Jacobi–Lie bracket.*

We will forego any analytical technicalities that may arise in making this identification.

The interested reader may consult Ebin and Marsden [EM70] for an approach to the analytical issues that arise in the volume-preserving case. The corresponding issues for the full diffeomorphism group remain an active field of current research.

4 EPDiff: Euler–Poincaré for diffeomorphisms

4.1 The n-dimensional EPDiff equation

Eulerian geodesic motion of a fluid in n dimensions is generated as an EP equation via Hamilton's principle, when the Lagrangian is given by the kinetic energy. The kinetic energy defines a norm $\|\mathbf{u}\|^2$ for the Eulerian fluid velocity, represented by the contravariant vector function $\mathbf{u}(\mathbf{x},t)$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. The choice of the kinetic energy as a positive functional of fluid velocity $\mathbf{u}$ is a modelling step that depends upon the physics of the problem being studied. We shall choose the kinetic-energy Lagrangian,

$$\ell = L_{\mathfrak{g}} = \frac{1}{2}\|u\|^2_{Q_{op}} = \frac{1}{2}\int \mathbf{u}\cdot\mathbf{m}\,\mathrm{d}V \quad \text{with} \quad \mathbf{m} := Q_{op}\mathbf{u}\,. \qquad (11)$$

This Lagrangian may also be expressed as the L^2 pairing,

$$\ell = \frac{1}{2}\Big\langle u\,,\,m\Big\rangle = \frac{1}{2}\int \mathbf{u}\cdot Q_{op}\mathbf{u}\,\mathrm{d}V\,, \tag{12}$$

where, in a coordinate basis, the components of the vector field u and the 1-form density m are defined by

$$u = u^j\frac{\partial}{\partial x^j} = \mathbf{u}\cdot\nabla \quad \text{and} \quad m = m_i \mathbf{d}x^i\otimes \mathbf{d}V = \mathbf{m}\cdot\mathbf{dx}\otimes\mathbf{d}V\,.$$

We use the same font for a quantity and its dual. In particular, italic font denotes vector field u and 1-form density m, and bold denotes vector $\mathbf{u}$ and covector $\mathbf{m}$. In eqns (11) and (12), the positive-definite, symmetric operator Q_{op} defines the norm $\|\mathbf{u}\|$, for appropriate (homogeneous, say, or periodic) boundary conditions. Conversely, the spatial velocity vector $\mathbf{u}$ is obtained by convolution of the momentum covector $\mathbf{m}$ with the **Green's function** for the operator Q_{op}. This Green's function G is defined by the vector equation

$$Q_{op}G = \delta(\mathbf{x})\,,$$

in which $\delta(\mathbf{x})$ is the Dirac measure and G satisfies appropriate boundary conditions. Consequently,

$$\mathbf{u}(\mathbf{x}) = (G * \mathbf{m})(\mathbf{x}) = \int G(\mathbf{x}, \mathbf{x}')\mathbf{m}(\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,. \tag{13}$$

For more discussion of Green's functions for linear differential operators, see [Tay96].

Remark 8 *An analogy exists between the kinetic energy in eqn (11) based on the norm $\|u\|_{Q_{op}}$ and the kinetic energy for the rigid body. In this analogy, the spatial velocity vector field u corresponds to body angular velocity, the operator Q_{op} to moment of inertia, and G to its inverse.*

Remark 9 *As defined earlier, the* **EPDiff equation** *is the Euler–Poincaré equation (8) for the Eulerian geodesic motion of a fluid with respect to norm $\|\mathbf{u}\|$. Its explicit form is given in the notation of Hamilton's principle by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \ell}{\delta u} + \mathrm{ad}^*_u \frac{\delta \ell}{\delta u} = 0\,, \quad \textit{in which} \quad \ell[u] = \frac{1}{2}\|\mathbf{u}\|^2\,. \tag{14}$$

Definition 8 *The variational derivative of ℓ is defined by using the L^2* **pairing** *between vector fields and 1-form densities as*

$$\delta\ell[u] = \left\langle \frac{\delta \ell}{\delta u}\,,\, \delta u \right\rangle = \int \frac{\delta \ell}{\delta \mathbf{u}} \cdot \delta\mathbf{u}\, \mathrm{d}V\,. \tag{15}$$

Computing EPDiff: The **variational derivative** with respect to the vector field u is the **one-form density** of momentum given as in eqn (11),

$$\frac{\delta \ell}{\delta u} = \frac{\delta \ell}{\delta \mathbf{u}} \cdot d\mathbf{x} \otimes \mathrm{d}V = m\,, \tag{16}$$

which has vector components given by

$$\frac{\delta \ell}{\delta \mathbf{u}} = Q_{op}\mathbf{u} = \mathbf{m}\,. \tag{17}$$

In addition, ad* is the dual of the vector-field ad-operation (minus the vector-field commutator) with respect to the L^2 pairing,

$$\langle \mathrm{ad}^*_u\, m\,,\, v\rangle = \langle m\,,\, \mathrm{ad}_u v\rangle\,, \tag{18}$$

where u and v are vector fields. The notation $\mathrm{ad}_u\, v$ from eqn (5) denotes the adjoint action of the **right Lie algebra** of $\mathrm{Diff}(\mathcal{D})$ on itself. The pairing in eqn (18) is the L^2 pairing. Hence, upon integration by parts, one finds

$$
\begin{aligned}
\langle \mathrm{ad}^*_u\, m\,,\, v\rangle &= \langle m\,,\, \mathrm{ad}_u v\rangle \\
&= -\int m_i\Big(u^j\frac{\partial v^i}{\partial x^j} - v^j\frac{\partial u^i}{\partial x^j}\Big)\mathrm{d}V \\
&= \int \Big(\frac{\partial}{\partial x^j}\big(m_i u^j\big) + m_j\frac{\partial u^j}{\partial x^j}\Big)v^i\,\mathrm{d}V\,,
\end{aligned}
$$

for homogeneous boundary conditions. In a coordinate basis, the preceding formula for $\mathrm{ad}^*_u\, m$ has the **coordinate expression** in $\mathbb{R}^n$,

$$
\Big(\mathrm{ad}^*_u\, m\Big)_i \mathrm{d}x^i\otimes \mathrm{d}V = \Big(\frac{\partial}{\partial x^j}(m_i u^j) + m_j\frac{\partial u^j}{\partial x^i}\Big)\,\mathrm{d}x^i\otimes\mathrm{d}V\,. \tag{19}
$$

In this notation, the abstract EPDiff equation (14) may be written explicitly in Euclidean coordinates as a partial differential equation for a covector function $\mathbf{m}(\mathbf{x},t):\ R^n\times R^1\to R^n$.

Namely, the **EPDiff equation** is given in Euclidean coordinates as

$$\frac{\partial}{\partial t}\mathbf{m} + \underbrace{\mathbf{u}\cdot\nabla\mathbf{m}}_{\text{Convection}} + \underbrace{(\nabla\mathbf{u})^T\cdot\mathbf{m}}_{\text{Stretching}} + \underbrace{\mathbf{m}(\text{div}\,\mathbf{u})}_{\text{Expansion}} = 0\,. \tag{20}$$

Here, one denotes $(\nabla\mathbf{u})^T\cdot\mathbf{m} = \sum_j m_j \nabla u^j$. To explain the terms in underbraces, we rewrite EPDiff as preservation of the one-form density of momentum along the characteristic curves of the velocity. In vector coordinates, this is

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{m}\cdot\mathrm{d}\mathbf{x}\otimes\mathrm{d}V\Big) = 0 \quad \text{along} \quad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{u} = G*\mathbf{m}\,. \tag{21}$$

This form of the EPDiff equation also emphasizes its non-locality, since the velocity is obtained from the momentum density by convolution against the Green's function G of the operator Q_{op}, as in eqn (13).

One may check that the **characteristic form** of EPDiff in eqn (21) recovers its Eulerian form by computing directly the result that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{m}\cdot \mathrm{d}\mathbf{x}\otimes \mathrm{d}V\Big)\\
&\quad = \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t}\cdot \mathrm{d}\mathbf{x}\otimes \mathrm{d}V + \mathbf{m}\cdot \mathrm{d}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\otimes \mathrm{d}V + \mathbf{m}\cdot \mathrm{d}\mathbf{x}\otimes\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}V\right)\\
&\quad = \left(\frac{\partial}{\partial t}\mathbf{m} + \mathbf{u}\cdot\nabla\mathbf{m} \;+ \nabla\mathbf{u}^T\cdot\mathbf{m} + \mathbf{m}(\mathrm{div}\,\mathbf{u})\right)\cdot \mathrm{d}\mathbf{x}\otimes \mathrm{d}V = 0\,, \qquad (22)
\end{aligned}$$

along

$$\frac{d\mathbf{x}}{\mathrm{d}t} = \mathbf{u} = G*\mathbf{m}\,.$$

This calculation explains the terms convection, stretching and expansion in the under-braces in eqn (20).

One may check that the **characteristic form** of EPDiff in eqn (21) recovers its Eulerian form by computing directly the result that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{m}\cdot\mathbf{dx}\otimes\mathrm{d}V\Big)\\
&\quad=\frac{\mathbf{dm}}{\mathrm{d}t}\cdot\mathbf{dx}\otimes\mathrm{d}V+\mathbf{m}\cdot\mathrm{d}\frac{\mathbf{dx}}{\mathrm{d}t}\otimes\mathrm{d}V+\mathbf{m}\cdot\mathbf{dx}\otimes\Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}V\Big)\\
&\quad=\Big(\frac{\partial}{\partial t}\mathbf{m}+\mathbf{u}\cdot\nabla\mathbf{m}\ +\nabla\mathbf{u}^T\cdot\mathbf{m}+\mathbf{m}(\mathrm{div}\,\mathbf{u})\Big)\cdot\mathbf{dx}\otimes\mathrm{d}V=0\,,
\end{aligned}\tag{22}$$

along

$$\frac{d\mathbf{x}}{\mathrm{d}t}=\mathbf{u}=G*\mathbf{m}\,.$$

This calculation explains the terms convection, stretching and expansion in the under-braces in eqn (20).

Remark 10 *In 2D and 3D, the EPDiff equation (20) may also be written equivalently in terms of the operators div, grad and curl as,*

$$\frac{\partial}{\partial t}\mathbf{m}-\mathbf{u}\times\ \mathrm{curl}\,\mathbf{m}+\nabla(\mathbf{u}\cdot\mathbf{m})+\mathbf{m}(\mathrm{div}\,\mathbf{u})=0\,.\tag{23}$$

4.2 Variational derivation of EPDiff

The EPDiff equation (20) may be derived by following the proof of the EP reduction theorem leading to the Euler–Poincaré equations for right-invariance in the form of eqn (14). Following the calculation for the present right invariant case in the continuum notation yields

$$\begin{aligned}
\delta \int_a^b l(u)\mathrm{d}t &= \int_a^b \left\langle \frac{\delta l}{\delta u}, \delta u \right\rangle \mathrm{d}t = \int_a^b \left\langle \frac{\delta l}{\delta u}\,,\, \frac{\mathrm{d}v}{\mathrm{d}t} - \mathrm{ad}_u v \right\rangle \mathrm{d}t \\
&= \int_a^b \left\langle \frac{\delta l}{\delta u}\,,\, \frac{\mathrm{d}v}{\mathrm{d}t} \right\rangle \mathrm{d}t - \int_a^b \left\langle \frac{\delta l}{\delta u}\,,\, \mathrm{ad}_u v \right\rangle \mathrm{d}t \\
&= -\int_a^b \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta l}{\delta u} + \mathrm{ad}_u^* \frac{\delta l}{\delta u}\,,\, v \right\rangle \mathrm{d}t\,,
\end{aligned}$$

where, as in (10), we have set

$$\delta u = \frac{\mathrm{d}v}{\mathrm{d}t} - \mathrm{ad}_u v\,, \tag{24}$$

for the variation of the right invariant vector field u and $\langle \cdot\,,\,\cdot \rangle$ is the pairing between elements of the Lie algebra and its dual.

In our case, $\langle \cdot \, , \, \cdot \rangle$ is the L^2 pairing between vector fields and 1-form densities in eqn (15), written in components as

$$\left\langle \frac{\delta l}{\delta u}, \delta u \right\rangle = \int \frac{\delta l}{\delta u^i} \, \delta u^i \, \mathsf{d}V \, .$$

In our case, $\langle \cdot\,,\,\cdot \rangle$ is the L^2 pairing between vector fields and 1-form densities in eqn (15), written in components as

$$\Big\langle \frac{\delta l}{\delta u}, \delta u \Big\rangle = \int \frac{\delta l}{\delta u^i}\,\delta u^i\,\mathrm{d}V\,.$$

This L^2 pairing yields the component form of the **EPDiff equation** and its **Noether Theorem** as

$$\begin{aligned}
\int_a^b \Big\langle \frac{\delta l}{\delta u}, \delta u \Big\rangle \mathrm{d}t &= \int_a^b \mathrm{d}t \int \frac{\delta l}{\delta u^i}\Big(\frac{\partial v^i}{\partial t} + u^j\frac{\partial v^i}{\partial x^j} - v^j\frac{\partial u^i}{\partial x^j}\Big)\,\mathrm{d}V \\
&= -\int_a^b \mathrm{d}t \int \Big\{\frac{\partial}{\partial t}\frac{\delta l}{\delta u^i} + \frac{\partial}{\partial x^j}\Big(\frac{\delta l}{\delta u^i}u^j\Big) + \frac{\delta l}{\delta u^j}\frac{\partial u^j}{\partial x^i}\Big\} v^i\,\mathrm{d}V \\
&\quad + \int_a^b \mathrm{d}t \int \Big\{\frac{\partial}{\partial t}\Big(\frac{\delta l}{\delta u^i}v^i\Big) + \frac{\partial}{\partial x^j}\Big(\frac{\delta l}{\delta u^i}v^i u^j\Big)\Big\}\mathrm{d}V\,.
\end{aligned} \tag{25}$$

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This L^2 pairing yields the component form of the **EPDiff equation** and its **Noether Theorem** as

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&= - \int_a^b \mathrm{d}t \int \Big\{ \frac{\partial}{\partial t}\frac{\delta l}{\delta u^i} + \frac{\partial}{\partial x^j}\Big(\frac{\delta l}{\delta u^i} u^j \Big) + \frac{\delta l}{\delta u^j}\frac{\partial u^j}{\partial x^i} \Big\} v^i \, \mathrm{d}V \\
&\quad + \int_a^b \mathrm{d}t \int \Big\{ \frac{\partial}{\partial t}\Big(\frac{\delta l}{\delta u^i} v^i \Big) + \frac{\partial}{\partial x^j}\Big(\frac{\delta l}{\delta u^i} v^i u^j \Big) \Big\} \mathrm{d}V \,. \qquad (25)
\end{aligned}$$

When $\ell[u] = \frac{1}{2}\|u\|^2$, EPDiff describes geodesic motion on the diffeomorphisms with respect to the norm $\|u\|$.

4.3 Noether's theorem for EPDiff

Noether's theorem associates conservation laws to continuous symmetries of a Lagrangian. See, e.g., [Olv00] for a clear discussion of the classical theory. Momentum and energy conservation for the EPDiff equation in eqn (20) readily emerge from Noether's theorem, since the Lagrangian in eqn (11) admits space and time translations. That is, the action for EPDiff,

$$S = \int \ell[\mathbf{u}]\mathrm{d}t = \int \frac{1}{2}\|\mathbf{u}\|^2\mathrm{d}t\,,$$

is invariant under the following transformations,

$$x^j \to x'^j = x^j + c^j \quad \text{and} \quad t \to t' = t + \tau\,, \tag{26}$$

for constants τ and c^j, with $j = 1, 2, 3$. Noether's theorem then implies conservation of corresponding momentum components m_j, with $j = 1, 2, 3$, and energy E of the expected forms,

$$m_j = \frac{\delta\ell}{\delta u^j} \quad \text{and} \quad E = \frac{\delta\ell}{\delta u^j}u^j - \ell[\mathbf{u}]\,, \tag{27}$$

which may be readily verified.

Exercise 1 *Show that the EPDiff equation (14) may be written as*

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)\left(\mathbf{m}\cdot\mathbf{dx}\otimes\mathrm{d}V\right) = 0\,, \tag{28}$$

where $\mathcal{L}_{\mathbf{u}}$ is the Lie derivative with respect to the vector field with components $\mathbf{u} = G * \mathbf{m}$. How does the Lie-derivative form of EPDiff in eqn (28) differ from its characteristic form (21)? Hint*: compare the coordinate expression obtained from the dynamical definition of the Lie derivative with the corresponding expression obtained from its definition via Cartan's formula.*

Exercise 2 *Show that EPDiff in 1D may be written as*

$$m_t + um_x + 2mu_x = 0\,. \tag{29}$$

How does the factor of 2 arise in this equation?
Hint*: Take a look at eqn (20).*

Exercise 3 *Write the EPDiff equation in coordinate form (20) for (a) the L^2 norm and (b) the H^1 norm (L^2 norm of the gradient) of the spatial fluid velocity.*

Exercise 4 *Verify that the EPDiff equation (20) conserves the spatially integrated momentum and energy in eqn (27).* Hint*: for momentum conservation look at eqn (25) when* $v^j = c^j$ *for spatial translations.*

EPDiff solution behaviour

This lecture discusses the coherent particle-like properties of the unidirectional singular solutions of the EPDiff equation (29).

These singular solutions emerge from any smooth spatially confined initial velocity profile $u(x, 0)$.

After emerging, they dominate the evolution in interacting fully nonlinearly by exchanging momentum in elastic collisions.

The mechanism for their emergence is proven to be pulse steepening due to nonlinearity.

Several examples of the dynamics among singular solutions are given.

5 EPDiff in 1D: The CH equation

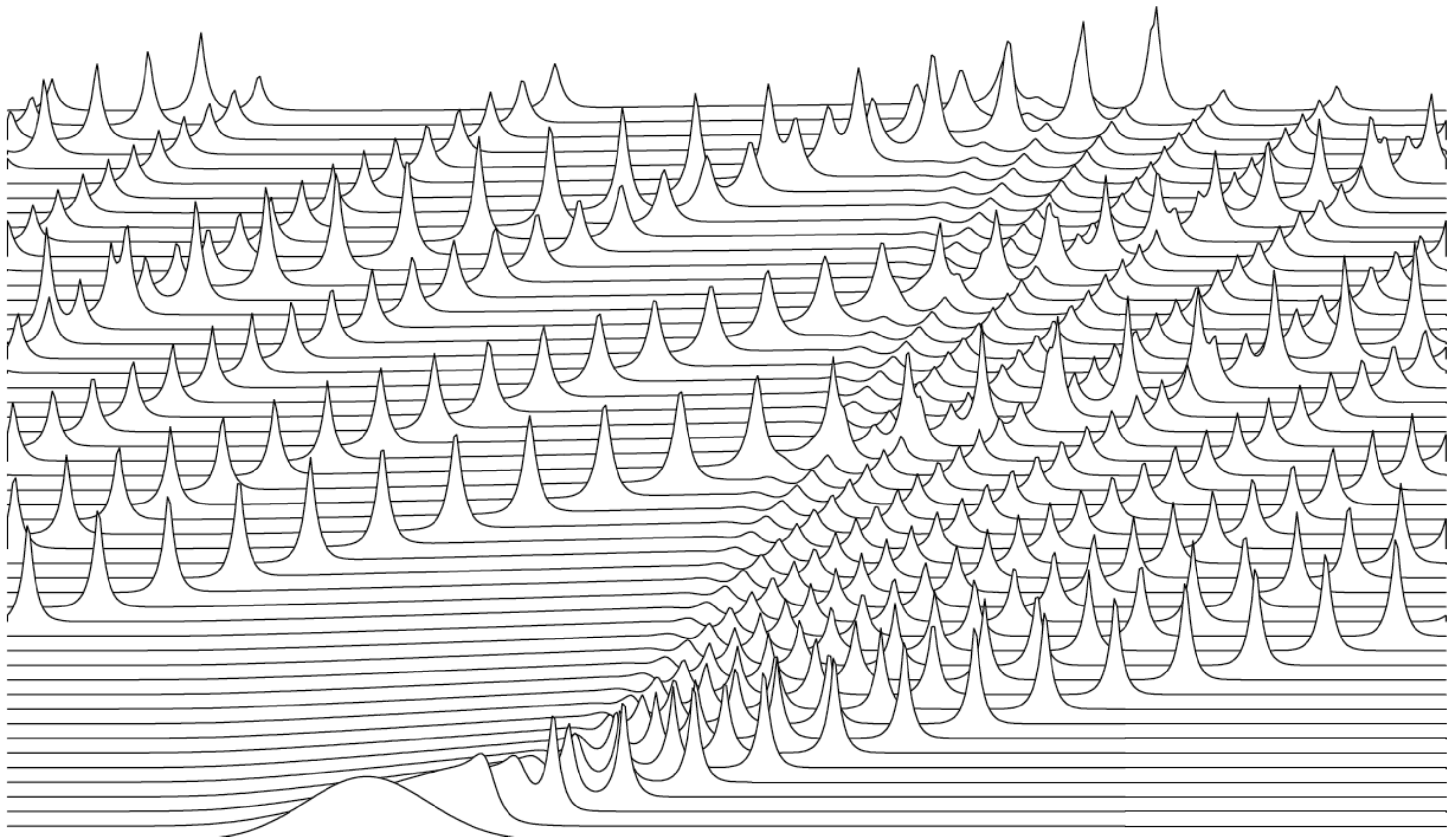

Figure 11: Under the evolution of the EPDiff equation (30), an ordered **wave train of peakons** emerges from a smooth localized initial condition (a Gaussian). The spatial profiles at successive times are offset in the vertical to show the evolution. The peakon wave train eventually wraps around the periodic domain, thereby allowing the leading peakons to overtake the slower peakons from behind in collisions that conserve momentum and preserve the peakon shape but cause phase shifts in the positions of the peaks, as discussed in [CH93].

Consider the following particular case of the EPDiff equation (29) in one spatial dimension,

$$m_t + u m_x + 2 m u_x = 0 \quad \text{with} \quad m = (1 - \alpha^2 \partial_x^2) u \,, \tag{30}$$

in which the fluid velocity u is a function of position x on the real line and time t. This equation governs geodesic motion on the smooth invertible maps (diffeomorphisms) of the real line with respect to the metric associated with the H^1 Sobolev norm of the fluid velocity given by

$$\|u\|^2_{H^1} = \int (u^2 + \alpha^2 u_x^2)\, \mathrm{d}x \,. \tag{31}$$

The **peakon** is the solitary travelling wave solution for the EPDIff equation (30),

$$u(x,t) = c\, \mathrm{e}^{-|x-ct|/\alpha} \,. \tag{32}$$

The peakon travelling wave moves at a speed equal to its maximum height, at which it has a sharp peak (jump in derivative).

The spatial velocity profile $e^{-|x|/\alpha}$ is the **Green's function** for the Helmholtz operator $(1-\alpha^2\partial_x^2)$ on the real line with vanishing boundary conditions at spatial infinity. In particular, it satisfies

$$(1-\alpha^2\partial_x^2)e^{-|x-ct|/\alpha} = 2\alpha\delta(x-ct)\,. \tag{33}$$

A novel feature of the EPDiff equation (30) is that it admits solutions representing a **wave train of peakons**

$$u(x,t) = \sum_{a=1}^{N} p_a(t)e^{-|x-q_a(t)|/\alpha}\,. \tag{34}$$

By eqn (33), this corresponds to a sum over delta functions representing the **singular solution** in momentum,

$$m(x,t) = 2\alpha\sum_{a=1}^{N} p_a(t)\,\delta(x-q_a(t))\,, \tag{35}$$

in which the **delta function** $\delta(x-q)$ is defined by

$$f(q) = \int f(x)\delta(x-q)\,\mathrm{d}x\,, \tag{36}$$

for an arbitrary smooth function f. Such a sum is an *exact solution* of the EPDiff equation (30) provided the time-dependent parameters $\{p_a\}$ and $\{q_a\}$, $a = 1, \ldots, N$, satisfy certain canonical Hamiltonian equations that will be discussed later.

Remark 11 *The peakon-train solutions of EPDiff are an* **emergent phenomenon***. A wave train of peakons emerges in solving the initial-value problem for the EPDiff equation (30) for essentially any spatially confined initial condition. An example of the emergence of a wave train of peakons from a Gaussian initial condition is shown in Figure 11.*

5.1 Steepening lemma: the peakon-formation mechanism

Lemma 1 (Steepening lemma [CH93])
Suppose the initial profile of velocity $u(0, x)$ has an inflection point at $x = \overline{x}$ to the right of its maximum, and otherwise it decays to zero in each direction sufficiently rapidly for the H^1 Sobolev norm of the fluid velocity in eqn (31) to be finite. Then, the negative slope at the inflection point will become vertical in finite time.

Remark 12 *Suppose the initial condition is anti-symmetric, so the inflection point at $u = 0$ is* fixed *and $\mathrm{d}\overline{x}/\mathrm{d}t = 0$, due to the symmetry $(u, x) \to (-u, -x)$ admitted by eqn (75). In this case, $M = 0$ and no matter how small $|s(0)|$ (with $s(0) < 0$) verticality $s \to -\infty$ develops at $\overline{x}$ in finite time.*

Remark 13 *The steepening lemma indicates that travelling wave solutions of the EPDiff equation (30) cannot have the sech^2 shape that appears for*

KdV solitons, since inflection points with sufficiently negative slope can lead to unsteady changes in the shape of the profile if inflection points are present.

Remark 14 *Numerical simulations show that the presence of an inflection point of negative slope in any confined initial velocity distribution triggers the steepening lemma as the* **mechanism** *for the formation of the peakons. Namely. the initial (positive) velocity profile "leans" to the right and steepens, then produces a peakon that is taller than the initial profile, so it propagates away to the right. This process leaves a profile behind with an inflection point of negative slope; so it repeats, thereby producing a wave train of peakons with the tallest and fastest ones moving rightward in order of height. Remarkably, this recurrent process produces only peakons.*

Remark 15 *The EPDiff equation (30) arises from a shallow water wave equation in the limit of zero linear dispersion in one dimension. As we shall see, the peakon solutions (35) for EPDiff generalize to higher dimensions and other kinetic energy norms.*

Exercise 5 *Verify that the EPDIff equation (30) preserves the* H^1 *norm (31).*

Exercise 6 *Verify that the peakon formula (32) provides the solitary travelling wave solution for the EPDIff equation (30).*

Exercise 7 *Verify formula (33) for the Green's function. Why is this formula useful in representing the travelling-wave solution of the EPDIff equation (30)?*

6 Shallow-water background for peakons

The EPDiff equation (30) whose solutions admit peakon wave trains (34) may be derived by taking the zero-dispersion limit of another equation obtained from Euler's fluid equations by using asymptotic expansions for shallow water waves [CH93]. Euler's equations for irrotational incompressible ideal fluid motion under gravity with a free surface have an asymptotic expansion for shallow water waves that involves two small parameters, ϵ and δ^2, with ordering $\epsilon \geq \delta^2$. These small parameters are $\epsilon = a/h_0$ (the ratio of wave amplitude to mean depth) and $\delta^2 = (h_0/l_x)^2$ (the squared ratio of mean depth to horizontal length, or wavelength).

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In one spatial dimension, EPDiff is the zero-dispersion limit of the Camassa–Holm (CH) equation for shallow water waves, which is the $b = 2$ case of the following **b-equation**, that results from the asymptotic expansion for shallow water waves,

$$m_t + c_0 u_x + u m_x + b\, m u_x - \gamma u_{xxx} = 0\,. \tag{37}$$

Here, $m = u - \alpha^2 u_{xx}$ is the momentum variable, and the constants α^2 and γ/c_0 are squares of length scales. At *linear* order in the asymptotic expansion for shallow water waves in terms of the small parameters ϵ and δ^2, one finds $\alpha^2 \to 0$, so that $m \to u$ in (37). In this case, the famous **Korteweg–de Vries** (KdV) soliton equation is recovered for $b = 2$,

$$u_t + 3uu_x = -c_0 u_x + \gamma u_{xxx} \,. \qquad (38)$$

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Any value of the parameter *except* $b = -1$ may be achieved in eqn (37) by an appropriate near-identity (normal form) transformation of the solution [DGH04]. The value $b = -1$ is disallowed in (37) because it cancels the leading-order nonlinearity and, thus, breaks the asymptotic ordering.

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Because of the relation $m = u - \alpha^2 u_{xx}$, the b-equation (37) is **non-local**. In other words, it is an integral-partial differential equation. In fact, after writing eqn (37) equivalently as,

$$(1 - \alpha^2 \partial_x^2)(u_t + uu_x) = -\,\partial_x\left(\frac{b}{2}\,u^2 + \frac{3-b}{2}\,\alpha^2 u_x^2\right) - c_0 u_x + \gamma u_{xxx} \,. \qquad (39)$$

The b-equation may be expressed in **hydrodynamic form** as

$$u_t + uu_x = -\,p_x\,, \tag{40}$$

with a 'pressure' p given by

$$p = G * \left(\frac{b}{2}\,u^2 + \frac{3-b}{2}\,\alpha^2 u_x^2 + c_0 u - \gamma u_{xx} \right), \tag{41}$$

in which the convolution kernel is the Green's function $G(x,y) = \frac{1}{2\alpha}\mathrm{e}^{-|x-y|/\alpha}$ for the Helmholtz operator $(1-\alpha^2\partial_x^2)$.

One sees the interplay between local and non-local linear dispersion in the b-equation by linearizing eqn (39) around $u=0$ to find its phase-velocity relation,

$$\frac{\omega}{k} = \frac{c_0 + \gamma\,k^2}{1+\alpha^{\,2}k^2}\,, \tag{42}$$

obtained for waves with frequency ω and wave number k. For $\gamma/c_0 > 0$, short waves and long waves travel in the same direction. Long waves travel faster than short ones (as required in shallow water) provided $\gamma/c_0 < \alpha^2$.

Then, the phase velocity lies in the interval $\omega/k \in (\gamma/\alpha^2, c_0]$. The parameters c_0 and γ represent linear wave dispersion, which modifies and may eventually balance the tendency for nonlinear waves to steepen and break. The parameter α, which introduces non-locality, also allows a balance leading to a stable wave shape, even in the absence of c_0 and γ.

The nonlinear effects of the parameter b on the solutions of eqn (37) were investigated in Holm and Staley [HS03], where b was treated as a bifurcation parameter. In the limiting case when the linear dispersion coefficients are absent, peakon solutions of eqn (37) are allowed theoretically for any value of b. However, they were found numerically to be stable only for $b > 1$. These coherent solutions are allowed, because the two nonlinear terms in eqn (37) may balance each other, even in the *absence* of linear dispersion. However, the instability of the peakons found numerically for $b < 1$ indicates that the relative strengths of the two nonlinearities will determine whether this balance can be maintained.

Exercise 8 *A solution* u *of the b-equation (37) with* $c_0 = 0$ *and* $\gamma = 0$ *vanishing at spatial infinity blows up in* H^1 *if and only if its first-order derivative blows up, that is, if wave breaking occurs.*

Lemma 2 (Steepening lemma for the b-equation with $b > 1$)
Suppose the initial profile of velocity $u(0, x)$ has an inflection point at $x = \overline{x}$ to the right of its maximum, and otherwise it decays to zero in each direction. Assume that the velocity at the inflection point remains finite. Then, the negative slope at the inflection point will become vertical in finite time, provided $b > 1$.

Proof. Consider the evolution of the slope at the inflection point $x = \overline{x}(t)$. Define $s = u_x(\overline{x}(t), t)$. Then, the b-equation (37) with $c_0 = 0$ and $\gamma = 0$ may be rewritten in hydrodynamic form as, cf. eqn (40),

$$u_t + uu_x = -\,\partial_x G * \left(\frac{b}{2}\, u^2 + \frac{3-b}{2}\, \alpha^2 u_x^2 \right). \tag{43}$$

The spatial derivative of this yields an equation for the evolution of s. Namely, using $u_{xx}(\overline{x}(t), t) = 0$ leads to

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}t} + s^2 &= -\,\partial_x^2 (G * p) \quad \text{with} \quad p := \left(\frac{b}{2} u^2(\overline{x}(t), t) + \frac{3-b}{2}\, \alpha^2 s^2 \right) \\
&= \frac{1}{\alpha^2}(1 - \,\alpha^2 \partial_x^2) G * p - \frac{1}{\alpha^2} G * p \\
&= \frac{1}{\alpha^2} p - \frac{1}{\alpha^2} G * p\,.
\end{aligned} \tag{44}$$

This calculation implies

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1-b}{2}s^2 - \frac{1}{2\alpha}\int_{-\infty}^{\infty} \mathrm{e}^{-|\overline{x}-y|/\alpha}\left(\frac{b}{2}u^2 + \frac{3-b}{2}\alpha^2 u_y^2\right)\mathrm{d}y + \frac{b}{2\alpha^2}u^2(\overline{x}(t),t)$$
$$\leq \frac{1-b}{2}s^2 + \frac{b}{2\alpha^2}u^2(\overline{x}(t),t)\,, \qquad (45)$$

where we have dropped the negative middle term in the last step. Then, provided $u^2(\overline{x}(t),t)$ remains finite, say less than a number M, we have

$$\frac{\mathrm{d}s}{\mathrm{d}t} \leq \left(\frac{1-b}{2}\right)s^2 + \frac{bM}{2\alpha^2}\,, \qquad (46)$$

which implies, for negative slope initially and $b > 1$, that the slope remains negative and becomes vertical in finite time.

Remark 16 *This proof of the steepening lemma for the b-equation identifies $b = 1$ as a special value.*

Remark 17 *One might wonder whether the dispersionless CH equation is the only shallow water b-equation that both possesses peakon solutions and* *is completely integrable as a Hamiltonian system. Mikhailov and Novikov [MN02] showed that among the b-equations only the cases* $b = 2$ *and* $b = 3$ *are completely integrable as Hamiltonian systems. The case* $b = 3$ *is the Degasperis–Processi equation, whose peakon solutions are studied in [DHH03].*

Remark 18 *Hereafter, we specialize the b-equation (37) to the case* $b = 2$*. If, in addition,* $c_0 = 0$ *and* $\gamma = 0$*, then the b-equation specializes to EPDiff.*

6.1 Hamiltonian dynamics of EPDiff peakons

Upon substituting the peakon solution expressions (34) for velocity u and eqn (35) for momentum m into the EPDiff equation,

$$m_t + um_x + 2mu_x = 0\,, \quad \text{with} \quad m = u - \alpha^2 u_{xx}\,, \tag{47}$$

one finds **Hamilton's canonical equations** for the dynamics of the discrete set of peakon parameters $p_a(t)$ and $q_a(t)$. Namely,

$$\dot{q}_a(t) = \frac{\partial H_N}{\partial p_a} \quad \text{and} \quad \dot{p}_a(t) = -\frac{\partial H_N}{\partial q_a}\,, \tag{48}$$

for $a = 1, 2, \ldots, N$, with Hamiltonian given by [CH93],

$$H_N = \frac{1}{2} \sum_{a,b=1}^{N} p_a\, p_b\, \mathrm{e}^{-|q_a - q_b|/\alpha}\,. \tag{49}$$

The first canonical equation in eqn (48) implies that the peaks at the positions $x = q^a(t)$ in the peakon-train solution (34) move with the flow of the fluid velocity u at those positions, since $u(q^a(t), t) = \dot{q}^a(t)$. This means the positions $q^a(t)$ are **Lagrangian coordinates** frozen into the flow of EPDiff. Thus, the singular momentum solution ansatz (35) is the map from Lagrangian coordinates to Eulerian coordinates (that is, the **Lagrange-to-Euler map**) for the momentum.

Remark 19 *The peakon wave train (35) forms a finite-dimensional invariant manifold of solutions of the EPDiff equation. On this invariant manifold of solutions for the partial differential equation (47), the dynamics turns out to be canonically Hamiltonian as in eqn (48). A later lecture will explain that the canonical Hamiltonian structure of the peakon solutions arises because the solution ansatz (35) for momentum m is a* **momentum map***.*

6.2 Pulsons for other Green's functions

The Hamiltonian H_N in eqn (49) depends on G, the Green's function for the relation $u = G * m$ between velocity u and momentum m. For the Helmholtz operator on the real line this Green's function is given by eqn (33) as $G(x) = \mathrm{e}^{-|x|/\alpha}/2\alpha$. However, the singular momentum solution ansatz (35) is *independent* of this Green's function. Thus, we may conclude the following [FH01].

Proposition 1 *The singular momentum solution ansatz*

$$m(x,t) = \sum_{a=1}^{N} p_a(t)\,\delta(x - q_a(t))\,, \tag{50}$$

for EPDiff,

$$m_t + um_x + 2mu_x = 0\,, \quad \textit{with} \quad u = G * m\,, \tag{51}$$

provides a **finite-dimensional invariant manifold** *of solutions governed by canonical Hamiltonian dynamics,* **for any choice** *of the Green's function G relating velocity u and momentum m.*

Proof. The singular momentum solution ansatz (50) is *independent* of the Green's function G.

Remark 20 *The pulson singular solutions (50) of the EPDiff equation (51) form a finite-dimensional invariant symplectic manifold, on which the EPDiff solution dynamics is governed by a canonical Hamiltonian system for the conjugate pairs of variables* $(q_a,\, p_a)$ *with* $a = 1, 2, \dots, N$*. Perhaps surprisingly, these singular solutions will turn out to emerge from any smooth confined initial distribution of momentum.*

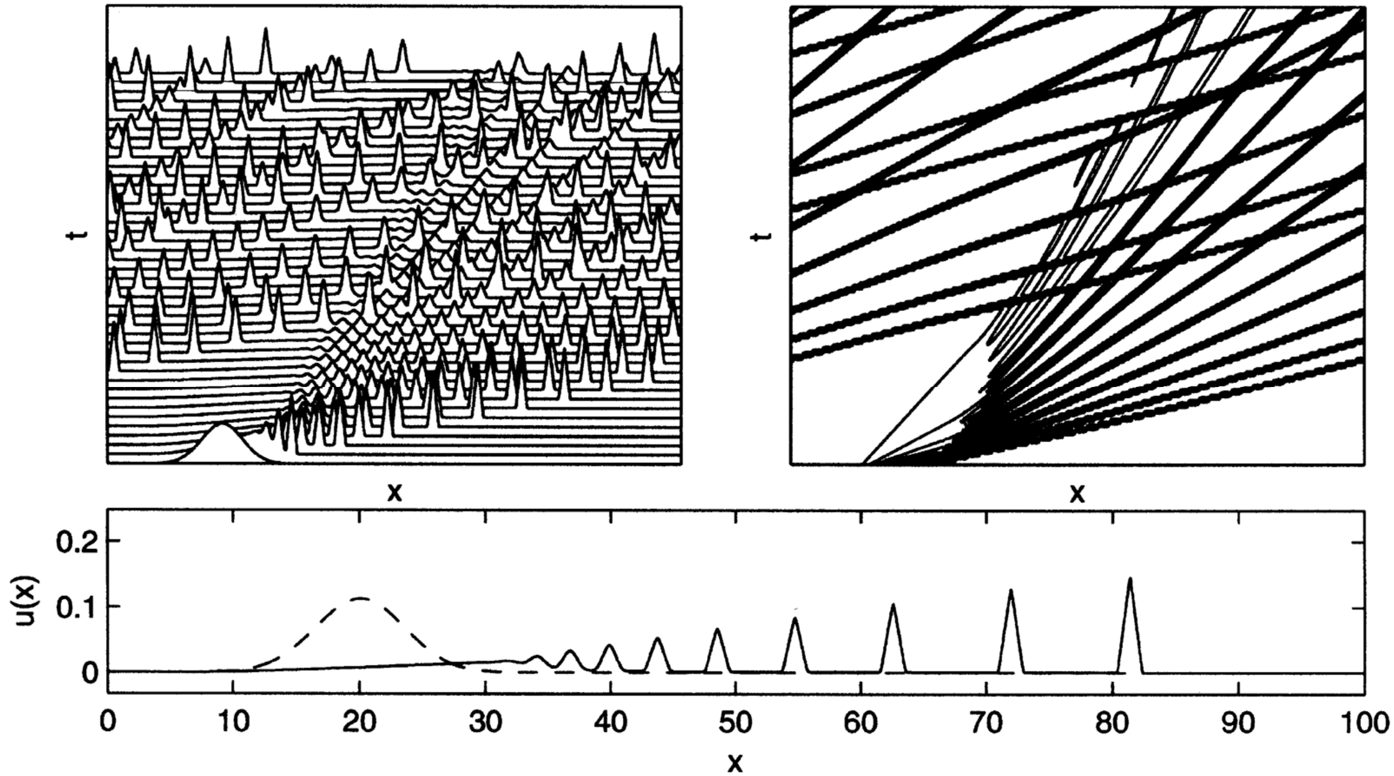


Figure 12: When the Green's function G has a triangular profile, a train of triangular pulsons emerges from a Gaussian initial velocity distribution as it evolves under the EPDiff equation (30). The upper panels show the collisions that occur as the faster triangular pulsons overtake the slower ones as they cross and re-cross the periodic domain. The upper left panel shows the progress of the pulsons by by showing offsets of the velocity profile at equal time intervals. The upper right panel shows the pulson paths obtained by plotting their elevation topography.

The fluid velocity solutions corresponding to the singular momentum ansatz (50) for eqn (51) are the **pulsons**. A pulson wave train is defined by the sum over N velocity profiles determined by the Green's function G, as

$$u(x,t) = \sum_{a=1}^{N} p_a(t) G\Big(x, q_a(t)\Big) \,. \tag{52}$$

A solitary travelling wave solution for the pulson is given by

$$u(x,t) = cG(x,ct) = cG(x-ct) \quad \text{with} \quad G(0) = 1 \,, \tag{53}$$

where one finds $G(x,ct) = G(x-ct)$, provided the Green's function G is translation-invariant.

For EPDiff (51) with any choice of the Green's function G, the singular momentum solution ansatz (50) results in a finite-dimensional invariant manifold of exact solutions. The $2N$ parameters $p_a(t)$ and $q_a(t)$ in these pulson-train solutions of EPDiff satisfy Hamilton's canonical equations

$$\frac{\mathrm{d}q_a}{\mathrm{d}t} = \frac{\partial H_N}{\partial p_a} \quad \text{and} \quad \frac{\mathrm{d}p_a}{\mathrm{d}t} = -\frac{\partial H_N}{\partial q_a} \,, \tag{54}$$

with N-particle Hamiltonian,

$$H_N = \frac{1}{2} \sum_{a,b=1}^{N} p_a \, p_b \, G(q_a, q_b) \,. \tag{55}$$

The canonical equations for the parameters in the pulson train define an invariant manifold of singular momentum solutions and provide a phase-space description of geodesic motion with respect to the cometric (inverse metric) given by the Green's function G. Mathematical analysis and numerical results for the dynamics of these pulson solutions are given in [FH01] whose results show how the results of collisions of pulsons (52) depend upon the *shape* of their travelling wave profile. The effects of the travelling-wave pulse shape

$$u(x - ct) = cG(x - ct)$$

on the multipulson collision dynamics are reflected in the Hamiltonian (55) that governs this dynamics. For example, see Figure 12, in which the pulsons are *triangular*.

Exercise 9 *Verify the hydrodynamic form of the b-equation in eqn (39).*

Exercise 10 *Verify that the b-equation (37) with* $c_0 = 0$ *and* $\gamma = 0$ *admits peakon-train solutions of the form (34) for any value of* b*.*

Exercise 11 *Verify that the b-equation (37) with* $c_0 = 0$ *and* $\gamma = 0$ *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{H^1} = (b-2)\int u_x^3\,\mathrm{d}x\,,$$

for any value of b *and for solutions that vanish sufficiently rapidly at spatial infinity that no endpoint contributions arise upon integration by parts.*

Exercise 12 *Prove a steepening lemma for the b-equation (37) with* $c_0 = 0$ *and* $\gamma = 0$ *that avoids the assumption that* $u^2(\overline{x}(t), t)$ *remains finite. That is, establish a necessary and sufficient condition depending only on the initial data for blow-up to occur in finite time. How does this condition depend on the value of* b*? Does this steepening lemma hold for every value of* $b > 1$*?*

Exercise 13 *Are the equations of peakon dynamics for the b-equation (37) with* $c_0 = 0$ *and* $\gamma = 0$ *canonically Hamiltonian for every value of* b*?* Hint*: try* $b = 3$*.*

7 Peakons and pulsons

7.1 Pulson–Pulson interactions

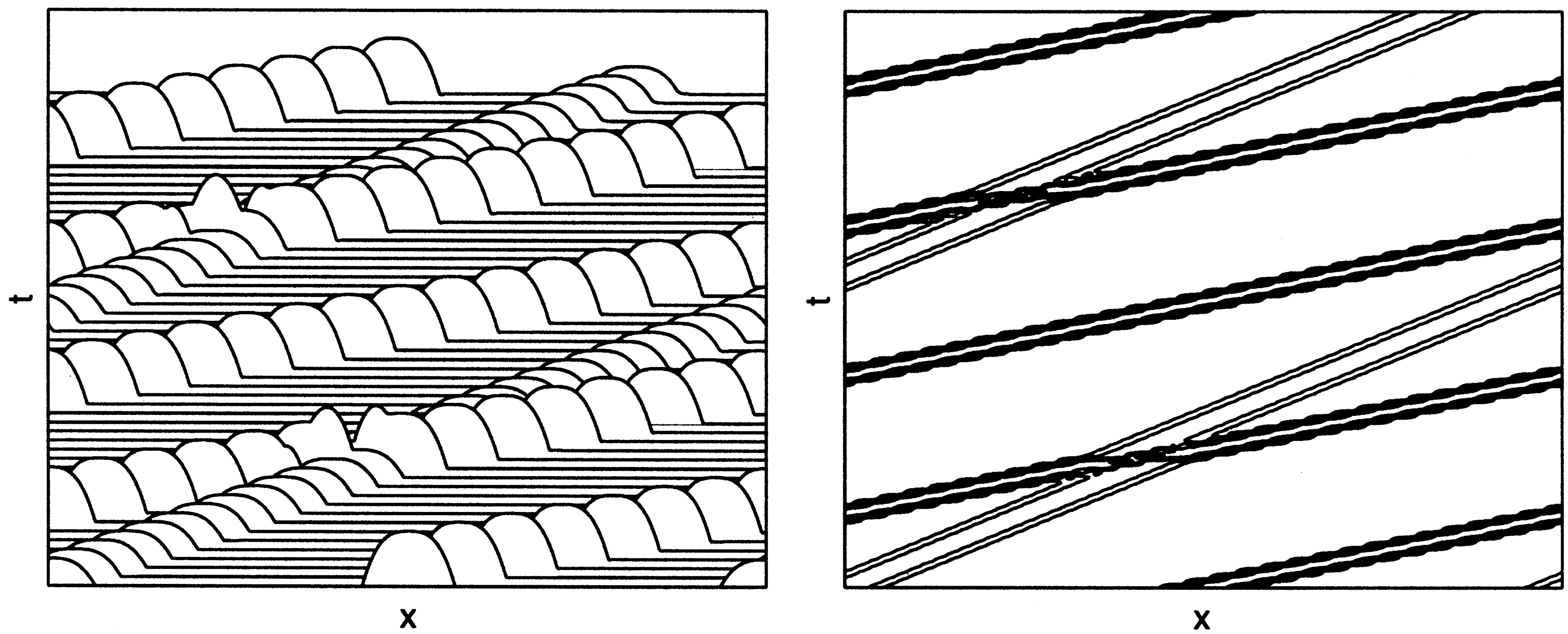


Figure 13: This is the velocity profile for an overtaking collision of H2 pulsons [FH01].

The solution of EPDiff in 1D

$$\partial_t m + u m_x + 2 u_x m = 0\,, \tag{56}$$

with $u = G * m$ for the momentum $m = Q_{op} u$ is given for the interaction of only two pulsons by the sum of delta functions in eqn (50) with $N = 2$,

$$m(x,t) = \sum_{i=1}^{2} p_i(t)\,\delta(x - q_i(t))\,. \tag{57}$$

The parameters satisfy the finite dimensional geodesic canonical Hamiltonian equations (48), in which the Hamiltonian for $N = 2$ is given by

$$H_{N=2}(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + p_1 p_2\, G(q_1 - q_2)\,. \tag{58}$$

Conservation laws and reduction to quadrature

Provided the Green's function G is symmetric under spatial reflections, $G(-x) = G(x)$, the two-pulson Hamiltonian system conserves the total momentum

$$P = p_1 + p_2\,. \tag{59}$$

Conservation of P ensures integrability, by Liouville's theorem, and reduces the 2-pulson system to quadratures. To see this, we introduce sum and difference variables as

$$P = p_1 + p_2\,,\quad Q = q_1 + q_2\,,\quad p = p_1 - p_2\,,\quad q = q_1 - q_2\,. \tag{60}$$

In these variables, the Hamiltonian (58) becomes

$$H(q,p,P) = \frac{1}{4}(P^2 - p^2)\Big(1 - G(q)\Big)\,. \tag{61}$$

Likewise, the 2-pulson equations of motion transform to sum and difference variables as

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -2\frac{\partial H}{\partial Q} = 0\,,\qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = 2\frac{\partial H}{\partial P} = P(1 + G(q))\,,$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -2\frac{\partial H}{\partial q} = \frac{1}{2}(p^2 - P^2)G'(q)\,,\qquad \frac{\mathrm{d}q}{\mathrm{d}t} = 2\frac{\partial H}{\partial p} = -\,p(1 - G(q))\,.$$

Eliminating p^2 between the formula for H and the equation of motion for q yields

$$\begin{aligned}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 &= P^2\Big(1-G(q)\Big)^2 - 4H\Big(1-G(q)\Big)\\ &=: Z(G(q);P,H) \geq 0\,,\end{aligned} \tag{62}$$

which rearranges into the following quadrature,

$$\mathrm{d}t = \frac{\mathrm{d}G(q)}{G'(q)\sqrt{Z(G(q);P,H)}}\,. \tag{63}$$

For the peakon case, we have $G(q) = \mathrm{e}^q$ so that $G'(q) = G(q)$ and the quadrature (63) simplifies to an elementary integral. Having obtained $q(t)$ from the quadrature, the momentum difference $p(t)$ is found from eqn (61) via the algebraic expression

$$p^2 = P^2 - \frac{4H}{1-G(q)}\,, \tag{64}$$

in terms of q and the constants of motion P and H. Finally, the sum $Q(t)$ is found by a further quadrature.

Upon writing the quantities H and P as

$$H = c_1 c_2, \quad P = c_1 + c_2, \quad \frac{1}{2}c_1^2 + \frac{1}{2}c_2^2 = \frac{1}{2}P^2 - H\,, \tag{65}$$

in terms of the asymptotic speeds of the pulsons, c_1 and c_2, we find the relative momentum relation,

$$p^2 = (c_1 + c_2)^2 - \frac{4c_1 c_2}{1 - G(q)}\,. \tag{66}$$

This equation has several implications for the qualitative properties of the 2-pulson collisions.

Definition 9 *Overtaking, or rear-end, pulson collisions satisfy* $c_1 c_2 > 0$*, while head-on pulson collisions satisfy* $c_1 c_2 < 0$*.*

The pulson order $q_1 < q_2$ is preserved in an overtaking, or rear-end, collision. This follows, as

Proposition 2 (Preservation of pulson order) *For overtaking, or rear-end, collisions, the 2-pulson dynamics preserves the sign condition*

$$q = q_1 - q_2 < 0 \,.$$

Proof. Suppose the peaks were to overlap in an overtaking collision with $c_1 c_2 > 0$, thereby producing $q = 0$ during a collision. The condition $G(0) = 1$ implies the second term in eqn (66) would diverge if this overlap were to occur. However, such a divergence would contradict $p^2 \geq 0$.

Consequently, seen as a collision between two 'particles' with initial speeds c_1 and c_2 that are initially well separated, the separation $q(t)$ reaches a non-zero distance of closest approach q_{min} in an overtaking, or rear-end, collision that may be expressed in terms of the pulse shape, as follows.

Corollary 1 (Minimum separation distance)
The minimum separation distance reachable in two-pulson collisions with $c_1 c_2 > 0$ is given by,

$$1 - G(q_{min}) = \frac{4 c_1 c_2}{(c_1 + c_2)^2} \,. \tag{67}$$

Proof. Set $p^2 = 0$ in eqn (66).

Proposition 3 (Head-on collisions admit $q \to 0$)
The 2-pulson dynamics allows the overlap $q \to 0$ in head-on collisions.

Proof. Because $p^2 \geq 0$, the overlap $q \to 0$ implying $g \to 1$ is only possible in eqn (66) for $c_1c_2 < 0$. That is, for the head-on collisions.

Remark 21 (Divergence of head-on momentum)
Equation (66) implies that $p^2 \to \infty$ diverges when $q \to 0$ in head-on collisions. As we shall discuss, this signals the development of a vertical slope in the velocity profile of the solution at the moment of collision.

7.2 Pulson–anti-pulson interactions

Head-on pulson–anti-pulson collision

In a **completely anti-symmetric** head-on collision of a pulson and anti-pulson, one has $p_1 = -p_2 = p/2$ and $q_1 = -q_2 = q/2$ (so that $P = 0$ and $Q = 0$). In this case, the quadrature formula (63) reduces to

$$\pm(t - t_0) = \frac{1}{\sqrt{-4H}} \int_{q(t_0)}^{q(t)} \frac{dq\,'}{\Big(1 - G(q\,')\Big)^{1/2}}\,, \tag{68}$$

and the second constant of motion in eqn (61) satisfies

$$-4H = p^2\Big(1 - G(q)\Big) \geq 0\,. \tag{69}$$

After the collision, the pulson and anti-pulson separate and travel apart in opposite directions; so that asymptotically in time $g(q) \to 0$, $p \to 2c$, and

$H \to -c^2$, where c (or $-c$) is the asymptotic speed (and amplitude) of the pulson (or anti-pulson). Setting $H = -c^2$ in eqn (69) gives a relation for the pulson–anti-pulson (p, q) phase trajectories for any kernel,

$$p = \pm \frac{2c}{\Big(1 - G(q)\Big)^{1/2}} \,. \tag{70}$$

Notice that p diverges (and switches branches of the square root) when $q \to 0^+$, because $G(0) = 1$. The convention of switching branches of the square root allows one to keep $q > 0$ throughout, so the particles retain their order. That is, the particles 'bounce' elastically at the moment when $q \to 0^+$ in the perfectly anti-symmetric head-on collision.

Remark 22 (Preservation of particle identity in collisions)
The relative separation distance $q(t)$ in pulson–anti-pulson collisions is determined by following a phase point along a level surface of the Hamiltonian H in the phase space with coordinates (q, p). Because H is quadratic, the relative momentum p has two branches on such a level surface, as indicated by the $\pm$ sign in eqn (70). At the pulson–anti-pulson collision point, both $q \to 0^+$ and either $1/p \to 0^+$ or $p \to 0^+$, so following a phase point through a collision requires that one must choose a convention for which branch of the level surface is taken after the collision. Taking the convention that p changes sign (corresponding to a **bounce***), but q does not change sign (so the* **particles keep their identity***) is convenient, because it allows the phase points to be followed more easily through multiple collisions. This choice is also consistent with the pulson–pulson and anti-pulson–anti-pulson collisions. In these other* **rear-end collisions***, as implied by eqn (66), the separation distance always remains positive and again the particles retain their identity.*

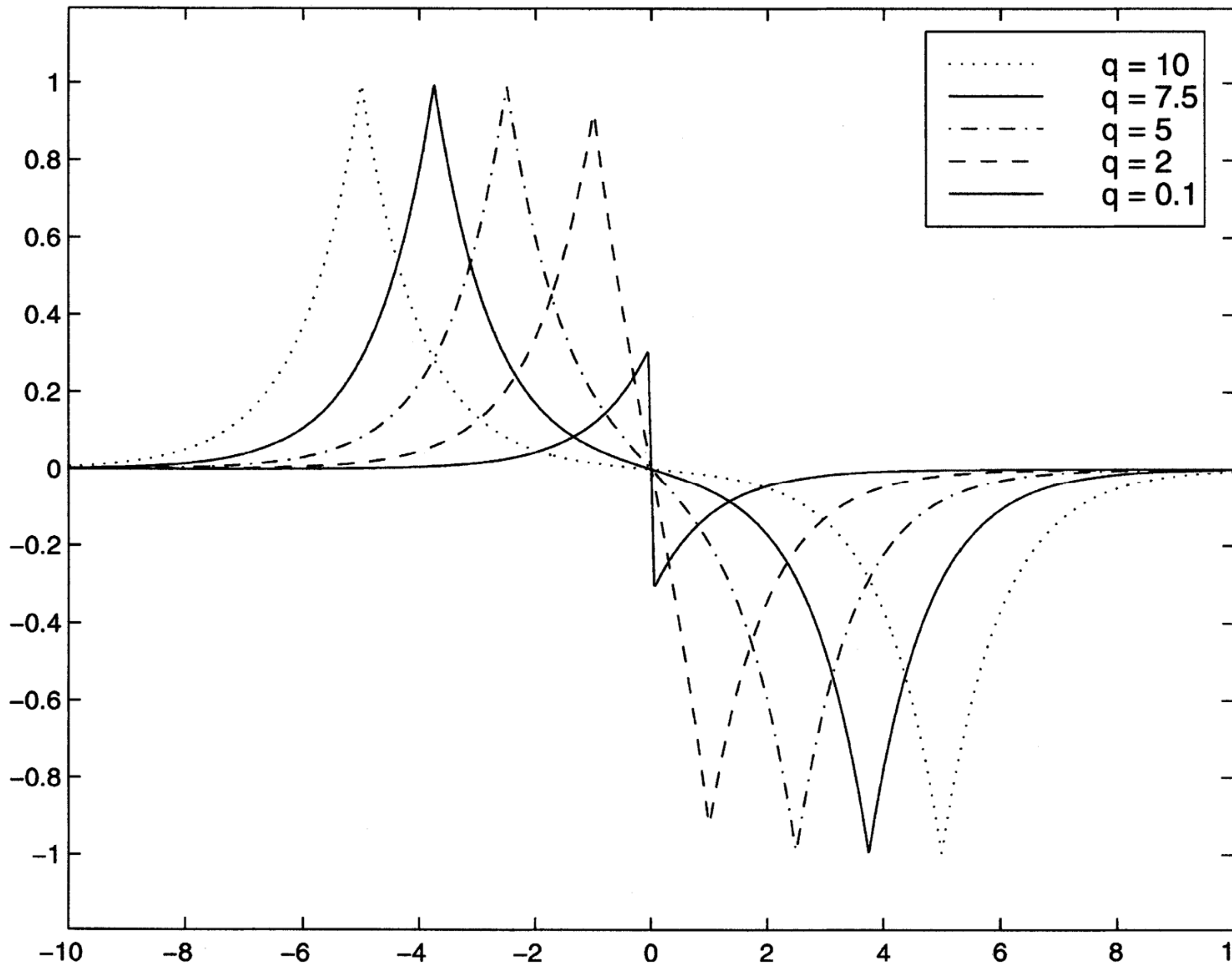


Figure 14: This is the velocity profile (71) for the peakon-antipeakon head-on collision as a function of separation between the peaks [FH01].

Theorem 2 (Pulson–anti-pulson exact solution)
The exact analytical solution for the pulson–anti-pulson collision for any symmetric G may be written as a function of position x and the separation between the pulses q for any pulse shape or kernel $G(x)$ as

$$u(x,q) = \frac{c}{\Big(1-G(q)\Big)^{1/2}}\Big[G(x+q/2) - G(x-q/2)\Big], \qquad (71)$$

where c is the pulson speed at sufficiently large separation and the dynamics of the separation $q(t)$ is given by the quadrature (68) with $\sqrt{-4H} = 2c$.

Proof. The solution for the velocity $u(x,t)$ in the head-on pulson–anti-pulson collision may be expressed in this notation as

$$u(x,t) = \frac{p}{2}G(x+q/2) - \frac{p}{2}G(x-q/2)\,. \qquad (72)$$

In using eqn (70) to eliminate p this solution becomes eqn (71).

Exercise 14 *According to eqn (68), how much time is required for the head-on pulson–anti-pulson collision, when $G(q) = \mathrm{e}^{-q^2/2}$ is a Gaussian?*

Exercise 15 *For the case that $G(x) = \mathrm{e}^{-|x|}$, which is Green's function for the Helmholtz operator in 1D with $\alpha = 1$, show that solution (72) for the peakon–anti-peakon collision yields*

$$q = -\log \operatorname{sech}^2(ct)\,, \quad p = \frac{\pm 2c}{\tanh(ct)}\,, \tag{73}$$

so the peakon–anti-peakon collision occurs at time $t = 0$ and eqn (72) results in

$$\begin{aligned} m(x,t) &= u - \alpha^2 u_{xx} \\ &= \frac{2c}{\tanh(ct)}\Big[\delta\Big(x - \frac{1}{2}q(t)\Big) - \delta\Big(x + \frac{1}{2}q(t)\Big)\Big]\,. \end{aligned} \tag{74}$$

Discuss the behaviour of this solution. What happens to the slope and amplitude of the peakon velocity just at the moment of impact?

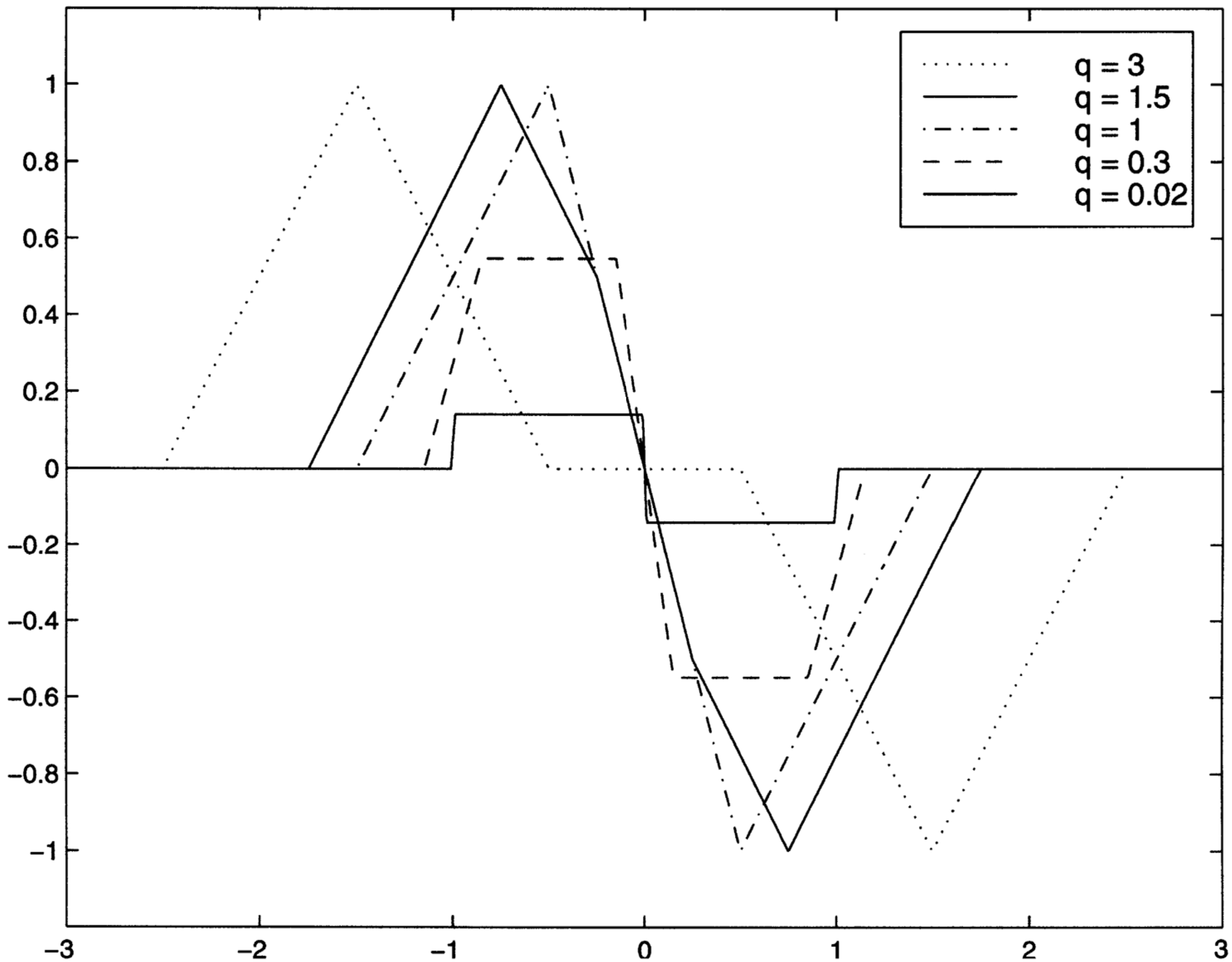


Figure 15: Velocity profile (71) for the head-on collision of the triangular peakon–anti-peakon pair as a function of separation between the peaks [FH01].

Lecture #2, Integrability of EPDiff in 1D

This lecture explains the noncanonical Hamiltonian properties of the CH equation (75) in one spatial dimension. In fact, the CH equation has two compatible Hamiltonian structures, so it is **bi-Hamiltonian**. In this situation, Magri's lemmas for bi-Hamiltonian PDE in 1D imply systematically that CH arises as a different compatibility condition for an **isospectral eigenvalue problem** and a linear evolution equation for the corresponding eigenfunctions in the case when $G(x) = \mathrm{e}^{-|x|/\alpha}$. The properties of being bi-Hamiltonian and possessing an associated isospectral problem are ingredients for proving the one-dimensional CH equation (75) is **completely integrable** as a Hamiltonian system and is solvable by the **inverse scattering transform (IST) method**.

In the previous lecture, we discussed the CH equation for unidirectional shallow-water waves derived in [CH93], as a special case of the b-equation (37) with $b = 2$,

$$m_t + um_x + 2mu_x = -c_0u_x + \gamma u_{xxx}\,, \qquad m = u - \alpha^2 u_{xx}\,. \qquad (75)$$

This partial differential equation (PDE) describes shallow-water dynamics at quadratic order in the asymptotic expansion for unidirectional shallow-water waves on a free surface under gravity. The previous lecture discussed its elastic particle-collision solution properties in the dispersionless case for which the linear terms on the right side of eqn (75) are absent. These elastic-collision solution properties hold for any Green's function $G(x)$ in the convolution relation $u = G * m$ between velocity u and momentum m. For the CH equation $G(x) = \mathrm{e}^{-|x|/\alpha}$ is the Green's function for the 1D Helmholtz operator on the real line with homogeneous boundary conditions.

8 The CH equation is bi-Hamiltonian

The CH equation is **bi-Hamiltonian**. This means that eqn (75) may be written in two compatible Hamiltonian forms, namely as

$$m_t = -B_2\,\frac{\delta H_1}{\delta m} = -B_1\,\frac{\delta H_2}{\delta m}\,, \tag{76}$$

where B_1 and B_2 are Poisson operators. For the CH equation, the pairs of Hamiltonians and Poisson operators are given by

$$\begin{aligned} H_1 &= \frac{1}{2}\int (u^2 + \alpha^2 u_x^2)\ \mathrm{d}x\,, \\ B_2 &= \partial_x m + m\partial_x + c_0\partial_x + \gamma\,\partial_x^3\,, \end{aligned} \tag{77}$$

$$\begin{aligned} H_2 &= \frac{1}{2}\int u^3 + \alpha^2 u\,u_x^2 + c_0 u^2 - \gamma\,u_x^2\ \mathrm{d}x\,, \\ B_1 &= \partial_x - \alpha^2\partial_x^3\,. \end{aligned} \tag{78}$$

These bi-Hamiltonian forms restrict properly to those for KdV when $\alpha^2 \to 0$, and to those for EPDiff when $c_0, \gamma \to 0$. **Compatibility** of B_1 and B_2 is assured, because $(\partial_x m + m\partial_x)$, ∂_x and ∂_x^3 are all mutually compatible

Hamiltonian operators. That is, any linear combination of these operators defines a Poisson bracket,

$$\{f, h\}(m) = -\int \frac{\delta f}{\delta m}(c_1 B_1 + c_2 B_2)\frac{\delta h}{\delta m}\,\mathrm{d}x\,, \tag{79}$$

as a bilinear skew-symmetric operation that satisfies the Jacobi identity. (In general, the sum of the Poisson brackets would fail to satisfy the Jacobi identity.)

Moreover, no further deformations of these Hamiltonian operators involving higher-order partial derivatives would be compatible with B_2, as shown in [Olv00]. This fact was already known in the literature for KdV, see [Fuc96].

8.1 Magri's lemmas

The property of **compatibility** of the two Hamiltonian operators for a bi-Hamiltonian equation enables the construction under certain conditions of an infinite hierarchy of Poisson-commuting Hamiltonians. The property of compatibility was used by Magri [Mag78] in proving the following important pair of lemmas (see also [Olv00] for a clear discussion of Magri's lemmas):

Lemma 3 (Magri 1978) *If B_1 and B_2 are compatible Hamiltonian operators, with B_1 non-degenerate, and if*

$$B_2\frac{\delta H_1}{\delta m} = B_1\frac{\delta H_2}{\delta m} \quad \textit{and} \quad B_2\frac{\delta H_2}{\delta m} = B_1\mathcal{K}\,, \tag{80}$$

for Hamiltonians H_1, H_2, and some function $\mathcal{K}$, then there exists a third Hamiltonian functional H such that $\mathcal{K} = \delta H/\delta m$.

To prove the existence of an infinite hierarchy of Hamiltonians, H_n, $n = 1, 2, \ldots,$ related to the two compatible Hamiltonian operators B_1, B_2, we need to check that the following two conditions hold:

(i) There exists an infinite sequence of functions $\mathcal{K}_1$, $\mathcal{K}_2$, ... satisfying

$$B_2\mathcal{K}_n = B_1\mathcal{K}_{n+1}\,; \tag{81}$$

(ii) There exist two functionals H_1 and H_2 such that

$$\mathcal{K}_1 = \frac{\delta H_1}{\delta m}\,, \qquad \mathcal{K}_2 = \frac{\delta H_2}{\delta m}\,. \tag{82}$$

It then follows from Lemma 3 that there exist functionals H_n such that

$$\mathcal{K}_n = \frac{\delta H_n}{\delta m}\,, \quad \text{for all} \quad n \geq 1\,. \tag{83}$$

Lemma 4 (Magri 1978) *Let $\{\,\cdot\,,\,\cdot\,\}_1$ and $\{\,\cdot\,,\,\cdot\,\}_2$ denote the Poisson brackets defined, respectively, by B_1 and B_2, which are assumed to be compatible Hamiltonian operators. Let H_1, H_2, ... be an infinite sequence of Hamiltonian functionals constructed from eqns (81) and*

(83). Then, these Hamiltonian functionals mutually commute under both Poisson brackets:

$$\{ H_m, H_n \}_1 = \{ H_m, H_n \}_2 = 0\,, \quad \textit{for all} \quad m,n \geq 1\,. \tag{84}$$

Definition 10 *A set of Hamiltonians that Poisson-commute among themselves is said to be* **in involution**.

Remark 23 *The condition for a canonical Hamiltonian system with N degrees of freedom to be* **completely integrable** *is that it possess N constants of motion in involution. The bi-Hamiltonian property is important because it produces the corresponding condition for an infinite-dimensional system. The infinite-dimensional case introduces additional questions, such as the* **completeness** *of the infinite set of independent constants of motion in involution. However, such questions are beyond our present scope.*

8.2 Applying Magri's lemmas

The bi-Hamiltonian property of eqn (75) allows one to construct an infinite number of Poisson-commuting conservation laws for it by applying Magri's lemmas. According to [Mag78], these conservation laws may be constructed for non-degenerate B_1 by defining the transpose operator $R^T = B_1^{-1}B_2$ that leads from the variational derivative of one conservation law to the next,

$$\frac{\delta H_n}{\delta m} = R^T \frac{\delta H_{n-1}}{\delta m}, \quad n = -1, 0, 1, 2, \ldots . \tag{85}$$

The operator $R^T = B_1^{-1}B_2$ recursively takes the variational derivative of H_{-1} to that of H_0, to that of H_1, then to that of H_2, etc. The next steps are not so easy for the integrable CH hierarchy, because each application of the recursion operator introduces an additional convolution integral into the sequence. Correspondingly, the **recursion operator** $R = B_2B_1^{-1}$ leads to a hierarchy of commuting flows, defined by $\mathcal{K}_{n+1} = R\mathcal{K}_n$, for $n =$

$0, 1, 2, \dots,$

$$
\begin{aligned}
m_t^{(n+1)} &= \mathcal{K}_{n+1}[m] = -B_1 \frac{\delta H_n}{\delta m} \\
&= -B_2 \frac{\delta H_{n-1}}{\delta m} = B_2 B_1^{-1} \mathcal{K}_n[m] \,. \qquad (86)
\end{aligned}
$$

The first three flows in the 'positive hierarchy' when $c_0, \gamma \to 0$ are

$$
m_t^{(1)} = 0 \,, \quad m_t^{(2)} = -\, m_x \,, \quad m_t^{(3)} = -\,(m\partial + \partial m)u \,, \qquad (87)
$$

the third being EPDiff. The next flow is too complicated to be usefully written here. However, by Magri's construction, all of these flows commute with the other flows in the hierarchy, so they each conserve H_n for $n = 0, 1, 2, \dots$.

The recursion operator can also be continued for negative values of n. The conservation laws generated in this way do not introduce convolutions, but care must be taken to ensure the conserved densities are integrable. All the Hamiltonian densities in the negative hierarchy are expressible in terms of

m only and do not involve u. Thus, for instance, the second Hamiltonian in the negative hierarchy of EPDiff is given by

$$m_t = B_1 \frac{\delta H_{-1}}{\delta m} = B_2 \frac{\delta H_{-2}}{\delta m}\,, \tag{88}$$

which gives

$$H_{-2} = \frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\alpha^2}{4}\frac{m_x^2}{m^{5/2}} - \frac{2}{\sqrt{m}}\right]. \tag{89}$$

The flow defined by eqn (88) is

$$m_t = -(\partial - \alpha^2\partial^3)\left(\frac{1}{2\sqrt{m}}\right). \tag{90}$$

For $m = u - \alpha^2 u_{xx}$, this flow is similar to the Dym equation,

$$u_{xxt} = \partial^3\left(\frac{1}{2\sqrt{u_{xx}}}\right)\,, \tag{91}$$

which is also a completely integrable soliton equation [AS06].

9 The CH equation is isospectral

The squared-eigenfunction trick. The isospectral eigenvalue problem associated with eqn (75) may be found by using the recursion relation of the bi-Hamiltonian structure, following a standard technique due to Gelfand and Dorfman [GD79]. Let us introduce a spectral parameter λ and multiply by λ^n the nth step of the recursion relation (86), then taking the sum yields

$$B_1 \sum_{n=0}^{\infty} \lambda^n \frac{\delta H_n}{\delta m} = \lambda B_2 \sum_{n=0}^{\infty} \lambda^{(n-1)} \frac{\delta H_{n-1}}{\delta m}\,, \tag{92}$$

or, by introducing the **squared-eigenfunction** ψ^2

$$\psi^2(x,t;\lambda) := \sum_{n=-1}^{\infty} \lambda^n \frac{\delta H_n}{\delta m}\,, \tag{93}$$

one finds, formally,

$$B_1 \psi^2(x,t;\lambda) = \lambda B_2 \psi^2(x,t;\lambda)\,. \tag{94}$$

This is a third-order eigenvalue problem for the squared-eigenfunction ψ^2, which turns out to be equivalent to a second-order **Sturm–Liouville problem** for ψ.

Proposition 4 *If ψ satisfies*

$$\lambda\left(\frac{1}{4}-\alpha^2\partial_x^2\right)\psi=\left(\frac{c_0}{4}+\frac{m(x,t)}{2}+\gamma\,\partial_x^2\right)\psi\,, \tag{95}$$

then ψ^2 is a solution of eqn (94).

Proof. This is is straightforward computation.

Now, assuming that λ will be independent of time, we seek, in analogy with the KdV equation, an evolution equation for ψ of the form,

$$\psi_t=a\psi_x+b\psi\,, \tag{96}$$

where a and b are functions of u and its derivatives. These functions are determined from the requirement that the **compatibility condition** $\psi_{xxt}=$

ψ_{txx} between eqns (95) and (96) implies eqn (75). Cross-differentiation shows

$$b = -\frac{1}{2}a_x\,, \quad \text{and} \quad a = -\,(\lambda + u)\,. \tag{97}$$

Consequently,

$$\psi_t = -\,(\lambda + u)\psi_x + \frac{1}{2}u_x\psi\,, \tag{98}$$

is the desired evolution equation for the eigenfunction ψ.

Summary of the isospectral property of the CH eqn

The Gelfand–Dorfman theory [GD79] determines the isospectral problem for integrable equations via the squared-eigenfunction approach.
Its bi-Hamiltonian property implies that the nonlinear shallow-water wave CH eqn (75) arises as a compatibility condition for two linear equations.

These are the **isospectral eigenvalue problem**,

$$\lambda\left(\frac{1}{4} - \alpha^2\partial_x^2\right)\psi = \left(\frac{c_0}{4} + \frac{m(x,t)}{2} + \gamma\,\partial_x^2\right)\psi\,, \tag{99}$$

and the **evolution equation** for the eigenfunction ψ,

$$\psi_t = -(u+\lambda)\,\psi_x + \frac{1}{2}u_x\,\psi\,. \tag{100}$$

Compatibility of these linear equations ($\psi_{xxt} = \psi_{txx}$) together with isospectrality

$$\mathrm{d}\lambda/\mathrm{d}t = 0\,,$$

imply eqn (75).

Remark 24 *The isospectral eigenvalue problem (99) for the nonlinear CH water-wave equation (75) restricts to the isospectral problem for KdV (namely, the Schrödinger equation) when* $\alpha^2 \to 0$*. The evolution equation (100) for the isospectral eigenfunctions in the cases of KdV and CH are identical. The isospectral eigenvalue problem and the evolution equation for its eigenfunctions are two linear equations whose compatibility implies a nonlinear equation for the unknowns in the KdV and CH equations. This formulation for the KdV equation led to the famous method of the* **inverse scattering transform (IST)** *for the solution of its initial-value problem, reviewed, e.g., in [AS06]. The CH equation also admits the IST solution approach, but for a different isospectral eigenvalue problem that limits to the Schrödinger equation when* $\alpha^2 \to 0$*. The isospectral eigenvalue problem (99) for CH arises in the study of the fundamental oscillations of a non-uniform string under tension.*

EPDiff is the dispersionless case of CH

In the dispersionless case $c_0 = 0 = \gamma$, the shallow-water equation (75) becomes the 1D geodesic equation EPDiff(H^1)

$$m_t + um_x + 2mu_x = 0\,, \qquad m = u - \alpha^2 u_{xx}\,. \tag{101}$$

The solitary travelling-wave solution of 1D EPDiff (101) in this dispersionless case is the **peakon**,

$$u(x,t) = c\,G(x-ct) = \frac{c}{2\alpha}\,\mathrm{e}^{-|x-ct|/\alpha}\,.$$

The EPDiff equation (30) may also be written as a conservation law for momentum,

$$\partial_t m = -\partial_x\Big(um + \frac{1}{2}u^2 - \frac{\alpha^2}{2}u_x^2\Big)\,. \tag{102}$$

Its isospectral problem forms the basis for completely integrating the EPDiff equation as a Hamiltonian system and, thus, for finding its soliton solutions. Remarkably, the isospectral problem (99) in the dispersionless case $c_0 =$

$0 = \Gamma$ has a **purely discrete spectrum** on the real line and the N-soliton solutions for this equation may be expressed as a **peakon wave train**,

$$u(x,t) = \sum_{i=1}^{N} p_i(t) \mathrm{e}^{-|x-q_i(t)|/\alpha} \,. \tag{103}$$

As before, $p_i(t)$ and $q_i(t)$ satisfy the finite-dimensional geodesic motion equations obtained as Hamilton's canonical equations

$$\dot{q}_i = \frac{\partial H_N}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H_N}{\partial q_i} \,, \tag{104}$$

where the Hamiltonian is given by,

$$H_N = \frac{1}{2} \sum_{i,j=1}^{N} p_i \, p_j \, \mathrm{e}^{-|q_i-q_j|/\alpha} \,. \tag{105}$$

Thus, we have proved the following.

Theorem 3 *CH peakons are an integrable subcase of EPDiff pulsons in one dimension for the choice of the H^1 norm.*

Remark 25 *The discrete process of peakon creation via the steepening lemma 5.1 is consistent with the discreteness of the isospectrum for the eigenvalue problem (99) in the dispersionless case, when* $c_0 = 0 = \gamma$*. These discrete eigenvalues correspond in turn to the asymptotic speeds of the peakons. The discreteness of the isospectrum means that* only peakons *will emerge in the initial-value problem for EPDiff*(H^1) *in 1D.*

Constants of motion for integrable N-peakon dynamics

One may verify the integrability of the N-peakon dynamics by substituting the N-peakon solution (103) (which produces the sum of delta functions (35) for the momentum m) into the isospectral problem (99). This substitution reduces (99) to an $N \times N$ matrix eigenvalue problem.

In fact, the canonical equations (104) for the peakon Hamiltonian (105) may be written directly in Lax matrix form,

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, A] \quad \Longleftrightarrow \quad L(t) = U(t)L(0)U^{\dagger}(t)\,, \tag{106}$$

with $A = \dot{U}U^{\dagger}(t)$ and $UU^{\dagger} = Id$. Explicitly, L and A are $N \times N$ matrices with entries

$$L_{jk} = \sqrt{p_j p_k}\,\phi(q_i - q_j)\,, \quad A_{jk} = -2\sqrt{p_j p_k}\,\phi'(q_i - q_j)\,. \tag{107}$$

Here, $\phi'(x)$ denotes derivative with respect to the argument of the function ϕ, given by $\phi(x) = \mathrm{e}^{-|x|/2\alpha} = 2\alpha G(x/2)$. The Lax matrix L in eqn (106) evolves by time-dependent unitary transformations, which leave its spectrum invariant. Isospectrality then implies that the traces tr L^n, $n = 1, 2, \ldots, N$ of the powers of the matrix L (or, equivalently, its N eigenvalues) yield N constants of the motion. These turn out to be functionally independent, non-trivial and in involution under the canonical Poisson bracket. Hence, the canonically Hamiltonian N-peakon dynamics (104) is completely integrable in the finite-dimensional (Liouville) sense.

Exercise 16 *Verify that the compatibility condition (equality of cross derivatives $\psi_{xxt} = \psi_{txx}$) obtained from the eigenvalue equation (99) and the evolution equation (100) do indeed yield the CH shallow-water wave equation (75) when the eigenvalue λ is constant.*

Exercise 17 *Show that the peakon Hamiltonian H_N in (105) may be expressed as a function of the invariants of the matrix L, as*

$$H_N = -\text{tr}\, L^2 + 2(\text{tr}\, L)^2 \,. \tag{108}$$

Show that evenness of H_N implies

1. *The N coordinates q_i, $i = 1, 2, \ldots, N$ keep their initial ordering.*

2. *The N conjugate momenta p_i, $i = 1, 2, \ldots, N$ keep their initial signs.*

This means that no difficulties arise, either due to the non-analyticity of $\phi(x)$, or the sign in the square roots in the Lax matrices L and A.

Exercise 18 (Hunter–Saxton equation) *Retrace the progress of this lecture for the EPDiff equation*

$$m_t + um_x + 2mu_x = 0\,, \quad \textit{with} \quad m = -u_{xx}\,. \tag{109}$$

This integrable Hamiltonian partial differential equation arises in the theory of liquid crystals. Its peakon solutions are the compactly supported triangles in Figure 12 and Figure 15. It may also be regarded as the $\alpha \to \infty$ limit of the CH equation. For more results and discussion of this equation, see [HZ94].

EPDiff in n dimensions

This lecture discusses the n-dimensional generalization of the one-dimensional singular solutions of the EP equation studied in the previous lecture. Much of the one-dimensional structure persists in higher dimensions. For example, the parameters defining the singular solutions of EPDiff in n dimensions still obey canonical Hamiltonian equations. This is understood by identifying the singular solution ansatz as a cotangent-lift momentum map for the left action of the diffeomorphisms on the lower-dimensional support set of the singular solutions.

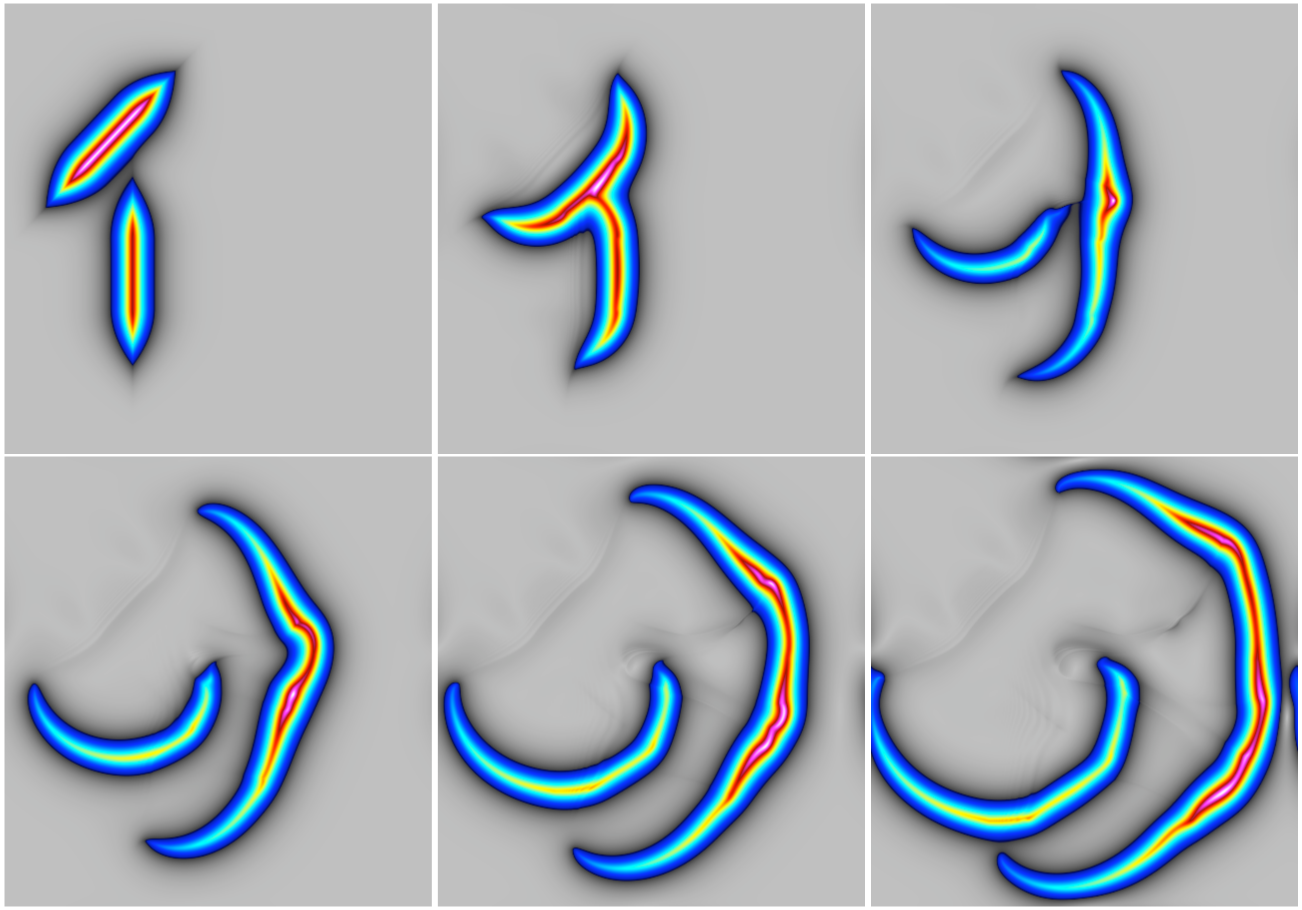


Figure 16: A single collision is shown involving reconnection as the faster peakon segment initially moving Southeast along the diagonal expands, curves and obliquely overtakes the slower peakon segment initially moving rightward (East). This reconnection illustrates one of the collision rules for the strongly two-dimensional EPDiff flow.

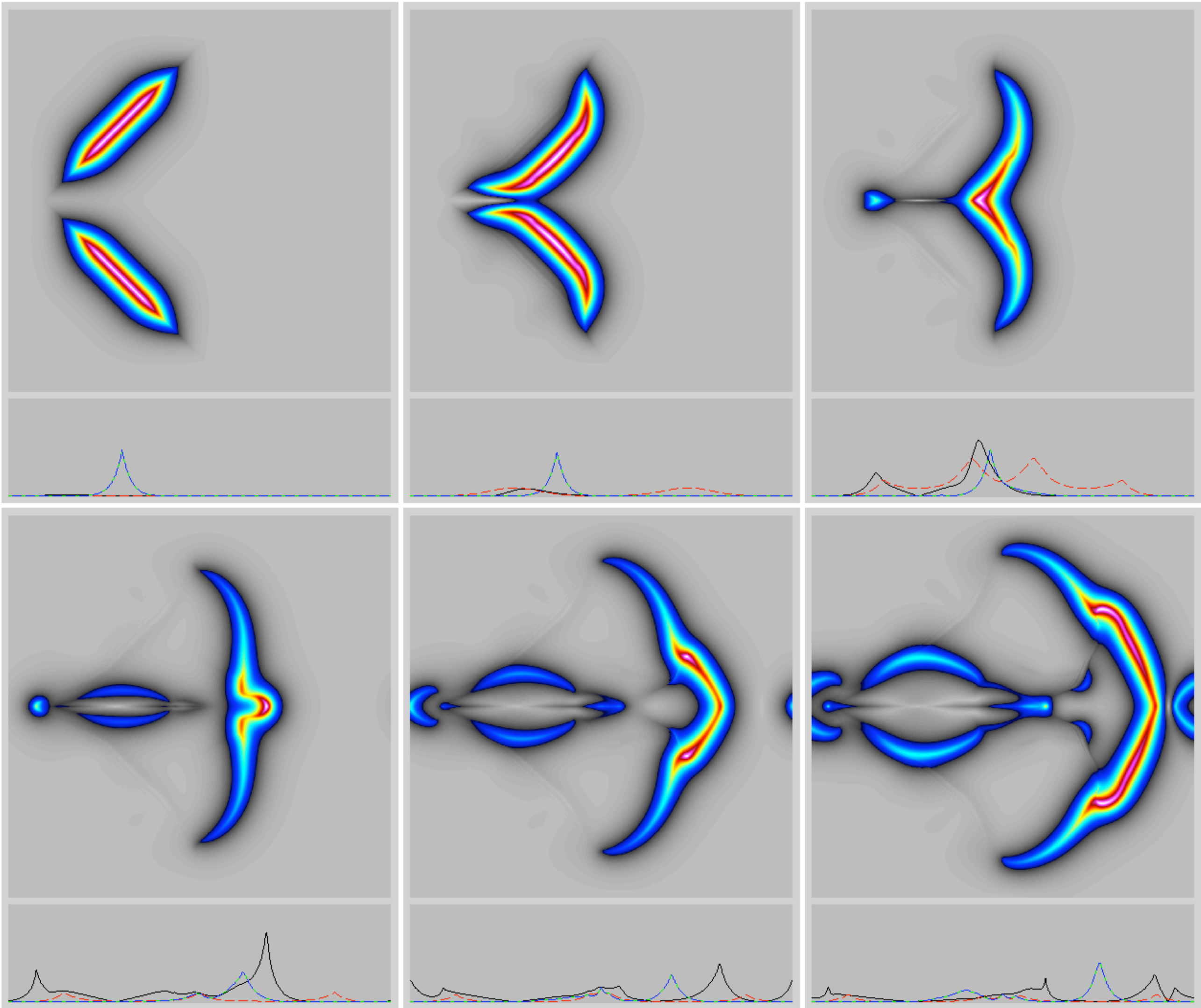


Figure 17: The convergence of two peakon segments moving with reflection symmetry generates considerable acceleration along the midline, which continues to build up after the initial collision.

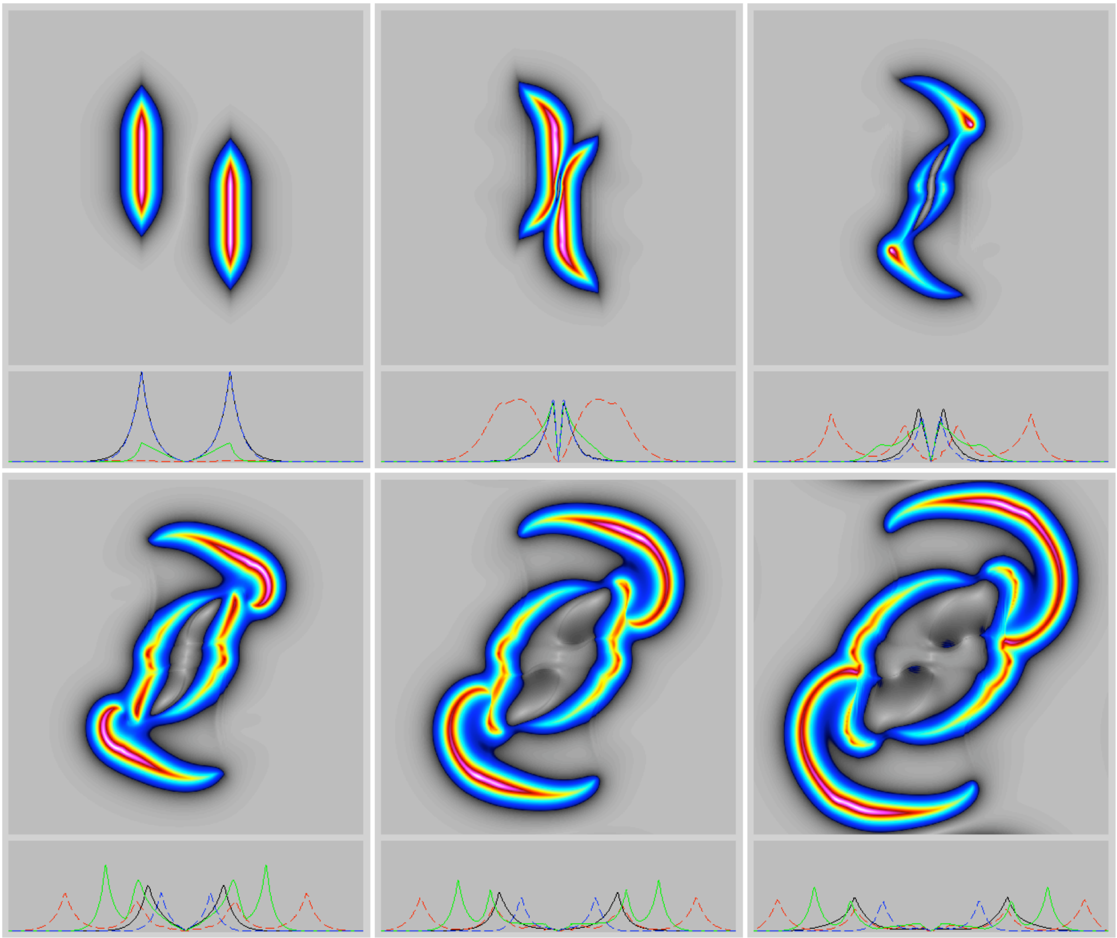


Figure 18: The head-on collision of two offset peakon segments generates considerable complexity. Some of this complexity is due to the process of annihilation and recreation that occurs in the 1D antisymmetric head-on collisions of a peakon with its reflection, the antipeakon, as shown in Figure 12.3. Other aspects of it involve flow along the crests of the peakon segments as they stretch.

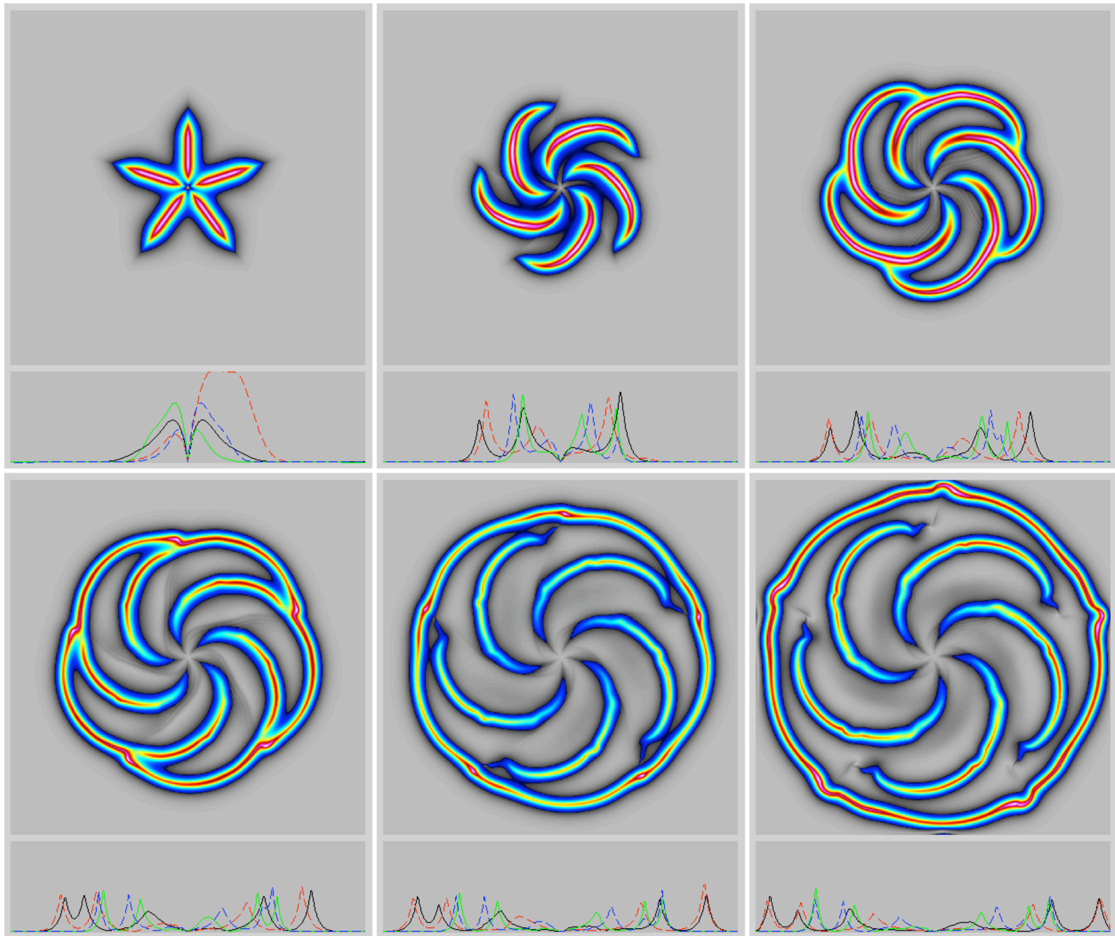


Figure 19: The overtaking collisions of these rotating peakon segments with five-fold symmetry produces many reconnections (mergers), until eventually one peakon ring surrounds five curved peakon segments. If the evolution were allowed to proceed further, reconnections would tend to produce additional concentric peakon rings.

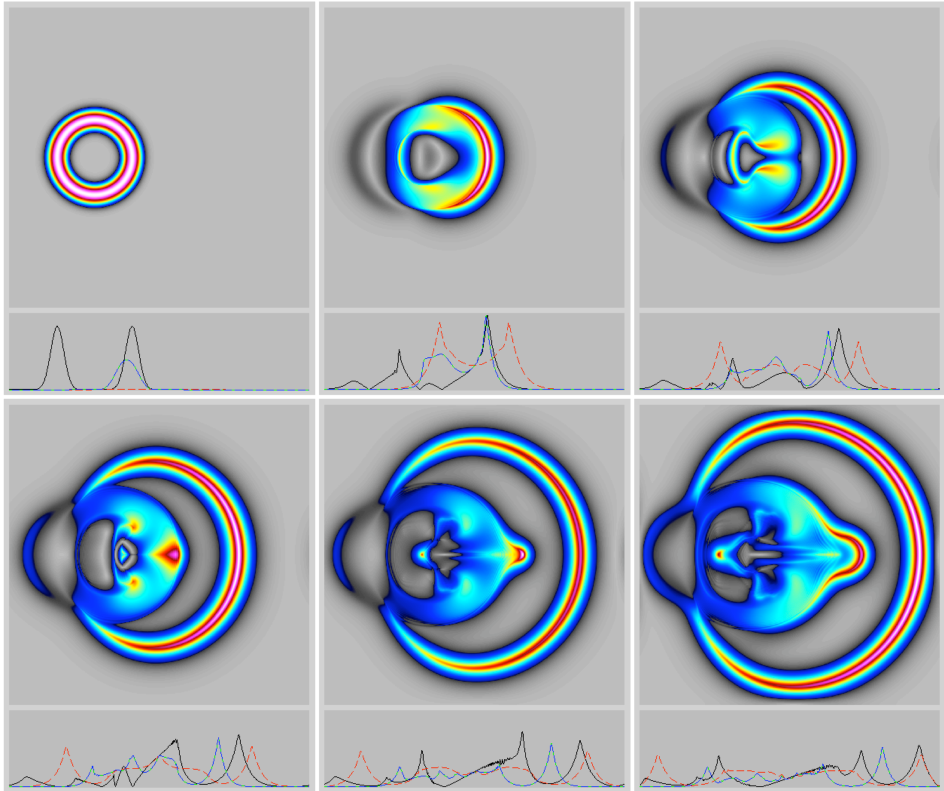


Figure 20: A circular peakon ring initially undergoes uniform rightward translational motion along the x axis. The right outer side of the ring produces diverging peakon curves, which slow as they propagate outward. The left inner side of the ring, however, produces converging peakon segments, which accelerate as they converge. They collide along the midline, then develop into divergent peakon curves still moving rightward that overtake the previous ones and collide with them from behind. These overtaking collisions impart momentum, but they apparently do not produce reconnections.

10 EPDiff solutions in higher dimensions

10.1 n-dimensional EPDiff equation

Eulerian geodesic motion of an ideal continuous fluid in n dimensions is generated as an EP equation via Hamilton's principle, when the Lagrangian is given by the kinetic energy. The kinetic energy defines a norm $\|u\|^2$ for the Eulerian fluid velocity, $\mathbf{u}(\mathbf{x},t): R^n \times R^1 \to R^n$. As mentioned earlier, the choice of the kinetic energy as a positive functional of fluid velocity $\mathbf{u}$ is a modelling step that depends upon the physics of the problem being studied. Following our earlier procedure, as in eqns (11) and (14), we shall choose the Lagrangian,

$$\|\mathbf{u}\|^2 = \int \mathbf{u} \cdot Q_{op}\mathbf{u}\, \mathrm{d}^n x = \int \mathbf{u} \cdot \mathbf{m}\, \mathrm{d}^n x \,, \tag{110}$$

so that the positive-definite, symmetric, operator Q_{op} defines the norm $\|\mathbf{u}\|$, under integration by parts for appropriate boundary conditions and the EPDiff equation for Eulerian geodesic motion of a fluid emerges,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta \ell}{\delta \mathbf{u}} + \mathrm{ad}^*_{\mathbf{u}}\frac{\delta \ell}{\delta \mathbf{u}} = 0\,, \quad \text{with} \quad \ell[\mathbf{u}] = \frac{1}{2}\|\mathbf{u}\|^2\,. \tag{111}$$

Legendre transforming to the Hamiltonian side

The corresponding Legendre transform yields the following invertible relations between momentum and velocity,

$$\mathbf{m} = Q_{op}\mathbf{u} \quad \text{and} \quad \mathbf{u} = G * \mathbf{m}\,, \tag{112}$$

where G is the **Green's function** for the operator Q_{op}, assuming appropriate boundary conditions (on $\mathbf{u}$) that allow inversion of the operator Q_{op} to determine $\mathbf{u}$ from $\mathbf{m}$.

The associated **Hamiltonian** is,

$$h[\mathbf{m}] = \langle \mathbf{m}\,,\,\mathbf{u}\rangle - \frac{1}{2}\|\mathbf{u}\|^2 = \frac{1}{2}\int \mathbf{m}\cdot G*\mathbf{m}\ \mathrm{d}^n x =: \frac{1}{2}\|\mathbf{m}\|^2\,, \tag{113}$$

which also defines a norm $\|\mathbf{m}\|$ via a convolution kernel G that is symmetric and positive, when the Lagrangian $\ell[\mathbf{u}]$ is a norm. As expected, the norm $\|\mathbf{m}\|$ given by the Hamiltonian $h[\mathbf{m}]$ specifies the velocity $\mathbf{u}$ in terms of its Legendre-dual momentum $\mathbf{m}$ by the variational operation,

$$\mathbf{u} = \frac{\delta h}{\delta \mathbf{m}} = G*\mathbf{m} \equiv \int G(\mathbf{x}-\mathbf{y})\,\mathbf{m}(\mathbf{y})\,\mathrm{d}^n y\,. \tag{114}$$

We shall choose the kernel $G(\mathbf{x}-\mathbf{y})$ to be translation-invariant (so Noether's theorem implies that total momentum $\mathbf{M} = \int \mathbf{m}\,\mathrm{d}^n x$ is conserved) and symmetric under spatial reflections (so that $\mathbf{u}$ and $\mathbf{m}$ have the same parity under spatial reflections).

After the Legendre transformation (113), the EPDiff equation (14) appears in its equivalent **Lie–Poisson Hamiltonian form**,

$$\frac{\partial}{\partial t}\mathbf{m} = \{\mathbf{m}, h\} = -\,\mathrm{ad}^*_{\delta h/\delta \mathbf{m}}\mathbf{m}\,. \tag{115}$$

Here the operation $\{\cdot\,,\,\cdot\}$ denotes the Lie–Poisson bracket dual to the (right) action of vector fields amongst themselves by vector-field commutation. That is,

$$\{f\,,\,h\} = -\left\langle \mathbf{m}\,,\,\left[\frac{\delta f}{\delta \mathbf{m}}\,,\,\frac{\delta h}{\delta \mathbf{m}}\right]\right\rangle\,. \tag{116}$$

For more details and additional background concerning the relation of classical EP theory to Lie–Poisson Hamiltonian equations, see [MR02, HMR98]. In a moment we will also consider the **momentum maps** for EPDiff.

10.2 Pulsons in n dimensions

The momentum for the one-dimensional pulson solutions (35) on the real line is supported at points via the Dirac delta measures in its solution ansatz,

$$m(x,t) = \sum_{i=1}^{N} p_i(t)\,\delta\Big(x - q_i(t)\Big)\,, \quad m \in \mathbb{R}\,. \tag{117}$$

We shall develop n-dimensional analogues of these one-dimensional pulson solutions for the Euler–Poincaré equation (23) by generalizing this solution ansatz to allow measure-valued n-dimensional vector solutions $\mathbf{m} \in \mathbb{R}^n$ for which the Euler–Poincaré momentum is supported on codimension-k *subspaces* $\mathbb{R}^{n-k}$ with integer $k \in [1,n]$. For example, one may consider the two-dimensional vector momentum $\mathbf{m} \in \mathbb{R}^2$ in the plane that is supported on one-dimensional curves (momentum fronts). Likewise, in three dimensions, one could consider two-dimensional momentum surfaces (sheets).

The corresponding vector momentum ansatz that we shall use is the following, cf. the pulsons (117),

$$\mathbf{m}(\mathbf{x},t)=\sum_{i=1}^{N}\int\mathbf{P}_i(s,t)\,\delta\Big(\,\mathbf{x}-\mathbf{Q}\,_i(s,t)\,\Big)\mathrm{d}s\,,\quad \mathbf{m}\in\mathbb{R}^n\,. \tag{118}$$

Here, $\mathbf{P}_i,\mathbf{Q}_i\in\mathbb{R}^n$ for $i=1,2,\dots,N$. For example, when $n-k=1$, so that $s\in\mathbb{R}$ is one-dimensional, the delta function in solution (118) supports an evolving family of vector-valued curves, called **momentum filaments**. (For simplicity of notation, we suppress the implied subscript i in the arclength s for each $\mathbf{P}_i$ and $\mathbf{Q}_i$.) The Legendre-dual relations (112) imply that the velocity corresponding to the momentum filament ansatz (118) is,

$$\mathbf{u}(\mathbf{x},t)=G*\mathbf{m}=\sum_{j=1}^{N}\int\mathbf{P}_j(s',t)\,G\Big(\,\mathbf{x}-\mathbf{Q}\,_j(s',t)\,\Big)\mathrm{d}s'\,. \tag{119}$$

Just as for the 1D case of the pulsons, we shall show that substitution of the n-D solution ansatz (118) and (119) into the EPDiff equation (20) produces canonical geodesic Hamiltonian equations for the n-dimensional vector parameters $\mathbf{Q}_i(s,t)$ and $\mathbf{P}_i(s,t)$, $i=1,2,\dots,N$.

Canonical Hamiltonian dynamics of momentum filaments

For definiteness in what follows, we shall consider the example of momentum filaments $\mathbf{m} \in \mathbb{R}^n$ supported on one-dimensional space curves in $\mathbb{R}^n$, so $s \in \mathbb{R}$ is the arclength parameter of one of these curves. This solution ansatz is reminiscent of the Biot–Savart Law for vortex filaments, although the flow is not incompressible. The dynamics of momentum surfaces, for $s \in \mathbb{R}^k$ with $k < n$, follow a similar analysis.

Substituting the momentum filament ansatz (118) for $s \in \mathbb{R}$ and its corresponding velocity (119) into the Euler–Poincaré equation (20), then integrating against a smooth test function $\phi(\mathbf{x})$ implies the following canonical equations (denoting explicit summation on $i, j \in 1, 2, \dots N$),

$$\frac{\partial}{\partial t}\mathbf{Q}_i(s,t) = \sum_{j=1}^{N} \int \mathbf{P}_j' \, G(\mathbf{Q}_i - \mathbf{Q}_j') \mathrm{d}s' = \frac{\delta H_N}{\delta \mathbf{P}_i}, \tag{120}$$

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{P}_i(s,t) &= -\sum_{j=1}^{N} \int \left(\mathbf{P}_i \cdot \mathbf{P}_j'\right) \frac{\partial}{\partial \mathbf{Q}_i} G(\mathbf{Q}_i - \mathbf{Q}_j') \, \mathrm{d}s' \\ &= -\frac{\delta H_N}{\delta \mathbf{Q}_i}, \end{aligned} \tag{121}$$

where $\mathbf{P}_i = \mathbf{P}_i(s,t)$, $\mathbf{P}_j' := \mathbf{P}_j(s',t)$ and

$$G(\mathbf{Q}_i - \mathbf{Q}_j') := G(\mathbf{Q}_i(s,t) - \mathbf{Q}_j(s',t)). \tag{122}$$

The dot product $\mathbf{P}_i \cdot \mathbf{P}_j'$ denotes the inner, or scalar, product of the two vectors $\mathbf{P}_i$ and $\mathbf{P}_j'$ in $T^*\mathbb{R}^n$. Thus, the solution ansatz (118) yields a closed set of **integro-partial-differential equations (IPDEs)** given by (120) and (121) for the vector parameters $\mathbf{Q}_i(s,t)$ and $\mathbf{P}_i(s,t)$ with $i = 1, 2 \dots N$. These equations are generated canonically by the Hamiltonian function

$H_N : (T^*\mathbb{R}^n)^N \to \mathbb{R}$ given by

$$H_N[\mathbf{P},\mathbf{Q}] = \frac{1}{2}\iint \sum_{i\,,\,j=1}^{N} \Big(\mathbf{P}_i \cdot \mathbf{P}_j'\Big)\, G\Big(\mathbf{Q}_i - \mathbf{Q}_j'\Big)\, \mathrm{d}s\, \mathrm{d}s' =: \frac{1}{2}\|\mathbf{P}\|^2 \,. \qquad (123)$$

This Hamiltonian arises by inserting the momentum ansatz (118) into the Hamiltonian (113) obtained from the Legendre transformation of the Lagrangian corresponding to the kinetic energy norm of the fluid velocity. Thus, the evolutionary IPDE system (120) and (121) represents canonically Hamiltonian geodesic motion on the space of curves in R^n with respect to the cometric given on these curves in eqn (123). The Hamiltonian $H_N = \frac{1}{2}\|\mathbf{P}\|^2$ in eqn (123) defines the norm $\|\mathbf{P}\|$ in terms of this cometric that combines convolution using the Green's function G and sum over filaments with the scalar product of momentum vectors in R^n.

Remark 26 *Note that the coordinate s is a Lagrangian label moving with the fluid, since*

$$\frac{\partial}{\partial t}\mathbf{Q}_i(s,t) = \mathbf{u}(\mathbf{Q}_i(s,t),t)\,.$$

Exercise 19 *Explain the meaning of the Hamiltonian equation (115) with Lie–Poisson bracket (116). Discuss the interpretation of* $\{\mathbf{m}, h\}$ *when* $\mathbf{m}$ *is a vector.*

Hint*: write* $\mathbf{m}(\mathbf{x}, t)$ *as a spatial integral by inserting a delta function,*

$$\mathbf{m}(\mathbf{x}, t) = \int_{\mathbb{R}^3} \mathbf{m}(\mathbf{y}, t)\, \delta(\mathbf{x} - \mathbf{y})\, \mathrm{d}^3 y\,.$$

Use this representation to show that the Lie–Poisson bracket (116) yields dynamics in the form of eqn (115),

$$\{\mathbf{m}, h\} = -\left\langle \mathrm{ad}^*_{\delta h/\delta \mathbf{m}} \mathbf{m}\,,\, \delta(\mathbf{x} - \mathbf{y}) \right\rangle = -\,\mathrm{ad}^*_{\delta h/\delta \mathbf{m}} \mathbf{m}\,.$$

11 Singular solution momentum map $\mathrm{J}_{\mathsf{Sing}}$

The momentum filament ansatz (118) reduces the solution of the geodesic EP PDE (20) in $n+1$ dimensions to the system of eqns (120) and (121) of $2N$ canonical evolutionary IPDEs. One can summarize the mechanism by which this process occurs, by saying that the map that implements the canonical $(\mathbf{Q},\mathbf{P})$ variables in terms of singular solutions is a (cotangent bundle) momentum map. Such momentum maps are Poisson maps; so the canonical Hamiltonian nature of the dynamical equations for $(\mathbf{Q},\mathbf{P})$ fits into a general theory that also provides a framework for suggesting other avenues of investigation.

Theorem 4 *The momentum ansatz* (118) *for measure-valued solutions of the* EPDiff *equation* (20)*, defines an equivariant momentum map*

$$\mathbf{J}_{\mathsf{Sing}} : T^*\,\mathsf{Emb}(S,\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)^*\,,$$

called the **singular solution momentum map** *in [HM04].*

Remark 27 *We shall explain the notation used in the theorem's statement in the course of its proof. By 'defines' one means that the momentum solution ansatz (118) expressing* $\mathbf{m}$ *(a vector function of spatial position* $\mathbf{x}$*) in terms of* $\mathbf{Q}, \mathbf{P}$ *(which are functions of* s*) can be regarded as a map from the space of* $(\mathbf{Q}(s), \mathbf{P}(s))$ *to the space of* $\mathbf{m}$*'s. This will turn out to be the Lagrange-to-Euler map for the fluid description of the singular solutions.*

Proof. For simplicity and without loss of generality, let us take $N = 1$ and so suppress the index a. That is, we shall take the case of an isolated singular solution. As the proof will show, this is not a real restriction. To set the **notation**, fix a k-dimensional manifold S with a given volume element and whose points are denoted $s \in S$. Let $\mathrm{Emb}(S, \mathbb{R}^n)$ denote the set of smooth embeddings $\mathbf{Q} : S \to \mathbb{R}^n$. (If the EPDiff equations are taken on a manifold M, replace $\mathbb{R}^n$ with M.) Under appropriate technical conditions, which we shall just treat formally here, $\mathrm{Emb}(S, \mathbb{R}^n)$ is a smooth manifold. (See, for example, [EM70] and [MH94] for a discussion and references.)

The tangent space $T_{\mathbf{Q}}\,\text{Emb}(S, \mathbb{R}^n)$ to $\text{Emb}(S, \mathbb{R}^n)$ at point $\mathbf{Q} \in \text{Emb}(S, \mathbb{R}^n)$ is given by the space of **material velocity fields**, namely the linear space of maps $\mathbf{V} : S \to \mathbb{R}^n$ that are vector fields over the map $\mathbf{Q}$. The dual space to this space will be identified with the space of one-form densities over $\mathbf{Q}$, which we shall regard as maps $\mathbf{P} : S \to (\mathbb{R}^n)^*$. In summary, the cotangent bundle $T^*\,\text{Emb}(S, \mathbb{R}^n)$ is identified with the space of pairs of maps $(\mathbf{Q}, \mathbf{P})$.

These give us the domain space for the singular solution momentum map. Now we consider the action of the symmetry group. Consider the group $G = \text{Diff}$ of diffeomorphisms of the space $\mathbb{R}^n$ in which the EPDiff equations are operating, concretely in our case this is $\mathbb{R}^n$. Let it act on $\mathbb{R}^n$ by **composition on the left**. Namely, for $\eta \in \text{Diff}(\mathbb{R}^n)$, we let

$$\eta \cdot \mathbf{Q} = \eta \circ \mathbf{Q}\,. \tag{124}$$

Now lift this action to the cotangent bundle $T^*\,\text{Emb}(S, \mathbb{R}^n)$ in the standard way. One may consult, for instance, [MR02] for this cotangent-lift construction. However, it is also given explicitly by the variational construction. This lifted action is a symplectic (and hence Poisson) action and has an equivariant momentum map. **This cotangent-lift momentum map for the left action (124) is precisely given by the ansatz (118).**

To see this, one only needs to recall and then apply the general formula for the momentum map associated with an action of a general Lie group G on a configuration manifold Q and cotangent lifted to T^*Q.

First let us recall the general formula. Namely, the momentum map is defined by $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ ($\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of G)

$$\mathbf{J}(\alpha_q) \cdot \xi = \left\langle \alpha_q, \xi_Q(q) \right\rangle , \qquad (125)$$

where $\alpha_q \in T_q^*Q$ and $\xi \in \mathfrak{g}$, where ξ_Q is the infinitesimal generator of the action of G on Q associated to the Lie algebra element ξ, and where

$\left\langle \alpha_q, \xi_Q(q) \right\rangle$ is the natural pairing of an element of T_q^*Q with an element of T_qQ.

Now we **apply formula (125)** to the special case in which the group G is the diffeomorphism group Diff($\mathbb{R}^n$), the manifold Q is Emb($S, \mathbb{R}^n$) and where the action of the group on Emb($S, \mathbb{R}^n$) is given by eqn (124). The Lie algebra of $G =$ Diff is the space $\mathfrak{g} = \mathfrak{X}$ of vector fields. Hence, its dual is naturally regarded as the space of one-form densities. The momentum map is thus a map $\mathbf{J} : T^*\,\text{Emb}(S, \mathbb{R}^n) \to \mathfrak{X}^*$.

With $\mathbf{J}$ given by (125), we only need to work out this formula. First, we shall work out the infinitesimal generators. Let $X \in \mathfrak{X}$ be a Lie algebra element. By differentiating the action (124) with respect to η in the direction of X at the identity element we find the **infinitesimal generator**

$$
X_{\text{Emb}(S,\mathbb{R}^n)}(\mathbf{Q}) = X \circ \mathbf{Q}\,.
$$

Thus, on taking α_q to be the cotangent vector $(\mathbf{Q},\mathbf{P})$, eqn (125) gives

$$\begin{aligned}\langle \mathbf{J}(\mathbf{Q},\mathbf{P}), X\rangle &= \langle (\mathbf{Q},\mathbf{P}), X\circ \mathbf{Q}\rangle \\ &= \int_S P_i(s) X^i(\mathbf{Q}(s))\mathrm{d}^k s\,.\end{aligned}$$

On the other hand, note that the right-hand side of eqn (118), when paired with the Lie algebra element X is

$$\begin{aligned}&\left\langle \int_S \mathbf{P}(s)\,\delta\left(\mathbf{x}-\mathbf{Q}(s)\right)\mathrm{d}^k s, X\right\rangle \\ &= \int_{\mathbb{R}^n}\int_S \Big(P_i(s)\,\delta\left(\mathbf{x}-\mathbf{Q}(s)\right)\mathrm{d}^k s\Big)\, X^i(\mathbf{x})\mathrm{d}^n x \qquad (126)\\ &= \int_S P_i(s) X^i(\mathbf{Q}(s)\mathrm{d}^k s\,.\end{aligned}$$

This shows that the expression given by eqn (118) is equal to $\mathbf{J}$ and so the result is proved.

Remark 28 *An important element left out in this proof so far is that it does not make clear that the momentum map is* **equivariant**, *a condition* *needed for the momentum map to be Poisson. The proof took care of this property automatically since* **momentum maps for cotangent-lifted actions are always equivariant** *and hence are Poisson.*

The proof has shown the following.

Corollary 2 *The singular solution momentum map defined by the singular solution ansatz (118), namely,*

$$\mathbf{J}_{\mathsf{Sing}} : T^* \,\mathsf{Emb}(S, \mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)^* \tag{127}$$

is a **Poisson map** *from the canonical Poisson structure on the cotangent space $T^* \,\mathsf{Emb}(S, \mathbb{R}^n)$ to the Lie–Poisson structure on $\mathfrak{X}(\mathbb{R}^n)^*$.*

This is the fundamental property of the singular solution momentum map. Some of its more sophisticated properties are outlined in [HM04].

Importance of equivariance Equivariance of a momentum map is important, because Poisson brackets among the components of an equivariant momentum map close among themselves and satisfy the Jacobi identity. That is, the following theorem holds.

Theorem 5 Equivariant momentum maps are Poisson.

Proof. A momentum map $J:\ P \longrightarrow \mathfrak{g}^*$ is equivariant, if

$$J \circ \Phi_{g(t)} = \mathrm{Ad}^*_{g(t)^{-1}} \circ J \,,$$

for any curve $g(t) \in G$. This means the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\ \Phi_{g(t)}\ } & P \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \mathfrak{g}^* & \xrightarrow{\ \mathrm{Ad}^*_{g(t)^{-1}}\ } & \mathfrak{g}^* \end{array}$$

The time derivative of the equivariance relation leads to the **infinitesimal equivariance** relation,

$$\{\langle J\,,\,\xi\rangle\,,\,\langle J\,,\,\eta\rangle\} = \Big\langle J\,,\,[\xi,\,\eta]\Big\rangle\,,$$

where ξ, $\eta \in \mathfrak{g}$ and $\{\,\cdot\,,\,\cdot\,\}$ denotes the Poisson bracket on the manifold P. This in turn implies that the momentum map preserves Poisson brackets in the sense that,

$$\{F_1 \circ J\,,\, F_2 \circ J\} = \{F_1\,,\, F_2\}_{LP} \circ J\,,$$

for all $F_1, F_2 \in \mathcal{F}(\mathfrak{g}^*)$, where $\{F_1\,,\, F_2\}_{LP}$ denotes the Lie-Poisson bracket for the appropriate left or right action of $\mathfrak{g}$ on P. That is, equivariance implies infinitesimal equivariance, which is sufficient for the momentum map to be Poisson.

Exercise 20 *Show that cotangent-lift momentum maps are equivariant.*

Pulling back the equations

Since the solution ansatz (118) has been shown in the preceding Corollary to be a **Poisson map**, the pull-back of the Hamiltonian from $\mathfrak{X}^*$ to $T^* \operatorname{Emb}(S, \mathbb{R}^n)$ gives equations of motion on the latter space that project to the equations on $\mathfrak{X}^*$. The functions $\mathbf{Q}^a(s,t)$ and $\mathbf{P}^a(s,t)$ in eqn (118) satisfy canonical Hamiltonian equations. The pull-back of the Hamiltonian $h[\mathbf{m}]$ defined in eqn (113) on the dual of the Lie algebra $\mathfrak{g}^*$, to $T^* \operatorname{Emb}(S, \mathbb{R}^n)$ is easily seen to be consistent with what we had defined before in eqn (123):

$$h[\mathbf{m}] \equiv \frac{1}{2}\langle \mathbf{m}\,,\, G * \mathbf{m}\rangle = \frac{1}{2}\langle\!\langle \mathbf{P}\,,\, G * \mathbf{P}\rangle\!\rangle = H_N[\mathbf{P}, \mathbf{Q}]\,. \tag{128}$$

Since the momentum map $\mathbf{J}_{\text{Sing}}$ is Poisson, the functions $\mathbf{Q}^a(s,t)$ and $\mathbf{P}^a(s,t)$ in eqn (118) satisfy canonical Hamiltonian equations.

Remark 29 *Recall that the coordinate $s \in \mathbb{R}^k$ labeling the functions in eqn (126) is a 'Lagrangian coordinate' in the sense that it does not evolve in time, but merely labels the solution.*

Remark 30 (Summary) *In concert with the Poisson nature of the singular solution momentum map, the singular solutions (118) in terms of* $\mathbf{Q}$ *and* $\mathbf{P}$ *satisfy Hamiltonian equations and also define an invariant solution set for the EPDiff equations. In fact, this invariant solution set is a singular coadjoint orbit for the diffeomorphism group.*

Exercise 21 *Show that the natural pairing relations preserve the stationary principle for the Lagrangian* $\ell[u]$ *under the cotangent lift of Diff*$(\mathbb{R}^n)$.

That is, state the conditions under which the stationary principle $\delta S = 0$ *for*

$$S = \int \ell[u]\,\mathrm{d}t$$

will produce equivalent equations of motion for the two expressions for the Lagrangian $\ell[u]$ *given by,*

$$\begin{aligned} \ell[u] &= \langle \mathbf{m}\,,\,\mathbf{u}\rangle - h[\mathbf{m}] &(129)\\ &= \langle\!\langle\,\mathbf{P}\,,\,\dot{\mathbf{Q}}\,\rangle\!\rangle - H_N[\mathbf{P},\mathbf{Q}]\,, &(130)\end{aligned}$$

Exercise 22 *Verify that the momentum map in (118) is the variation in $\boldsymbol{u}$ of the constrained action*

$$S = \int \ell[u]\,\mathrm{d}t + \int_0^1\!\!\int_{S^1} \Big[\boldsymbol{P}(s,t)\cdot\Big(\dot{\boldsymbol{Q}}(s,t) - \boldsymbol{u}(\boldsymbol{Q}(s,t),t)\Big)\Big]\,\mathrm{d}s\,\mathrm{d}t\,.$$

Hint: *Before taking variations in velocity $\boldsymbol{u}$, insert a delta function as in the momentum map proof in (126)*

$$\begin{aligned}&\int_0^1\!\!\int_{S^1} \boldsymbol{P}(s,t)\cdot\boldsymbol{u}(\boldsymbol{Q}(s,t),t)\,\mathrm{d}s\,\mathrm{d}t\\ &=\int_0^1\!\!\int_{S^1}\!\!\int_{\mathbb{R}^2} \boldsymbol{P}(s,t)\cdot\boldsymbol{u}(\boldsymbol{x},t)\,\delta\Big(\boldsymbol{x}-\boldsymbol{Q}(s,t)\Big)\mathrm{d}^2x\,\mathrm{d}s\,\mathrm{d}t\,.\end{aligned}$$

This yields the momentum map (118) upon varying S with respect to $\boldsymbol{u}$.

Exercise 23 *Find the system of equations defined by minimizing the alternative action,*

$$\mathcal{S} = \int_0^1 \ell(\boldsymbol{u})\,\mathrm{d}t + \frac{1}{2\sigma^2}\int_0^1\!\!\int_{S^1} \Big|\dot{\boldsymbol{Q}}(s,t) - \boldsymbol{u}(\boldsymbol{Q}(s,t),t)\Big|^2\,\mathrm{d}s\,\mathrm{d}t, \qquad \ell = \frac{1}{2}\Big\|\boldsymbol{u}(t)\Big\|_{\mathfrak{g}}^2,$$

where $\sigma^2 > 0$ is a constant parameter, the norm $\|\cdot\|_{\mathfrak{g}}$ defines a metric on the tangent space $\mathfrak{g}$ of the diffeomorphisms and $|\cdot|$ without subscript is the Euclidean metric for vectors. Do the equations for the minimizers of this action still admit the momentum-map relation in (118)?

Hint*: in minimizing this action, the quantity $\dot{\boldsymbol{Q}}(s,t) - \boldsymbol{u}(\boldsymbol{Q}(s,t))$ is minimized by imposing it as a* **penalty***, rather than constraining it to vanish exactly. Before taking variations of this alternative action, it is helpful to rewrite the penalty term equivalently as*

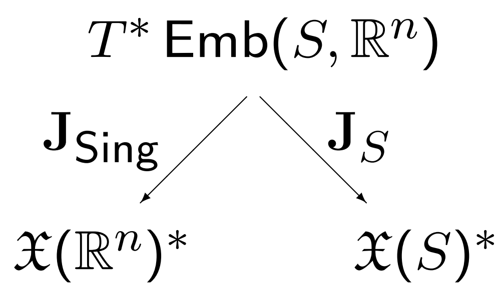
$$\int_0^1\!\!\int_{S^1}\left[\boldsymbol{P}(s,t)\cdot\Big(\dot{\boldsymbol{Q}}(s,t)-\boldsymbol{u}(\boldsymbol{Q}(s,t),t)\Big)-\frac{\sigma^2}{2}|\boldsymbol{P}|^2(s,t)\right]\mathrm{d}s\,\mathrm{d}t\,.$$

This form of the penalty term may be seen to be equivalent to the original one, upon taking stationary variations with respect to $\boldsymbol{P}$ to find the defining relation

$$\sigma^2\boldsymbol{P}(s,t)=\dot{\boldsymbol{Q}}(s,t)-\boldsymbol{u}\Big(\boldsymbol{Q}(s,t),t\Big)\,.$$

In the limit $\sigma^2 \to 0$ the penalty form of the action and its variational equations recovers the corresponding results from the constraint form.

12 The momentum map for right action



12.1 $\mathbf{J}_S$ and the Kelvin circulation theorem

The momentum map $\mathbf{J}_{\mathsf{Sing}}$ involves Diff$(\mathbb{R}^n)$, the **left action** of the diffeomorphism group on the space of embeddings Emb$(S, \mathbb{R}^n)$ by smooth maps of the target space $\mathbb{R}^n$, namely,

$$\mathrm{Diff}(\mathbb{R}^n): \;\; \mathbf{Q} \cdot \eta = \eta \circ \mathbf{Q}, \tag{131}$$

where, recall, $\mathbf{Q} : S \to \mathbb{R}^n$. As before, one identifies the cotangent bundle $T^* \,\mathrm{Emb}(S, \mathbb{R}^n)$ with the space of pairs of maps $(\mathbf{Q}, \mathbf{P})$, with $\mathbf{Q} : S \to \mathbb{R}^n$ and $\mathbf{P} : S \to T^*\mathbb{R}^n$.

The momentum map for right action Another cotangent-lift momentum map $\mathbf{J}_S$ exists, associated with the **right action** of the diffeomorphism group of S on the embeddings Emb$(S, \mathbb{R}^n)$ by smooth maps of the **Lagrangian labels** S (fluid particle relabeling by $\eta_r : S \to S$). This particle-relabelling action is given by

$$\text{Diff}(S): \;\; \mathbf{Q} \cdot \eta_r = \mathbf{Q} \circ \eta_r \,, \tag{132}$$

with parameter $r = 0$ at the identity. The infinitesimal generator of this right action is

$$X_{\text{Emb}(S,\mathbb{R}^n)}(\mathbf{Q}) = \frac{\text{d}}{\text{d}r}\bigg|_{r=0} \mathbf{Q} \circ \eta_r = T\mathbf{Q} \cdot X \,, \tag{133}$$

where $X \in \mathfrak{X}$ is tangent to the curve η_r at $r = 0$. Thus, again taking $N = 1$ (so we suppress the index a) and also letting α_q in the momentum map formula (125) be the cotangent vector $(\mathbf{Q}, \mathbf{P})$, one computes $\mathbf{J}_S$:

$$
\begin{aligned}
\langle \mathbf{J}_S(\mathbf{Q},\mathbf{P}), X\rangle &= \langle (\mathbf{Q},\mathbf{P}), T\mathbf{Q}\cdot X\rangle \\
&= \int_S P_i(s)\frac{\partial Q^i(s)}{\partial s^m}X^m(s)\,\mathsf{d}^k s \\
&= \int_S X\Big(\mathbf{P}(s)\cdot \mathsf{d}\mathbf{Q}(s)\Big)\,\mathsf{d}^k s \\
&= \Big(\int_S \mathbf{P}(s)\cdot \mathsf{d}\mathbf{Q}(s)\otimes \mathsf{d}^k s\,, X(s)\Big) \\
&= \langle\, \mathbf{P}\cdot \mathsf{d}\mathbf{Q}\,, X\,\rangle\,.
\end{aligned}
$$

Consequently, the momentum map formula (125) yields

$$
\mathbf{J}_S(\mathbf{Q},\mathbf{P}) = \mathbf{P}\cdot \mathsf{d}\mathbf{Q}\,, \tag{134}
$$

with the indicated pairing of the one-form density $\mathbf{P}\cdot\mathsf{d}\mathbf{Q}$ with the vector field X. We have set things up so that the following is true.

Proposition 5 *The momentum map $\mathbf{J}_S$ is preserved by the evolution equations (120) and (121) for $\mathbf{Q}$ and $\mathbf{P}$.*

Proof. It is enough to notice that the Hamiltonian H_N in eqn (123) is invariant under the cotangent lift of the **right action** of Diff(S), which amounts to the invariance of the integral over S with respect to reparametrization given by the change of variables formula. (Keep in mind that $\mathbf{P}$ includes a density factor.)

Remark 31

- *The result of Proposition 5 is similar to the* **Kelvin–Noether theorem** *for circulation* Γ *of an ideal fluid, which may be written as*

$$\Gamma = \oint_{c(s)} D(s)^{-1}\mathbf{P}(s) \cdot \mathrm{d}\mathbf{Q}(s)\,,$$

for each Lagrangian circuit $c(s)$*, where* D *is the mass density and* $\mathbf{P}$ *is again the canonical momentum density. This similarity should come as no surprise, because the Kelvin–Noether theorem for ideal fluids arises from invariance of Hamilton's principle under fluid-parcel relabelling by the same right action of the diffeomorphism group as in (132).*

- *Note that, being an equivariant momentum map, the map* $\mathbf{J}_S$*, as with* $\mathbf{J}_{\mathsf{Sing}}$*, is also a Poisson map. Hence, substituting the canonical Poisson bracket into relation* (134)*; that is, the relation*

$$\mathbf{M}(\mathbf{x}) = \sum_i P_i(\mathbf{x})\nabla Q^i(\mathbf{x})$$

yields the Lie–Poisson bracket on the space of $\mathbf{M} \in \mathfrak{X}^*$. *We use the different notations* $\mathbf{m}$ *and* $\mathbf{M}$ *because these quantities are analogous to the body and spatial angular momentum for rigid body mechanics. In fact, the quantity* $\mathbf{m}$ *given by the solution Ansatz* $\mathbf{m} = \mathbf{J}_{\mathsf{Sing}}(\mathbf{Q}, \mathbf{P})$ *gives the singular solutions of the EPDiff equations, while the expression*

$$\mathbf{M}(\mathbf{x}) = \mathbf{J}_S(\mathbf{Q}, \mathbf{P}) = \sum_i P_i(\mathbf{x}) \nabla Q^i(\mathbf{x})$$

is a conserved quantity.

- *In the language of fluid mechanics, the expression of* $\mathbf{m}$ *in terms of* $(\mathbf{Q}, \mathbf{P})$ *is an example of a* **Clebsch representation**, *which expresses the solution of the EPDiff equations in terms of canonical variables that evolve by standard canonical Hamilton equations. This has been known in the case of fluid mechanics for more than 100 years. For modern discussions of the Clebsch representation for ideal fluids, see, for example, [HK83, MW83, CM87].*

- *One last remark is in order. Namely, the special case in which* $S = M$ *is of course allowed. In this case,* $\mathbf{Q}$ *corresponds to the time evolution* *map η_t and $\mathbf{P}$ corresponds to its conjugate momentum. The quantity $\mathbf{m}$ corresponds to the spatial (dynamic) momentum density (that is, right translation of $\mathbf{P}$ to the identity), while $\mathbf{M}$ corresponds to the conserved 'body' momentum density (that is, left translation of $\mathbf{P}$ to the identity).*

Exercise 24 *To investigate the space-versus-body aspects discussed in the last of these remarks, derive the Euler-Poincaré equation (111) as an* **optimal control problem** *obtained by minimising the* **alternative action integral**,

$$S = \int L(u, w, \eta)\,\mathsf{d}t = \int l(u) + \frac{1}{2\sigma^2}\Big| w - \mathsf{Ad}_{\eta^{-1}}\, u \Big|^2_{\mathfrak{X}}\,\mathsf{d}t\,, \qquad (135)$$

where $w := \eta^{-1}\dot{\eta}$ is a left-invariant vector field under $\mathsf{Diff}(M)$. The second summand imposes a penalty that strengthens as $\sigma^2 \to 0$. This penalty function introduces a Riemannian structure that defines a norm $|\cdot|_{\mathfrak{X}}$ via the L^2 inner product $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{X}^* \times \mathfrak{X} \to \mathbb{R}$.

12.2 Brief summary

Emb$(S, \mathbb{R}^n)$ admits two group actions. These are: the group Diff(S) of diffeomorphisms of S, which acts by composition on the *right*; and the group Diff$(\mathbb{R}^n)$, which acts by composition on the *left*. The group Diff$(\mathbb{R}^n)$ acting from the left produces the singular solution momentum map, $\mathbf{J}_{\text{Sing}}$. The action of Diff(S) from the right produces the conserved momentum map,

$$\mathbf{J}_S : T^* \,\text{Emb}(S, \mathbb{R}^n) \to \mathfrak{X}(S)^* \,.$$

The two momentum maps may be assembled into a single figure as follows:

$$\begin{array}{ccc} & T^* \,\text{Emb}(S, M) & \\ \mathbf{J}_{\text{Sing}} \swarrow & & \searrow \mathbf{J}_S \\ \mathfrak{X}(M)^* & & \mathfrak{X}(S)^* \end{array}$$

13 Numerical simulations of EPDiff in 2D

Many open problems and other future applications remain for the EPDiff equation. For example, its analysis requires development of additional methods for PDEs. In particular, while its smooth solutions satisfy a local existence theorem that is analogous to the famous Ebin–Marsden theorem for the Euler fluid equations [EM70], its singular solutions inevitably emerge from smooth initial conditions in its initial-value problem. The implications of this observation are discussed briefly in [HM04], where it is conjectured that these singular solutions may arise from incompleteness of the geodesic flows on the diffeomorphisms. This conjecture emphasizes the opportunities for future analysis of the emergence of measure-valued solutions from smooth initial conditions in nonlinear non-local PDEs. We close this lecture by giving a few examples of the evolutionary behavior of EPDiff singular solutions in simple two-dimensional situations from [HS04].

Figure 21 shows the results for the EPDiff equation when a straight peakon segment of finite length and transverse profile $u(x) = \mathrm{e}^{-|x|/\alpha}$ is created initially moving rightward (East). In adjusting to the condition of zero speed at its ends and the finite speed in its interior, the initially straight segment expands outward as it propagates and curves into a peakon "bubble." This adjustment and change of shape requires propagation along the wave crest. (Indeed, the wave crest gets longer.)

Figure 22 shows an initially straight segment whose velocity distribution is exponential in the transverse direction, $u(x) = \mathrm{e}^{-|x|/\alpha}$, but the width α is 5 times wider than the lengthscale in the EPDiff equation. This initial velocity distribution evolves under EPDiff to separate into a train of curved peakon 'bubbles,' each of width α. This example illustrates the emergent property of the peakon solutions in two dimensions.

Figure 23 shows an oblique wave-front collision that produces reconnections for the EPDiff equation in two dimensions. Figure 23 shows a single oblique overtaking collision, as a faster expanding peakon wave front overtakes a slower one and reconnects with it at the collision point via flow along the wave crest.

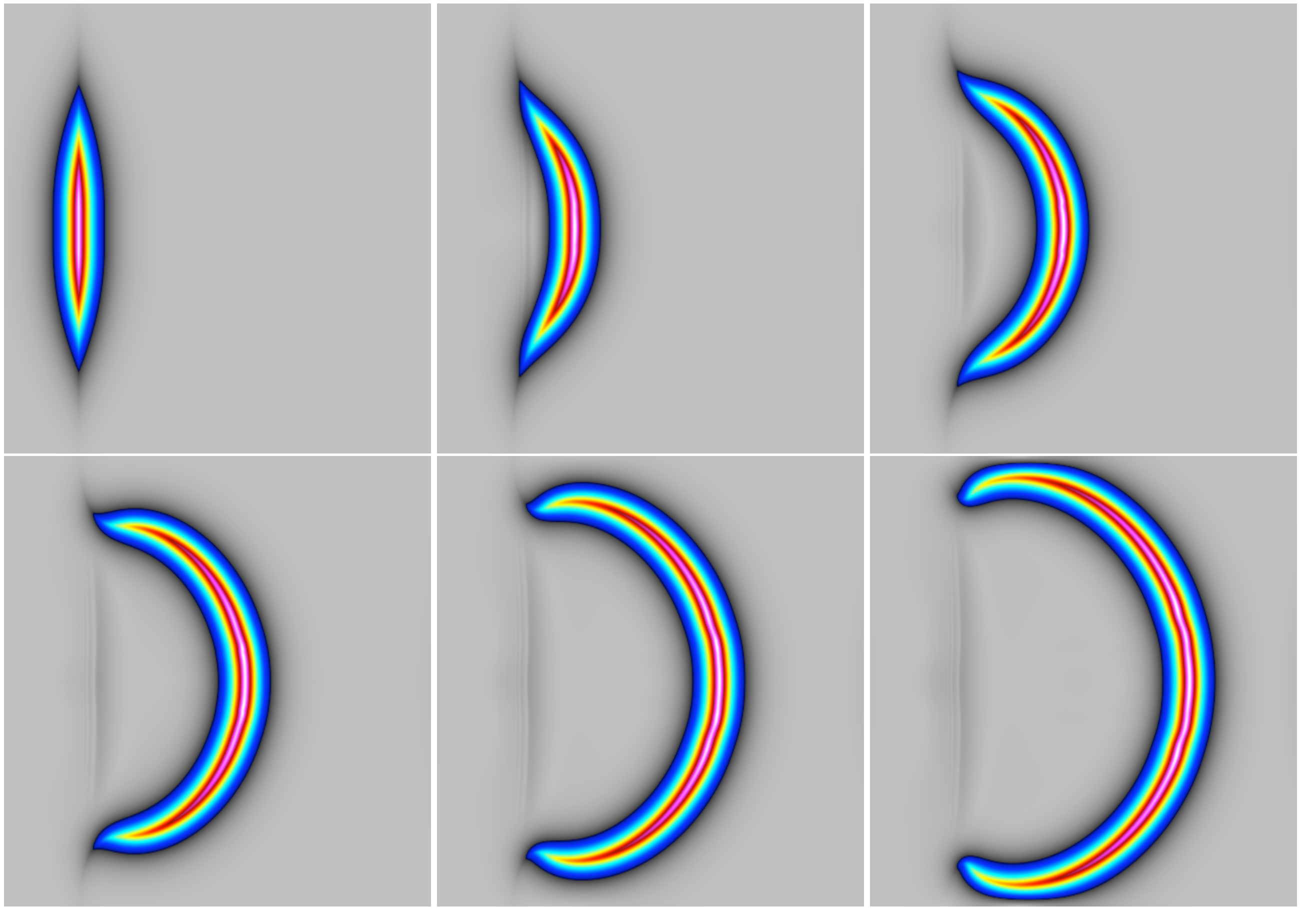


Figure 21: A peakon segment of finite length is initially moving rightward (East). Because its speed vanishes at its ends and it has fully two-dimensional spatial dependence, it expands into a peakon 'bubble' as it propagates. (The colors indicate speed: red is highest, yellow is less, blue low, grey zero.)

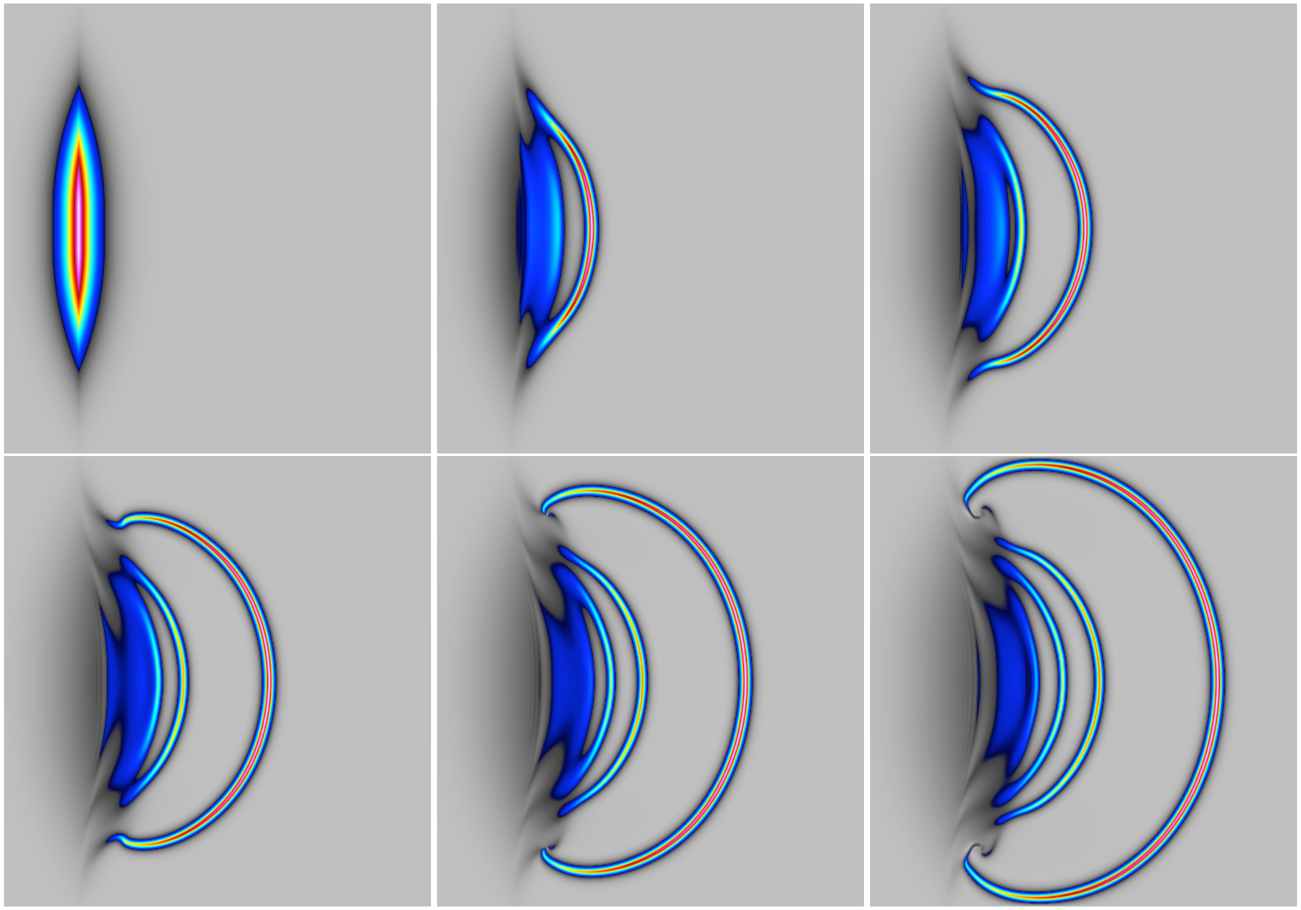


Figure 22: An initially straight segment of velocity distribution whose exponential profile is wider than the width of the peakon solution will break up into a **train** of curved peakon 'bubbles'. This example illustrates the emergent property of the peakon solutions in two dimensions.

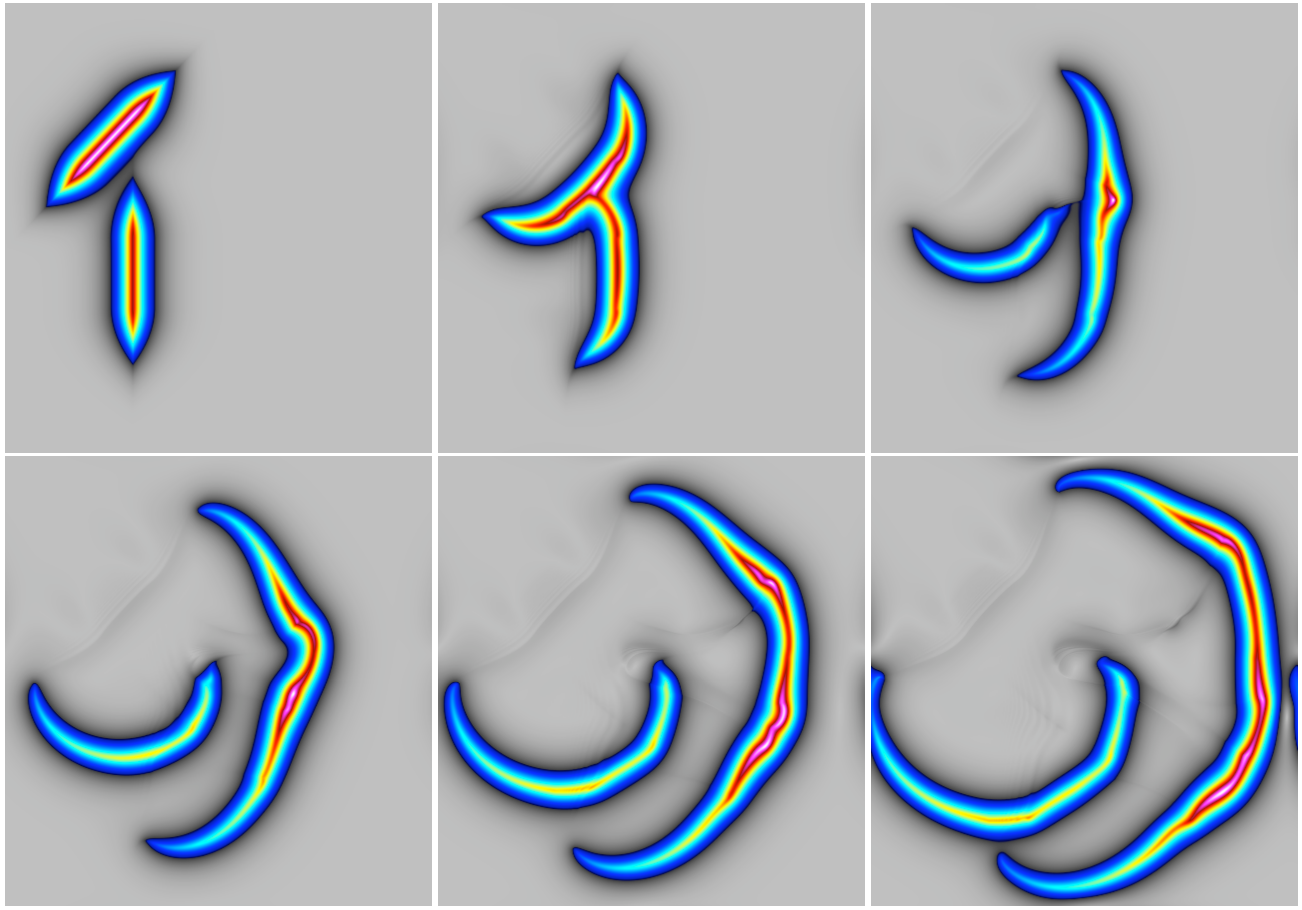


Figure 23: A single collision is shown involving reconnection as the faster peakon segment initially moving Southeast along the diagonal expands, curves and obliquely overtakes the slower peakon segment initially moving rightward (East). This reconnection illustrates one of the collision rules for the strongly two-dimensional EPDiff flow.

Nonlinear wave reconnection. The phenomenon of nonlinear wave reconnection is also observed in Nature. For example, it may be seen in the images taken from the Space Shuttle of trains of internal waves in the South China Sea shown in Figures 24 and 25. These transbasin oceanic internal waves are some of the most impressive wave fronts seen in Nature. About 200 kilometres in length and separated by about 75 kilometres, they are produced every twelve hours by the tide through the Luzon strait between Taiwan and the Phillipines. They may be observed as they propagate and interact with each other and with geographic features. The characteristic property of these strongly nonlinear wavefronts is that they reconnect when two of them collide transversely, as seen in Figures 23–25.

Figures 26–29 show additional collision configurations of peakon segments moving in the plane [HS04].

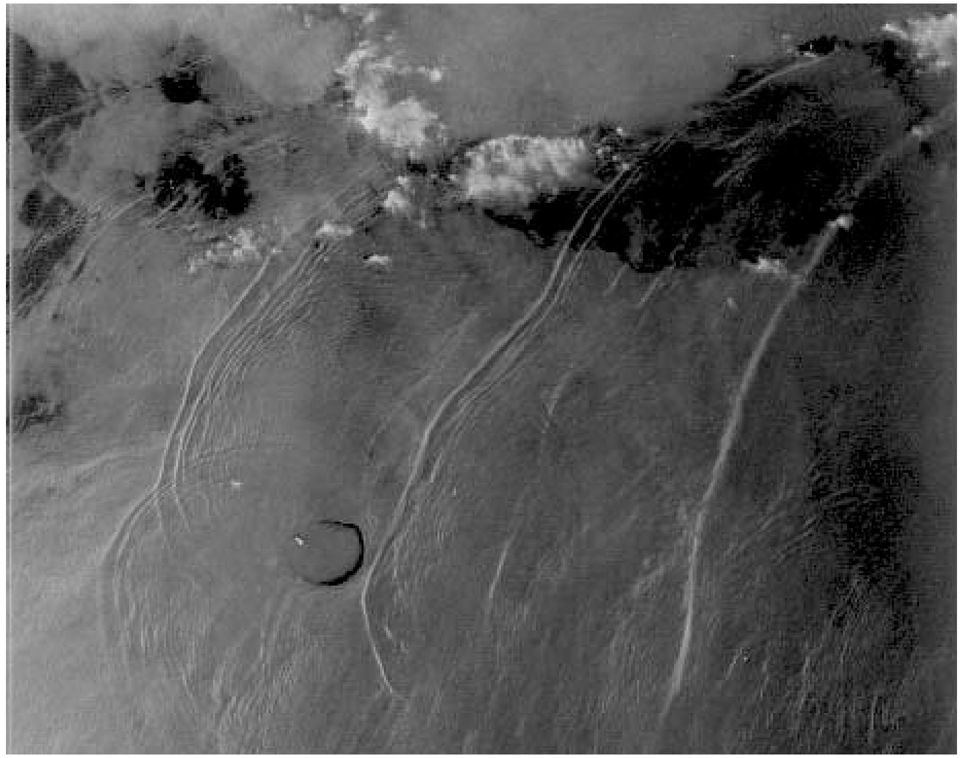


Figure 24: Satellite image using synthetic aperture radar (SAR) of internal wave fronts propagating westward in the South China Sea. A multiwave merger occurs in the region West of the Dong-Sha atoll, which is about 40 km in diameter. An expanded view of this nonlinear wave merger is shown in Figure 25. SAR images from A. Liu, private communication.

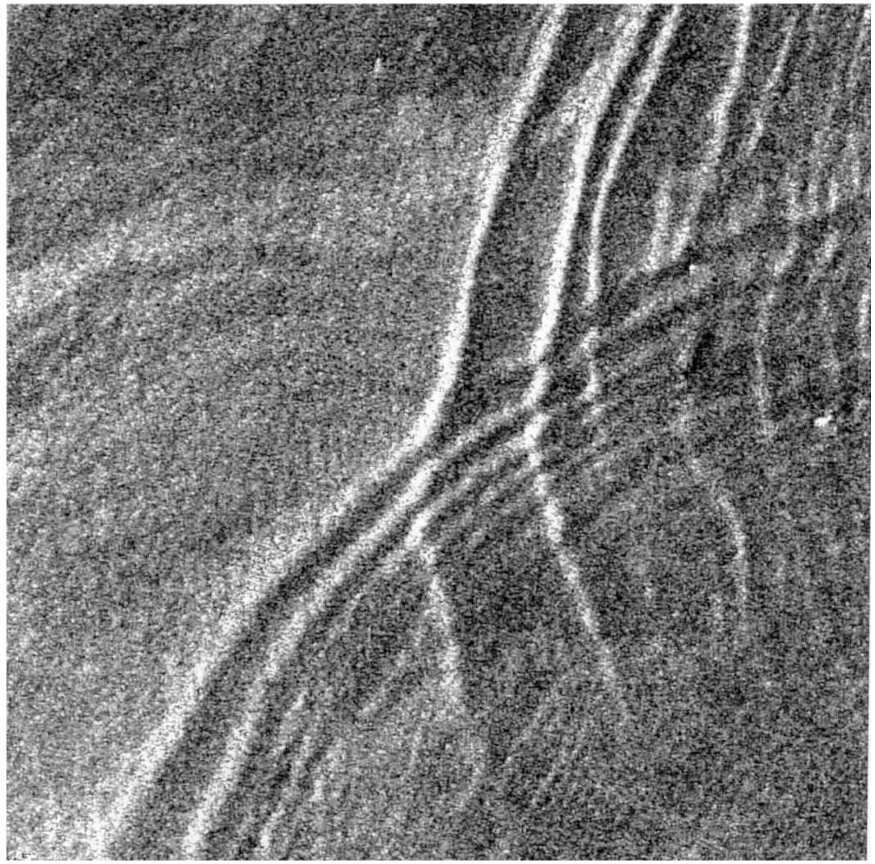


Figure 25: SAR image of nonlinear internal waves West of DongSha in the South China Sea shows merger upon collision due to flow along the wave crests. This sort of merger with flow along the crests of the waves is also seen in the numerical simulations of EPDiff in the plane shown in Figure 23.

Solutions to selected exercises

Solution to Exercise 24

To avoid confusion with earlier notation $\eta \in \text{Diff}(S)$ for the action of diffeomorphisms on an embedded submanifold S, we denote elements of $\text{Diff}(M)$ as $g \in \text{Diff}(M)$. The cross-derivative identities for $\dot{g} = gw$ and $g' = g\xi$ yield the standard formula for variations of the left-invariant velocity,

$$\dot{g}' = g'w + gw' = \dot{g}\xi + g\dot{\xi} \quad \Longrightarrow \quad w' = \dot{\xi} + \text{ad}_w\, \xi\,, \tag{136}$$

where prime ($'$) denotes variational derivative and $w' = \delta w$ is the variation in w inherited from the variation in g, $g' = \delta g$. This formula will be substituted into the variation of the action integral in eqn (135) given by

$$0 = \delta S = \delta\!\int L(u,w,\eta)\,\mathrm{d}t = \int \Big\langle \frac{\partial l}{\partial u},\, u' \Big\rangle + \Big\langle p,\, w' - (\mathrm{Ad}_{g^{-1}}\, u)' \Big\rangle\, dt \tag{137}$$

where the momentum 1-form density, p, dual to the vector field w, is given by

$$p := \frac{\delta L}{\delta w} = \frac{1}{\sigma^2}\Big(w - \mathrm{Ad}_{g^{-1}}\, u\Big) \tag{138}$$

and the pairing by the L^2 inner product $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{X}^* \times \mathfrak{X} \longrightarrow \mathbb{R}$ is induced by the variational-derivative operation from the Riemannian structure introduced by the penalty term. Formula (136) gives the variation ω' in eqn (137) in terms of the vector field $\xi = g^{-1}g' \in \mathfrak{X}$. One calculates the other variation as

$$(\mathrm{Ad}_{g^{-1}}\,\Omega)' = (g^{-1}\Omega g)' = \mathrm{Ad}_{g^{-1}}\, u' + \mathrm{ad}_{(\mathrm{Ad}_{g^{-1}}\, u)}\, \xi\,.$$

Hence, the variation of the action integral in (137) becomes

$$
\begin{aligned}
0 = \delta S &= \int \Big\langle \frac{\delta l}{\delta u}\,,\, u' \Big\rangle + \Big\langle p\,,\, \dot\xi + \mathrm{ad}_w\,\xi - \mathrm{ad}_{(\mathrm{Ad}_{g^{-1}}u)}\,\xi - \mathrm{Ad}_{g^{-1}}\,u') \Big\rangle\, dt \\
&= \int \Big\langle \frac{\delta l}{\delta u} - \mathrm{Ad}^*_{g^{-1}}\,p\,,\, u' \Big\rangle - \Big\langle \dot p - \mathrm{ad}^*_\omega\,\pi + \mathrm{ad}^*_{(\mathrm{Ad}_{g^{-1}}u)}\,p\,,\, \xi \Big\rangle\, dt\,,
\end{aligned}
$$

where we assume endpoint terms may be ignored when integrating by parts. Requiring the coefficients of the independent variations to vanish yields the expressions we seek,

$$
\begin{aligned}
m := \frac{\delta l}{\delta u} &= \mathrm{Ad}^*_{g^{-1}}\,p\,, \\
\dot p - \mathrm{ad}^*_w\,p &= -\,\mathrm{ad}^*_{(\mathrm{Ad}_{g^{-1}}u)}\,p\,.
\end{aligned}
\tag{139}
$$

The first of these relates the momenta $p, m \in \mathfrak{X}^*$ dual to the vector fields $w, u \in \mathfrak{X}$ exactly as the spatial and body angular momenta are related for the rigid body.

The variational eqns (139) imply, when paired with a fixed vector field

$\xi \in \mathfrak{X}$, that

$$
\begin{aligned}
\frac{d}{dt}\langle m\,,\,\xi\rangle &= \frac{d}{dt}\Big\langle \mathsf{Ad}^*_{g^{-1}}\,p\,,\,\xi\Big\rangle \\
\text{On taking } \tfrac{d}{dt}\,\mathsf{Ad}^*_{g^{-1}} &= \Big\langle \mathsf{Ad}^*_{g^{-1}}\Big(\dot p - \mathsf{ad}^*_w\,p\Big)\,,\,\xi\Big\rangle \\
\text{On using } p\text{-eqn (139)} &= -\Big\langle \mathsf{Ad}^*_{g^{-1}}\Big(\mathsf{ad}^*_{(\mathsf{Ad}_{g^{-1}}u)}\,p\Big)\,,\,\xi\Big\rangle \\
\text{On using Ad \& ad definitions} &= -\Big\langle p\,,\,\mathsf{ad}_{(\mathsf{Ad}_{g^{-1}}u)}\Big(\mathsf{Ad}_{g^{-1}}\,\xi\Big)\Big\rangle \\
\text{On rearranging} &= -\Big\langle p\,,\,\mathsf{Ad}_{g^{-1}}\Big(\mathsf{ad}_u\,\xi\Big)\Big\rangle \\
\text{On taking duals} &= -\Big\langle \mathsf{ad}^*_u\Big(\mathsf{Ad}^*_{g^{-1}}\,p\Big)\,,\,\xi\Big\rangle \\
\text{On substituting the definition of } m &= -\langle \mathsf{ad}^*_u\,m\,,\,\xi\rangle\,.
\end{aligned}
$$

This recovers EPDiff, the Euler-Poincaré equation,

$$
\frac{d}{dt}\frac{\delta l}{\delta u} = -\,\mathsf{ad}^*_u\,\frac{\delta l}{\delta u}\,, \quad \text{or} \quad \frac{d}{dt}\Big(\mathsf{Ad}^*_g\,\frac{\delta l}{\delta u}\Big) = 0\,.
$$

Thus, using a penalty term in the action integral to impose the action of $\mathsf{Ad}_{g^{-1}}$ on vector fields as a 'soft constraint' when $\sigma^2 > 0$ yields EPDiff dynamics for coadjoint motion on the L^2 dual, $\mathfrak{X}^*$, of the right-invariant Lie-algebra, $\mathfrak{X}$.

Equation (139) for p may also be written as, cf. eqn (138),

$$\dot{p} - \sigma^2 \, \mathsf{ad}^*_p \, p = 0 \, . \tag{140}$$

Since ad^* and Lie derivative with respect to a vector field are the same for 1-form densities, this relation for the evolution of the left-invariant momentum density may be interpreted as

$$\frac{d}{dt}\Big(\mathbf{p}\cdot d\mathbf{x}\otimes dV\Big) = 0 \quad \text{along} \quad \frac{d\mathbf{x}}{dt} = -\sigma^2 \mathbf{p} \, . \tag{141}$$

In Euclidean components, this is

$$\partial_t p_i = \sigma^2 \frac{\partial}{\partial x^j}\left(p_i p^j + \frac{1}{2}\delta^j_i |\mathbf{p}|^2 \right) \, ,$$

which implies conservation of the integrated left linear momentum,

$$\frac{d}{dt}\int p_i(\mathbf{x},t)\, d^3x = 0 \, ,$$

for homogeneous boundary conditions. This is the analogue for EPDiff of the conservation of spatial angular momentum for the rigid body.

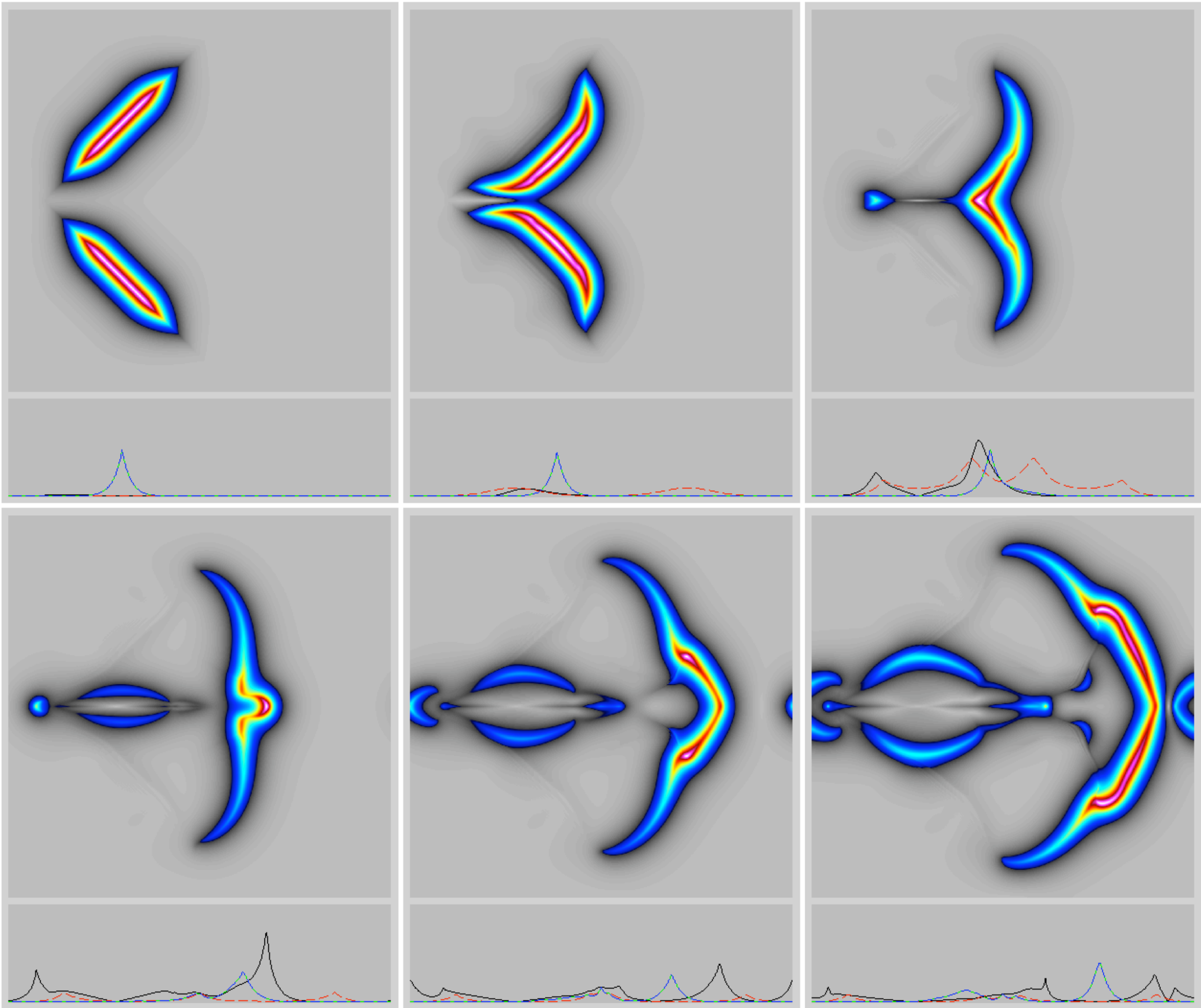


Figure 26: The convergence of two peakon segments moving with reflection symmetry generates considerable acceleration along the midline, which continues to build up after the initial collision.

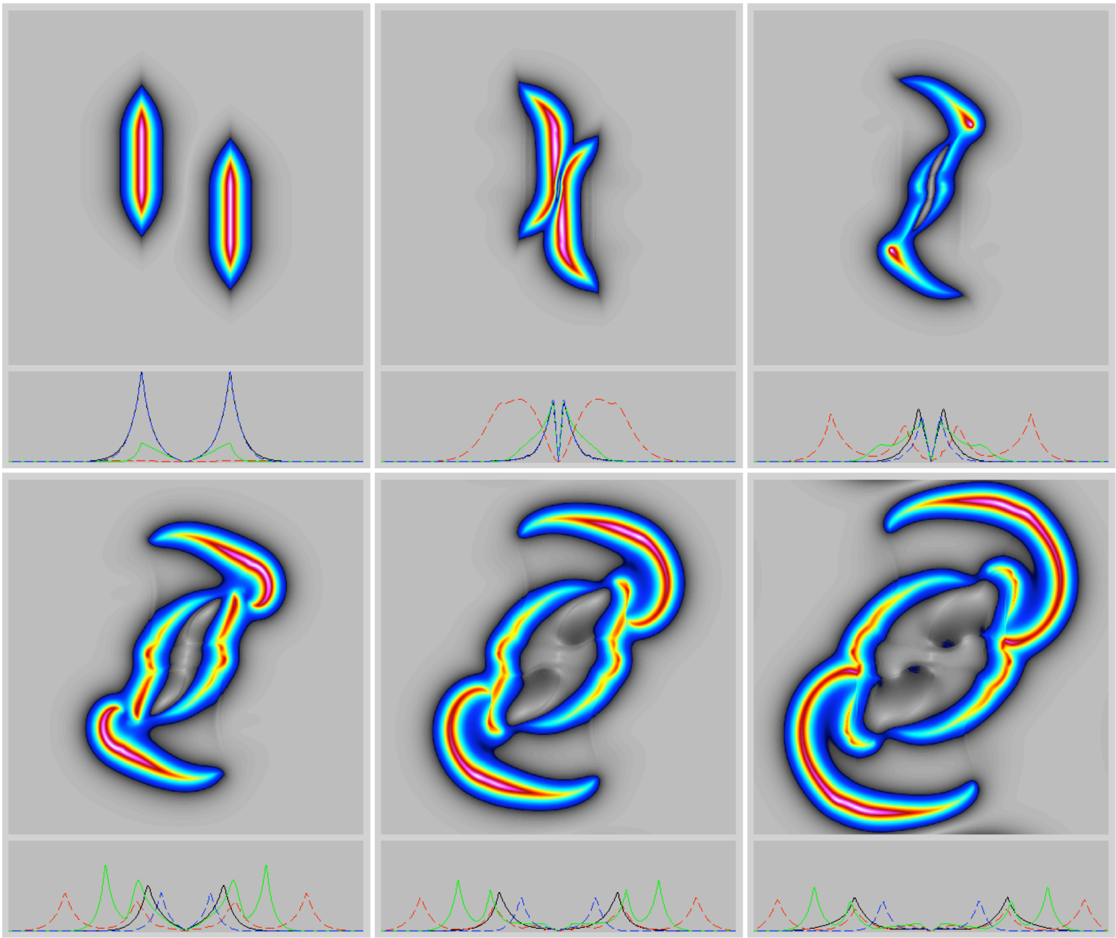


Figure 27: The head-on collision of two offset peakon segments generates considerable complexity. Some of this complexity is due to the process of annihilation and recreation that occurs in the 1D antisymmetric head-on collisions of a peakon with its reflection, the antipeakon, as shown in Figure 12.3. Other aspects of it involve flow along the crests of the peakon segments as they stretch.

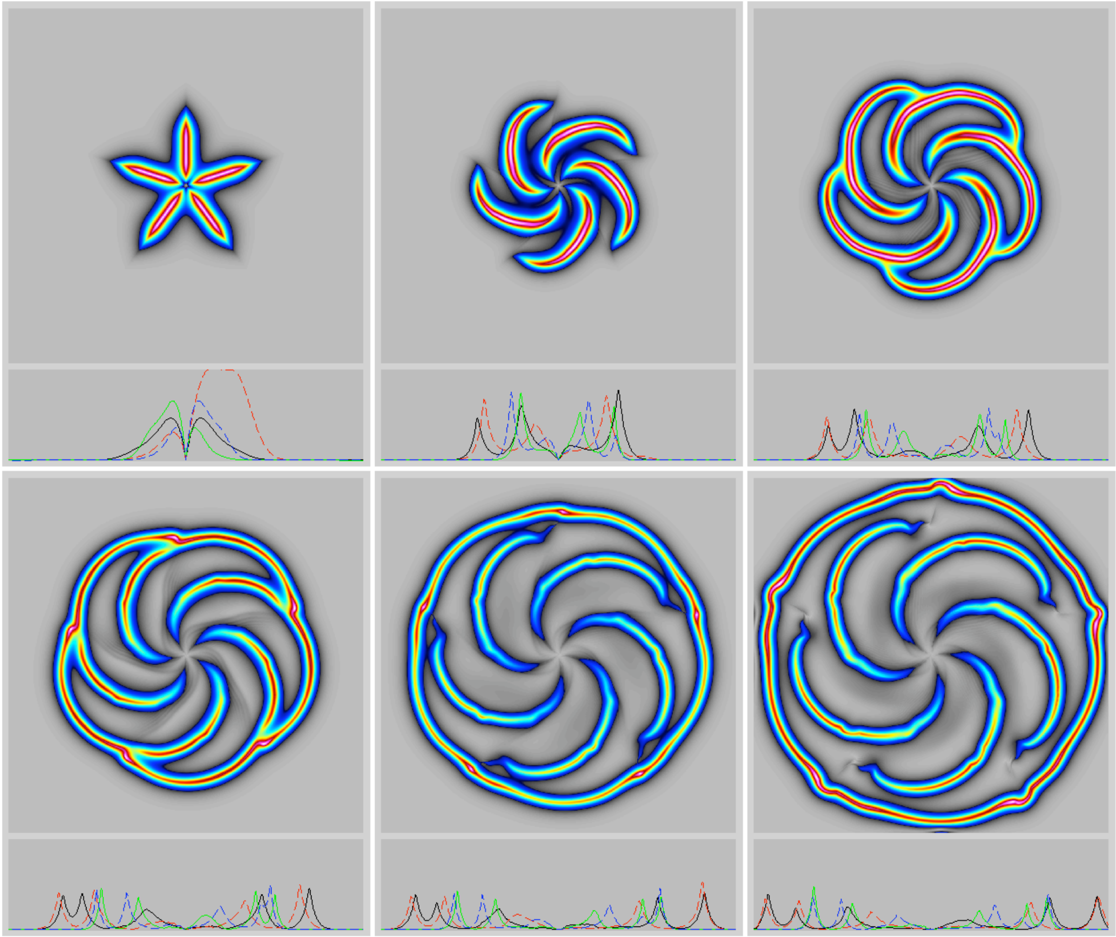


Figure 28: The overtaking collisions of these rotating peakon segments with five-fold symmetry produces many reconnections (mergers), until eventually one peakon ring surrounds five curved peakon segments. If the evolution were allowed to proceed further, reconnections would tend to produce additional concentric peakon rings.

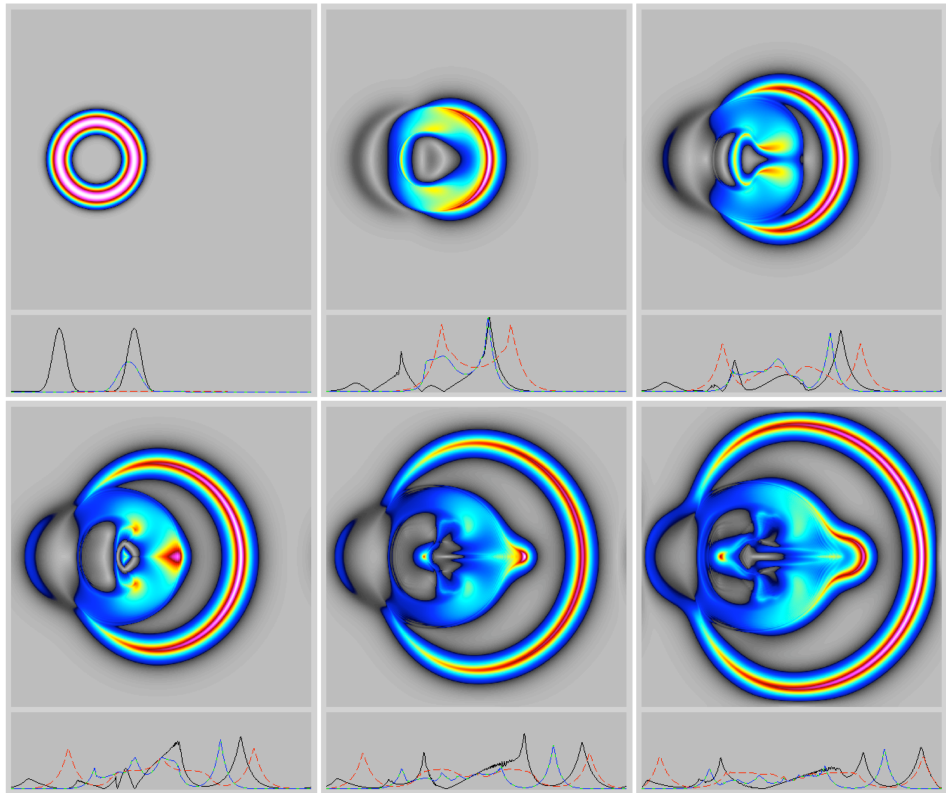


Figure 29: A circular peakon ring initially undergoes uniform rightward translational motion along the x axis. The right outer side of the ring produces diverging peakon curves, which slow as they propagate outward. The left inner side of the ring, however, produces converging peakon segments, which accelerate as they converge. They collide along the midline, then develop into divergent peakon curves still moving rightward that overtake the previous ones and collide with them from behind. These overtaking collisions impart momentum, but they apparently do not produce reconnections.

Lecture #3, Computational anatomy: Contour matching using EPDiff

This lecture explains that EPDiff is the perfect tool to realize D'Arcy Thompson's concept of comparing shapes, upon choosing a norm that measures the differences between anatomical forms defined by contours.

14 Introduction to computational anatomy (CA)

Morphology and computational anatomy

Computational anatomy (CA) must measure and analyse a range of variations in shape, or appearance, of highly deformable biological structures. The problem statement for CA was formulated long ago in a famous book by D'Arcy Thompson [Tho92]

> In a very large part of morphology, our essential task lies in the **comparison of related forms** rather than in the precise definition of each. . . . This process of comparison, of recognizing in one form a definite permutation or deformation of another, . . . lies within the immediate province of mathematics and finds its solution in . . . the Theory of Transformations. **. . . I learnt of it from Henri Poincaré.**
> – D'Arcy Thompson, *On Growth and Form* (1917)

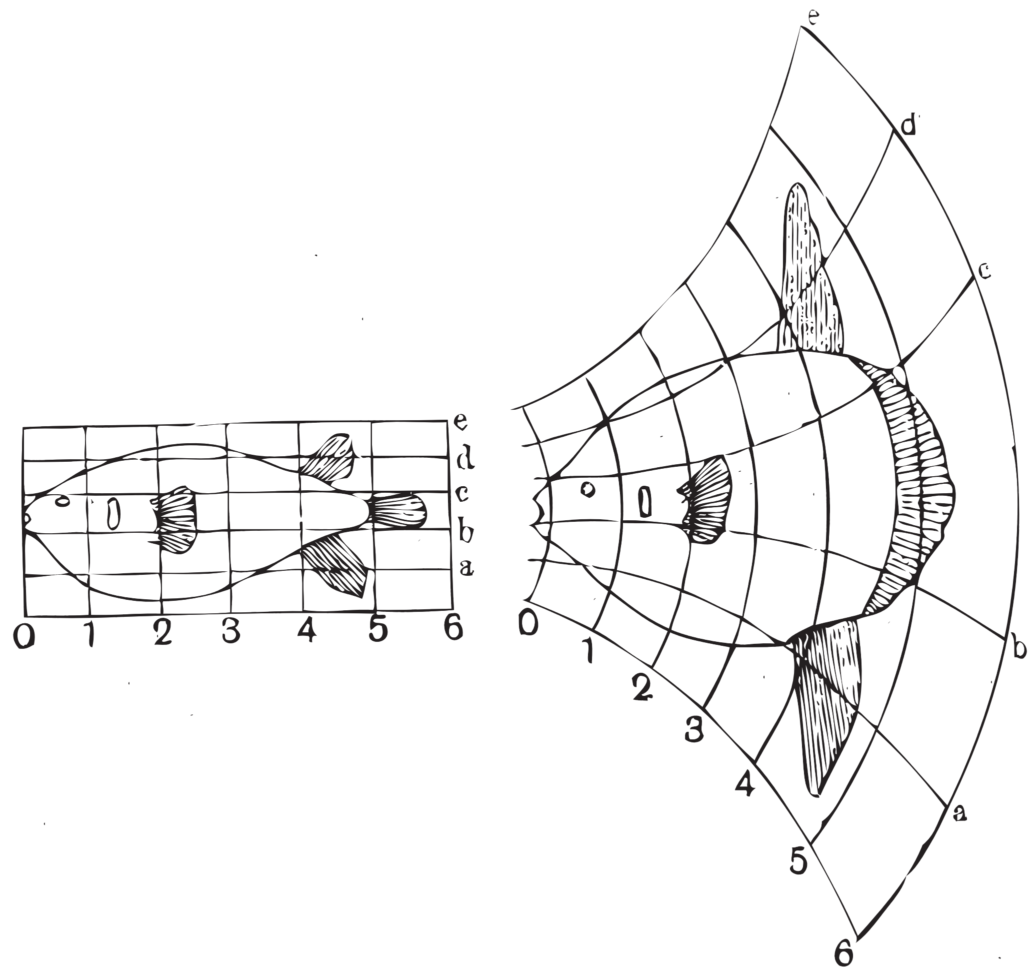


Figure 30: One of D'Arcy Thompson's illustrations of the transformation of two-dimensional shapes from one fish to another, from [Tho92].

D'Arcy Thompson's book [Tho92] examines the idea that the growth and form of all plants and animals can be explained by mathematical principles. His book also acts as a practical guide to understanding how flows of smooth invertible maps may be used to compare shapes. For example, his chapter on transformations contains remarkable diagrams showing how differences in the forms of, say, species of fish can be understood in terms of smooth invertible distortions of the reference coordinate systems onto which they are mapped. A fish is drawn on a square grid, which is then stretched, sheared or shifted so that the deformed image may be identified as that of a related species, as in Figures 30 and 31.

The flow generated by the EPDiff equation transforms one shape along a curve in the space of smooth invertible maps that takes it optimally into another with respect to the chosen norm. Its application to contours in biomedical imaging, for example, realizes D'Arcy Thompson's concept of quantifying growth and measuring other changes in shape, such as occurs in a beating heart, by providing the transformative mathematical path between the two shapes.

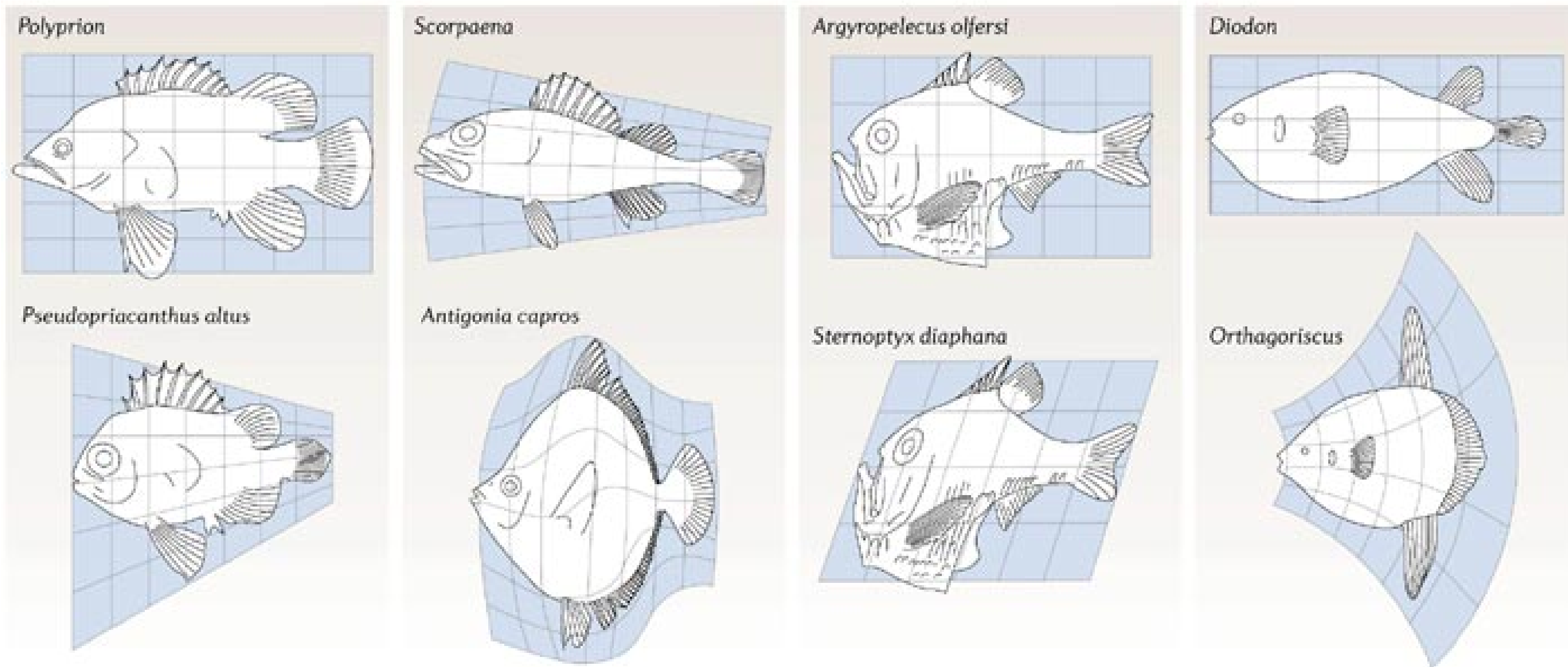


Figure 31: More illustrations by D'Arcy Thompson of the transformation of two-dimensional shapes from one fish to another, from [Tho92].

Computational anatomy (CA)

The pioneering work of Bookstein and Grenander first took up D'Arcy Thompson's challenge by introducing a method called **template matching** [Boo91, Gre81]. The past several years have seen an explosion in the use and development of template-matching methods in computer vision and medical imaging seem to be fulfilling D'Arcy Thompson's expectation. These methods enable the systematic measurement and comparison of anatomical shapes and structures in medical imagery. The mathematical theory of Grenander's deformable template models, when applied to these problems, involves smooth invertible maps (diffeomorphisms). See, e.g., [MTY02] for a review. In particular, the template-matching approach defines classes of Riemannian metrics on the tangent space of the diffeomorphisms and employs their projections onto specific landmark shapes, or image spaces, for the representation of CA data.

The problem for CA then becomes to determine the minimum distance between two images as specified in a certain representation space, V, on which the diffeomorphisms act. Metrics are written so that the optimal path in Diff satisfies an evolution equation. This equation turns out to be EPDiff, when V is a closed contour representing the shape of the image. A discussion of EPDiff and the application of its peakons and other singular solutions for matching templates defined by contours of image outlines appears in [HRTY04].

Objectives

This lecture introduces the variational formulation of template matching problems in computational anatomy. It makes the connection to the EPDiff evolution equation and discusses the relation of images in CA to the singular momentum map of the EPDiff equation. Then it draws some consequences of EPDiff for the outline matching problem in CA and gives a numerical example. The numerical example is reminiscent of the chapter in D'Arcy Thompson's book where the shapes of fish are related to each other by stretching one shape into another on a square grid.

This lecture also discusses how the Euler–Poincaré theory may be used to develop new perspectives in CA. In particular, these lectures discuss how CA may be informed by the concept of weak solutions, solitons and momentum maps for geodesic flows [HRTY04, CH93, HS04]. For example, among the geometric structures of interest in CA, the landmark points and image outlines may be identified with the singular solutions of the EPDiff equation.

The singular solutions are given by the momentum map J_{Sing} for EPDiff earlier. The momentum map J_{Sing} also yields the canonical Hamiltonian formulation of peakon dynamics. This evolution, in turn, provides a complete parameterization of the landmarks and image outlines by the **linear vector space** comprising their canonical positions and momenta. The singular momentum map J_{Sing} for EPDiff provides an isomorphism between the landmarks and outlines for images and the singular soliton solutions of the EPDiff equation. This isomorphism provides a dynamical paradigm for CA, and a representation of anatomical data in a linear vector space.

Computational anatomy: Euler–Poincaré image matching

15 Overview of pattern matching

This lecture explains how the pattern-matching problem for images is governed by the Euler–Poincaré equations of geodesic motion for a Lagrangian given by a right-invariant norm on $(T\text{Diff} \times TN)/\text{Diff}$, where N is the manifold of images on which Diff acts.

Pattern matching is an important component of imaging science, and is fundamental in computational anatomy (computerized anatomical analysis of medical images). When comparing images, the purpose is to find an optimal deformation that aligns the images and matches their photometric properties. Diffeomorphic pattern matching methods have been developed to achieve both this objective and the additional goal of defining a (Riemannian) metric structure on spaces of deformable objects [DGM98, Tro98]. This approach has found many applications in medical imaging, where the objects of interest include images, landmarks, measures (supported on point sets) and currents (supported on curves and surfaces). These methods usually address the registration problem by solving a variational problem of the form

$$\text{Minimize }\left(d(\text{id}, g)^2 + \text{Error term}(g.n_{temp}, n_{targ})\right) \tag{142}$$

over all diffeomorphisms g, where n_{temp} and n_{targ} are the images being compared (usually referred to as the template and the target), $(g, n) \mapsto g.n$ is the action of diffeomorphisms on the objects and d is a right-invariant Riemannian distance on diffeomorphisms.

In problems formulated as in eqn (142), the error term breaks the metric aspects inherited from the distance d on the diffeomorphisms because the error term has an inherent template vs. target asymmetry. With the aim of designing a fully metric approach to the template-matching problem, the **metamorphosis approach** was formulated in [TY05]. The metamorphosis approach embraces what are called **morphing** and **warping** in computer graphics while endowing the composition of the two operations with a Riemannian variational structure. The metamorphosis approach provides interesting alternatives to the pattern-matching approach based on eqn (142), in the context of a metric framework. This lecture explains the Lagrangian formulation for metamorphosis of images developed in [HTY09] that includes the Riemannian formalism introduced in [TY05].

The discussion will be general enough to include a range of applications. Consider a manifold N that is acted upon by a Lie group G. The manifold N contains the **deformable objects** and G is the Lie group of deformations, which is taken to be the group of diffeomorphisms. (A few examples of the space N will be discussed later. In particular, N could be another Lie group.)

Definition 11 *A* **metamorphosis** *[TY05] is a pair of curves* $(g_t, \eta_t) \in G \times N$ *parameterized by time* t*, with* $g_0 = \mathrm{id}$*. Its* **image** *is the curve* $n_t \in N$ *defined by the action* $n_t = g_t.\eta_t$*. The quantities* g_t *and* η_t *are called the* **deformation component** *of the metamorphosis, and its* **template component**, *respectively. When* η_t *is constant, the metamorphosis reduces to standard template matching, which is a* **pure deformation**. *In the general case, the image is a composition of deformation and template variation.*

This lecture places the metamorphosis approach into a Lagrangian formulation, and applies the Euler–Poincaré variational framework to derive its evolution equations. Analytical questions about these equations (for example, the existence and uniqueness of their solutions) require additional assumptions on G and the space N of deformed objects that are beyond the scope of the present text. For analytical discussions of the equations in this lecture, see [HTY09].

The next section provides notation and definitions related to the problem of metamorphosis.

16 Notation and Lagrangian formulation

The letters η or n will be used to denote elements of N, the former being associated to the template component of a metamorphosis, and the latter to its image under the action of the group.

The variational problem we shall study optimizes over metamorphoses (g_t, η_t) by minimizing, for some Lagrangian L, the integral

$$\int_0^1 L(g_t, \dot{g}_t, \eta_t, \dot{\eta}_t)\mathrm{d}t\,, \tag{143}$$

with fixed endpoint conditions for the initial and final images n_0 and n_1 (with $n_t = g_t\eta_t$) and $g_0 = \mathrm{id}_G$ (so only the images are constrained at the endpoints, with the additional normalization $g_0 = \mathrm{id}$).

Let $\mathfrak{g}$ denote the Lie algebra of G and let $(g, U_g, \eta, \xi_\eta) \in TG \times TN$. We will consider Lagrangians defined on $TG \times TN$, that satisfy the following invariance conditions: there exists a function ℓ defined on $\mathfrak{g} \times TN$ such that

$$L(g, U_g, \eta, \xi_\eta) = \ell(U_g g^{-1}, g\eta, g\xi_\eta).$$

In other words, L is taken to be invariant under the right action of G on $G \times N$ defined by $(g, \eta)h = (gh, h^{-1}\eta)$.

For a metamorphosis (g_t, η_t), the following definitions

$$u_t = \dot{g}_t g_t^{-1}\,, \qquad n_t = g_t \eta_t\,, \quad \text{and} \quad \nu_t = g_t \dot{\eta}_t \tag{144}$$

lead by right invariance to an expression for the **reduced Lagrangian**

$$L(g_t, \dot{g}_t, \eta_t, \dot{\eta}_t) = \ell(u_t, n_t, \nu_t)\,. \tag{145}$$

Notation about Lie brackets and pairings

The Lie derivative with respect to a vector field X will be denoted $\pounds_X$. The Lie algebra of G is identified with the set of right-invariant vector fields $U_g = ug$, $u \in T_eG = \mathfrak{g}$, $g \in G$. If G acts on a set $\tilde{N}$, and $f : \tilde{N} \to \mathbb{R}$, one finds $\pounds_u f(\tilde{n}) = (\mathrm{d}/\mathrm{d}t) f(g_t \tilde{n})$ with $g_0 = \mathrm{id}$ and $\dot{g}_t(0) = u$.

The **Lie bracket** $[u, v]$ on $\mathfrak{g}$ is defined by

$$\pounds_{[u,v]} = -(\pounds_u \pounds_v - \pounds_v \pounds_u) \tag{146}$$

and the associated adjoint operator is $\mathrm{ad}_u\, v = [u, v]$. Letting $I_g(h) = ghg^{-1}$ and $Ad_v g = \pounds_v I_g(\mathrm{id})$ yields $\mathrm{ad}_u\, v = \pounds_u(\mathrm{Ad}_v)(\mathrm{id})$. When G is a group of diffeomorphisms, this defines

$$ad_u v = \mathrm{d}u\, v - \mathrm{d}v\, u\,, \tag{147}$$

as in eqn (6).

The **pairing** between a linear form l and a vector u will be denoted $\big\langle l,\, u\big\rangle$. Duality with respect to this pairing will be denoted with an asterisk $(\,\cdot\,)^*$.

When the Lie group G acts on a manifold $\tilde{N}$, the associated **diamond** operation ($\diamond$) (or dual action) is defined on $T\tilde{N}^* \times T\tilde{N}$ and takes values in $\mathfrak{g}^*$, so that

$$\diamond:\; T\tilde{N}^* \times T\tilde{N} \to \mathfrak{g}^* \,. \tag{148}$$

The diamond operation is defined in terms of the pairing $\big\langle\,\cdot\,,\;\cdot\,\big\rangle:\,\mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$, which in this notation is defined as

$$\big\langle \gamma \diamond \tilde{n},\, u\big\rangle = -\big\langle \gamma,\, u\tilde{n}\big\rangle_{T\tilde{N}^*}\,, \tag{149}$$

where $(\gamma,\,\tilde{n}) \in T\tilde{N}^* \times \tilde{N}$, $u\tilde{n} = \pounds_u\tilde{\eta}$ and the bracket $\big\langle\,\cdot\,,\;\cdot\,\big\rangle_{T\tilde{N}^*}$ denotes the pairing between TN and TN^*.

17 Symmetry-reduced Euler equations

We compute the symmetry-reduced Euler equations as stationarity conditions that extremalize the reduced action, defined in terms of the reduced Lagrangian by,

$$S_{red} := \int_0^1 \ell(u_t, n_t, \nu_t)\mathrm{d}t \,, \tag{150}$$

with respect to variations δu and $\omega = \delta n = \delta(g\eta)$ for fixed endpoint conditions n_0 and n_1. The variation $\delta\nu$ can be obtained from $n = g\eta$ and $\nu = g\dot{\eta}$ yielding

$$\dot{n} = \nu + un \quad \text{and} \quad \dot{\omega} = \delta\nu + u\omega + \delta un \,, \tag{151}$$

in which Lie algebra action is denoted by **concatenation from the left**. For example, $un = \pounds_u n$ denotes the Lie derivative of n along the vector field u, etc. The computations are performed in a local chart on TN in terms of which partial derivatives are taken.

Taking stationary variations of S_{red} yields

$$\int_0^1 \left(\left\langle \frac{\delta \ell}{\delta u}\,,\, \delta u_t \right\rangle + \left\langle \frac{\delta \ell}{\delta n}\,,\, \omega_t \right\rangle + \left\langle \frac{\delta \ell}{\delta \nu}\,,\, \dot{\omega}_t - u_t\omega_t - \delta u_t\, n_t \right\rangle \right) \mathrm{d}t = 0\,. \tag{152}$$

The δu-term yields the constant of motion,

$$\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n_t = 0\,. \tag{153}$$

A slight abuse of notation is allowed in writing $\delta\ell/\delta\nu \in T(TN)^*$ as a linear form on TN via $\big\langle \delta\ell/\delta\nu,\, z \big\rangle := \big\langle \delta\ell/\delta\nu,\, (0, z) \big\rangle$.

After an integration by parts in time, the ω-term in the variation equation (152) yields,

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \nu} + u_t \star \frac{\delta \ell}{\delta \nu} - \frac{\delta \ell}{\delta n} = 0\,, \tag{154}$$

with additional notation for the $\star$ operation, defined by

$$\left\langle u \star \frac{\delta \ell}{\delta \nu}\,,\, \omega \right\rangle := \left\langle \frac{\delta \ell}{\delta \nu}\,,\, u\omega \right\rangle. \tag{155}$$

The endpoint terms vanish in the integration by parts for the ω-term because δn vanishes at the endpoints for n_0 and n_1 fixed. These manipulations have proven the following.

Theorem 6 (Metamorphosis equations)

The **symmetry-reduced Euler equations** *associated with extremals of the reduced action $S_{\rm red}$ in eqn (150)*

$$\int_0^1 \ell(u_t, n_t, \nu_t)\mathrm{d}t$$

with fixed endpoint conditions n_0 and n_1 under variations of the right-invariant velocity (δu) and image ($\omega = \delta n$) defined in eqn (144) consist of the system of **metamorphosis equations**

$$\left.\begin{aligned} &\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n_t = 0\,, \\ &\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \nu} + u_t \star \frac{\delta \ell}{\delta \nu} = \frac{\delta \ell}{\delta n}\,, \\ &\dot{n}_t = \nu_t + u_t n_t\,. \end{aligned}\right\} \qquad (156)$$

Remark 32 *The quantity* $\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n$ *is the conserved momentum arising from Noether's theorem for right invariance of the Lagrangian. As we shall see, the special form of the endpoint conditions (fixed* n_0 *and* n_1*) ensures that this conserved momentum* vanishes identically.

Remark 32 *The quantity* $\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n$ *is the conserved momentum arising from Noether's theorem for right invariance of the Lagrangian. As we shall see, the special form of the endpoint conditions (fixed* n_0 *and* n_1*) ensures that this conserved momentum* vanishes identically*.*

18 Euler–Poincaré reduction

A dynamical system equivalent to the metamorphosis equations (156) may be obtained by using Euler–Poincaré reduction. In this setting, one takes variations in the group element (δg) and in the template ($\delta \eta$) instead of the velocity and the image. Set

$$\xi_t = \delta g_t g_t^{-1} \quad \text{and} \quad \varpi_t = g_t \delta \eta_t \,. \tag{157}$$

These definitions lead to expressions for δu, δn and $\delta\nu$. For the velocity one finds the **constrained variation**,

$$\delta u_t = \dot{\xi}_t + [\xi_t, u_t]\,. \tag{158}$$

This is a standard relation in Euler–Poincaré reduction. From the definition $n_t = g_t\eta_t$ in (144) one has the variational relation

$$\delta n_t = \delta(g_t\eta_t) = \varpi_t + \xi_t n_t\,. \tag{159}$$

From the definition $\nu_t = g_t\dot{\eta}_t$, one finds

$$\delta\nu_t = g_t\delta\dot{\eta}_t + \xi_t\nu_t\,, \tag{160}$$

and from $\varpi_t = g_t\delta\eta_t$ one observes

$$\dot{\varpi}_t = u_t\varpi_t + g_t\dot{\eta}_t\,, \tag{161}$$

which, in turn, yields

$$\delta\nu_t = \dot{\varpi}_t + \xi_t\nu_t - u_t\varpi_t\,. \tag{162}$$

The endpoint conditions for ξ and ϖ are computed, as follows. One starts at $t = 0$ with $g_0 = \mathrm{id}$ and $n_0 = g_0\eta_0 = \mathrm{cst}$, which implies $\xi_0 = 0$ and

$\varpi_0 = 0$. At $t = 1$, the relation $g_1\eta_1 =$ cst yields an endpoint condition on the variations at $t = 1$,

$$\delta(g_t\eta_t)\Big|_{t=1} = \xi_1 n_1 + \varpi_1 = 0\,. \tag{163}$$

The variation of the reduced action S_{red} in eqn (150) is now expressed as

$$\int_0^1 \left(\left\langle \frac{\delta \ell}{\delta u}\,,\, \dot{\xi}_t - \mathrm{ad}_{u_t}\,\xi_t \right\rangle + \left\langle \frac{\delta \ell}{\delta n_t}\,,\, \varpi_t + \xi_t n_t \right\rangle + \left\langle \frac{\delta \ell}{\delta \nu}\,,\, \dot{\varpi}_t + \xi_t \nu_t - u_t \varpi_t \right\rangle \right) \mathrm{d}t = 0\,.$$

In the integrations by parts in time to eliminate $\dot{\xi}_t$ and $\dot{\varpi}_t$, the endpoint terms sum to

$$\Big\langle (\delta\ell/\delta u)_1,\, \xi_1 \Big\rangle + \Big\langle (\delta\ell/\delta\nu)_1,\, \varpi_1 \Big\rangle\,.$$

Using the endpoint condition (163) on the variations at $t = 1$ allows the last term to be rewritten as

$$\Big\langle (\delta\ell/\delta\nu)_1,\, \varpi_1 \Big\rangle = -\Big\langle (\delta\ell/\delta\nu)_1,\, \xi_1 n_1 \Big\rangle = \Big\langle (\delta\ell/\delta\nu)_1 \diamond n_1,\, \xi_1 \Big\rangle\,.$$

One therefore obtains the stationarity relation at time $t = 1$,

$$\frac{\delta \ell}{\delta u}(1) + \frac{\delta \ell}{\delta \nu}(1) \diamond n_1 = 0\,. \tag{164}$$

After another integration by parts, the ξ-terms provide the evolution equation for $\delta\ell/\delta u$,

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} + \mathrm{ad}^*_{u_t}\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta n} \diamond n_t + \frac{\delta \ell}{\delta \nu} \diamond \nu_t = 0\,. \tag{165}$$

Likewise, the ϖ-terms provide the evolution equation for $\delta\ell/\delta\nu$,

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \nu} + u_t \star \frac{\delta \ell}{\delta \nu} - \frac{\delta \ell}{\delta n} = 0. \tag{166}$$

These additional manipulations have proven the following.

Theorem 7 (Metamorphosis dynamics)
The Euler–Poincaré equations associated with extremals of the reduced action S_{red} in eqn (150)

$$S_{\text{red}} = \int_0^1 \ell(u_t, n_t, \nu_t)\mathrm{d}t$$

with fixed endpoint conditions n_0 and n_1 under variations in the group element (δg) and in the template ($\delta\eta$) consist of the system of equations for **Euler–Poincaré metamorphosis dynamics**

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} + \mathrm{ad}^*_{u_t}\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta n}\diamond n_t + \frac{\delta \ell}{\delta \nu}\diamond \nu_t = 0\,,\\
&\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \nu} + u_t \star \frac{\delta \ell}{\delta \nu} - \frac{\delta \ell}{\delta n} = 0\,,\\
&\frac{\delta \ell}{\delta u}(1) + \frac{\delta \ell}{\delta \nu}(1)\diamond n_1 = 0\,,\\
&\dot{n}_t = \nu_t + u_t n_t\,.
\end{aligned}\right\} \qquad (167)$$

Proposition 6 *The dynamical system* (167) *is equivalent to eqn* (156).

Proof. The equivalence is obvious, since the two systems of equations characterize the same critical points of the reduced action obtained by different independent variations. However, an instructive proof can be given by rewriting the first equation in (167) as a **Kelvin–Noether theorem for images**,

$$\frac{\partial}{\partial t}\Big(\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n\Big) + \mathrm{ad}^*_{u_t}\Big(\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n\Big) = 0\,. \tag{168}$$

Indeed, any solution of (167) satisfies,

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(\frac{\delta \ell}{\delta u_t} + \frac{\delta \ell}{\delta \nu} \diamond n_t\Big) &= \frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} + \Big(\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \nu}\Big) \diamond n_t + \frac{\delta \ell}{\delta \nu} \diamond \dot{n}_t \\
&= \frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} + \Big(\frac{\delta \ell}{\delta n} - u_t \star \frac{\delta \ell}{\delta \nu}\Big) \diamond n_t + \frac{\delta \ell}{\delta \nu} \diamond (\nu_t + u_t n_t) \\
&= \frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta n} \diamond n_t + \frac{\delta \ell}{\delta \nu} \diamond \nu_t - \Big(u_t \star \frac{\delta \ell}{\delta \nu}\Big) \diamond n_t + \frac{\delta \ell}{\delta \nu} \diamond (u_t n_t) \\
&= -\,\mathrm{ad}^*_{u_t}\frac{\delta \ell}{\delta u} - \mathrm{ad}^*_{u_t}(\frac{\delta \ell}{\delta \nu} \diamond n_t).
\end{aligned}$$

In the last equation, we have used the fact that, for any $\alpha \in \mathfrak{g}$,

$$\begin{aligned}\left\langle \frac{\delta \ell}{\delta \nu} \diamond (un) - \left(u \star \frac{\delta \ell}{\delta \nu} \right) \diamond n \,,\, \alpha \right\rangle &= -\left\langle \frac{\delta \ell}{\delta \nu} \,,\, \alpha(un) - u(\alpha n) \right\rangle \\ &= \left\langle \frac{\delta \ell}{\delta \nu} \,,\, [u, \alpha] n \right\rangle \\ &= -\left\langle \frac{\delta \ell}{\delta \nu} \diamond n \,,\, [u, \alpha] \right\rangle \\ &= -\left\langle \mathrm{ad}^*_{u_t} \left(\frac{\delta \ell}{\delta \nu} \diamond n_t \right) \,,\, \alpha \right\rangle .\end{aligned}$$

Consequently, the first equation in the system (167) is equivalent to eqn (168), which in turn may be rewritten equivalently as,

$$\frac{\partial}{\partial t} \left(\mathrm{Ad}^*_{g_t} \left(\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n \right) \right) = 0 \,. \tag{169}$$

This equation combined with $(\delta \ell / \delta u)_1 + (\delta \ell / \delta \nu)_1 \diamond n_1 = 0$ implies the first equation in (156).

Exercise 25 *Show that these manipulations prove the claim of Remark 32 that the quantity*

$$\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n$$

is the conserved momentum arising from **Noether's theorem** *for right invariance of the reduced Lagrangian in eqn (145).*

Exercise 26 *Compute the Hamiltonian and Lie–Poisson bracket for the system (167) governing metamorphosis dynamics.*

19 Semidirect-product examples

19.1 Riemannian metric

A primary application of this framework can be based on the definition of a **Riemannian metric** on $G \times N$ that is invariant under the right action of G: $(g, \eta)h = (gh, h^{-1}\eta)$. The corresponding Lagrangian then takes the form

$$l(u, n, \nu) = \|(u, \nu)\|_n^2 \,. \tag{170}$$

The variational problem is now equivalent to the computation of geodesics for the canonical projection of this metric from $G \times N$ onto N. This framework was introduced in [MY01]. The evolution equations were derived and studied in [TY05] in the case $l(u, n, \nu) = |u|_{\mathfrak{g}}^2 + |\nu|_n^2$, for a given norm, $|.|_{\mathfrak{g}}$, on $\mathfrak{g}$ and a pre-existing Riemannian structure on N. This Riemannian metric on N incorporates the group actions. An example of its application is given below for images. First, though, let us discuss the semidirect-product case $G \circledS N$.

19.2 Semidirect product

Assume that N is a group and that for all $g \in G$, the action of g on N is a group homomorphism: For all $n, \tilde{n} \in N$, $g(n\tilde{n}) = (gn)(g\tilde{n})$ (for example, N can be a vector space and the action of G can be linear). Consider the **semidirect product** $G \circledS N$ with

$$(g, n)(\tilde{g}, \tilde{n}) = (g\tilde{g}, (g\tilde{n})n)\,, \tag{171}$$

and build on $G \circledS N$ a right-invariant metric constrained by its value $\|\ \|_{(\mathrm{id}_G, \mathrm{id}_N)}$ at the identity. Then, optimizing the geodesic energy in $G \circledS N$ between

(id_G, n_0) and (g_1, n_1) with fixed n_0 and n_1 and free g_1 yields a particular case of metamorphosis.

Right invariance for the metric on $G Ⓢ N$ implies

$$\|(U,\zeta)\|_{(g,n)} = \|(U\tilde{g}, (U\tilde{n})n + (g\tilde{n})\zeta\|_{(g\tilde{g},(g\tilde{n})n)}\,, \tag{172}$$

which yields, upon using $(\tilde{g},\tilde{n}) = (g^{-1}, g^{-1}n^{-1})$ and letting $u = Ug^{-1}$,

$$\begin{aligned} \|(U,\zeta)\|_{(g,n)} &= \|(u, (un^{-1})n + n^{-1}\zeta\|_{(\mathrm{id}_G,\mathrm{id}_N)} \\ &= \|(u, n^{-1}(\zeta - un)\|_{(\mathrm{id}_G,\mathrm{id}_N)}\,, \end{aligned}$$

which may be proved by using the identity

$$0 = u(n^{-1}n) = (un^{-1})n + n^{-1}(un)\,.$$

Consequently, the geodesic energy on $G Ⓢ N$ for a path of unit length is

$$\int_0^1 \|(u_t, n_t^{-1}(\dot{n}_t - u_t n_t)\|^2_{(\mathrm{id}_G,\mathrm{id}_N)}\,. \tag{173}$$

Optimizing this geodesic energy with fixed n_0 and n_1 is equivalent to solving the metamorphosis problem with

$$l(u,n,\nu) = \|(u, n^{-1}\nu)\|^2_{(\mathrm{id}_G,\mathrm{id}_N)}. \tag{174}$$

This turns out to be a particular case of the previous example. The situation is even simpler when N is a vector space. In this case, $n^{-1}n' = n' - n$ and one computes $(g, n)(\tilde{g}, \tilde{n}) = (g\tilde{g}, g\tilde{n} + n)$ so that

$$(\dot{g}, \dot{n})(\tilde{g}, \tilde{n}) = (\dot{g}\tilde{g}, \dot{g}\tilde{n} + \dot{n})\,.$$

Consequently,

$$(\dot{g}, \dot{n})(g^{-1}, g^{-1}n^{-1}) = (\dot{g}g^{-1}, \dot{g}g^{-1}n^{-1} + \dot{n}) = (u, -un + \dot{n}) = (u, \nu)$$

and the symmetry-reduced Lagrangian does not depend on n. The systems (156) and (167) take a very simple form when the group operation on N is additive. Namely, they become, respectively,

$$\left.\begin{aligned} &\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n_t = 0\,, \\ &\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \nu} + u_t \star \frac{\delta \ell}{\delta \nu} = 0\,, \\ &\dot{n}_t = \nu_t + u_t n_t\,, \end{aligned}\right\} \tag{175}$$

and the equivalent dynamical system

$$
\left.
\begin{aligned}
&\frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} + \mathrm{ad}^*_{u_t}\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond \nu_t = 0\,, \\
&\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \nu} + u_t \star \frac{\delta \ell}{\delta \nu} = 0\,, \\
&\frac{\delta \ell}{\delta u}(1) + \frac{\delta \ell}{\delta \nu}(1) \diamond n_1 = 0\,, \\
&\dot{n}_t = \nu_t + u_t n_t\,,
\end{aligned}
\right\}
\qquad (176)
$$

in which concatenation as in $u_t n_t$ denotes Lie algebra action, as before.

Thus, when N is a vector space, the evolution of the variable $n \in N$ decouples from the rest of the dynamical system (176).

Remark 33 *Even when N is not a vector space, metamorphoses that are obtained from the semidirect-product formulation are specific among general metamorphoses, because they satisfy the conservation of momentum property that comes with right invariance of the metric under the Lie group. This conservation equation may be written as, see eqn (169),*

$$\left(\frac{\delta \ell}{\delta u_t}, \frac{\delta \ell}{\delta \nu_t}\right) = \mathsf{Ad}^*_{(g_t, n_t)^{-1}}\left(\frac{\delta \ell}{\delta u_0}, \frac{\delta \ell}{\delta \nu_0}\right), \tag{177}$$

where the adjoint representation is associated with the semidirect-product Lie group action. This property (that we do not write in the general case) will be illustrated in an example below.

19.3 Image matching

Consider the case when N is a space of smooth functions from domain Ω to $\mathbb{R}$, that we will call **images**, with the action

$$(g, n) \mapsto n \circ g^{-1} . \tag{178}$$

A simple case of metamorphoses [MY01] can be obtained with the Lagrangian

$$\ell(u, \nu) = \|u\|_{\mathfrak{g}}^2 + \frac{1}{\sigma^2}\|\nu\|_{L^2}^2 . \tag{179}$$

If $w \in \mathfrak{g}$ and n is an image, then $wn = -\nabla n^T w$, so that

$$\left\langle \frac{\delta \ell}{\delta \nu} \diamond n \, , \, w \right\rangle = \left\langle \frac{\delta \ell}{\delta \nu} \, , \, \nabla n^T w \right\rangle . \tag{180}$$

Thus, since $\delta\ell/\delta\nu = 2\nu/\sigma^2$, the first equation in the system (175) is

$$L_{\mathfrak{g}} u_t := \frac{\delta \ell}{\delta u} = -\frac{1}{\sigma^2}\nu_t \nabla n_t \, , \tag{181}$$

where $L_{\mathfrak{g}}$ is the positive symmetric operator associated with the norm $\|u_t\|_{\mathfrak{g}}^2$ by

$$\|u_t\|_{\mathfrak{g}}^2 = \Big\langle L_{\mathfrak{g}} u_t,\, u_t \Big\rangle\,. \tag{182}$$

Now, $u \star (\delta\ell/\delta\nu)$ is defined by

$$\begin{aligned}
\left\langle u \star \left(\frac{\delta\ell}{\delta\nu}\right),\, \omega \right\rangle &= \left\langle \frac{\delta\ell}{\delta\nu},\, u\omega \right\rangle \\
&= -\left\langle \frac{\delta\ell}{\delta\nu},\, \nabla\omega^T u \right\rangle \\
&= -\frac{1}{\sigma^2}\Big\langle \nu,\, \nabla\omega^T u \Big\rangle \\
&= \frac{1}{\sigma^2}\Big\langle \mathrm{div}(\nu u),\, \omega \Big\rangle,
\end{aligned}$$

which yields the second equation in the system (175)

$$\dot{\nu}_t + \frac{1}{\sigma^2}\mathrm{div}(\nu_t u_t) = 0\,. \tag{183}$$

We denote $z = \nu/\sigma^2$ and rewrite the three equations in the system (175) as

$$\left.\begin{aligned} L_{\mathfrak{g}} u_t &= -z_t \nabla n_t \,, \\ \dot{z}_t + \mathrm{div}(z_t u_t) &= 0 \,, \\ \dot{n}_t + \nabla n_t^T u_t &= \sigma^2 z_t \,. \end{aligned}\right\} \qquad (184)$$

Existence and uniqueness of solutions for this system were proved in [TY05]. From a visual point of view, image metamorphoses are similar to what is usually called **morphing** in computer graphics. The evolution of the image over time, $t \mapsto n_t$, is a combination of deformations and image intensity variation. Algorithms and results for the solution of the boundary-value problem (minimize the Lagrangian between two images at the initial and final times) can be found in [MY01]. Two examples of minimizing geodesics between a pair of images are also provided in Figure 32.

Metamorphosis is an optimal-control matching problem that seeks geodesics on semidirect-product groups – this means that it fits perfectly into the Euler-Poincaré framework!

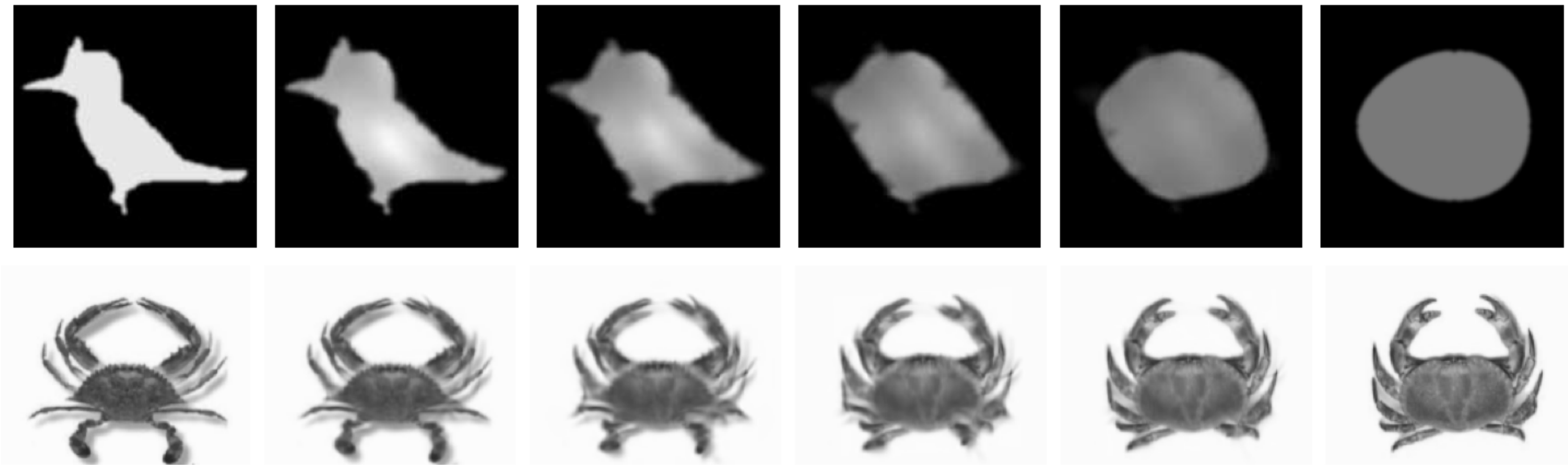


Figure 32: Geodesic flow on Diff$\circledS\mathcal{F}$ governs image morphing. The figure plots the morphing process in the (conjugate) $\mathcal{F}$-variable as a function of time between two pairs of images. The optimal trajectories for n_t are computed between the first and last images in each pair. The remaining images show n_t at intermediate points in time. (Figure from DD Holm, A Trouvé and L.Younes, *Quart Appl Math*, 2009.)

Image matching can also be seen from the semidirect-product viewpoint, since the action is linear and the Lagrangian takes the form (174) with $n^{-1}\nu = \nu$. This implies that the momentum in this case, given by the pair $(L_{\mathfrak{g}}u, z)$, is conserved in a fixed frame and the n-equation in (184) is absent. Working out the conservation equation $\mathrm{Ad}^*_{(g,n)}(L_{\mathfrak{g}}u, z) = \mathrm{const}$ in this case yields the equations

$$L_{\mathfrak{g}}u_t + z_t \nabla n_t = \mathrm{const}$$

and

$$z_t = \det(Dg_t^{-1}) z_0 \circ g_t^{-1} \,.$$

This last condition is the integrated form of the second equation in the eqn set (184), while the first equation of the set (184) evaluates the conservation law as $L_{\mathfrak{g}}u_t + z_t \nabla n_t = 0$.

Exercise 27 *Solve Exercise 23 again from the viewpoint of metamorphoses by writing the action A using the Lagrangian in Riemannian form given in eqn (179) as,*

$$A = \frac{1}{2}\int_0^1 \Big\| \boldsymbol{u}(t) \Big\|_{\mathfrak{g}}^2 \,\mathrm{d}t + \frac{1}{2\sigma^2}\int_0^1\!\!\int_{S^1} \Big| \boldsymbol{\nu}(s,t) \Big|^2 \,\mathrm{d}s\,\mathrm{d}t\,,$$

in which $\boldsymbol{\nu}(s,t)$ is defined by

$$\boldsymbol{\nu}(s,t) := \dot{\boldsymbol{Q}}(s,t) - \boldsymbol{u}(\boldsymbol{Q}(s,t),t) \in T\mathbb{R}^2\,,$$

as was done in Exercise 23.

Exercise 28 *Write the metamorphosis equations in the previous exercise when the integral $\int_{S^1}\mathrm{d}s$ over the continuous parameter s is replaced by a sum $\sum_s$ over a finite set of points in $\mathbb{R}^2$ known as the* **landmarks** *$Q_s(t) \in \mathbb{R}^2$ of the image.*

19.4 A special case of 1D metamorphosis: CH2 equations

In 1D, the evolutionary form of the system (167) or equivalently (184) becomes,

$$\partial_t m + u\partial_x m + 2m\partial_x u = -\rho\partial_x\rho\,, \quad \partial_t\rho + \partial_x(\rho u) = 0 \quad (185)$$

upon denoting

$$m = L_{\mathfrak{g}}u = (1-\partial_x^2)u \quad \text{and} \quad \rho = \sigma z\,.$$

Up to a minus sign in front of $\rho\partial_x\rho$ in the first equation that does not affect its integrability as a Hamiltonian system, this is the **two-component Camassa–Holm system** (CH2) studied in [CLZ05, Fal06, Kuz07]. The system (185) in our case is equivalent to the compatibility for $\mathrm{d}\lambda/\mathrm{d}t = 0$ of the two linear equations

$$\partial_x^2\psi \;+\; \Big(-\tfrac{1}{4} + m\lambda + \rho^2\lambda^2\Big)\psi = 0\,, \quad (186)$$

$$\partial_t\psi \;=\; -\Big(\frac{1}{2\lambda} + u\Big)\partial_x\psi + \tfrac{1}{2}\psi\partial_x u\,. \quad (187)$$

Because the eigenvalue λ in (186) is time independent, the evolution of the nonlinear semidirect-product system (185) is said to be **isospectral**. The second equation (187) is the evolution equation for the eigenfunction ψ. Thus, the semidirect-product system (185) for the metamorphosis of images in 1D is also completely integrable and possesses **soliton solutions for the CH2 system** that may be obtained by using the **inverse scattering transform method**. An identification of soliton dynamics in image matching of graphical structures, landmarks and image outlines for computational anatomy using the invariant subsystem of system (185) with $\rho = 0$ is found in [HRTY04].

19.5 Modified CH2 equations

The Euler–Poincaré system (167) for semidirect-product metamorphosis leads to an interesting modification of the CH2 equations (185) when

$G = \text{Diff}(\mathbb{R})$ and $N = \mathcal{F}(\mathbb{R})$ (smooth functions). These modified CH2 equations follow from a Lagrangian defined as a norm on $\text{Diff}(\mathbb{R})\circledS\mathcal{F}(\mathbb{R})$ in (179) given in this notation by

$$\begin{aligned} \ell(u,\rho) &= \tfrac{1}{2}\|u\|^2_{H^1} + \tfrac{1}{2}\|\rho\|^2_{H^{-1}} \\ &= \tfrac{1}{2}\|u\|^2_{L^2} + \tfrac{\alpha_1^2}{2}\|u_x\|^2_{L^2} + \tfrac{1}{2}\|(\overline{\rho}-\overline{\rho}_0)\|^2_{L^2} + \tfrac{\alpha_2^2}{2}\|\overline{\rho}_x\|^2_{L^2}\,, \end{aligned} \tag{188}$$

where α_1 and α_2 are constant length scales, $\overline{\rho}$ is defined in terms of ρ by

$$(1-\alpha_2^2\partial_x^2)\overline{\rho} = \rho + \overline{\rho}_0\,, \tag{189}$$

and $\overline{\rho}_0$ is the constant value of $\overline{\rho}$ as $|x|\to\infty$. Taking stationary variations of the reduced action $S_{\text{red}} = \int_0^1 \ell(u,\rho)\text{d}t$ yields

$$\begin{aligned} 0 = \delta S_{\text{red}} &= \int_0^1 \left(\left\langle \frac{\delta \ell}{\delta u}\,,\,\delta u \right\rangle + \left\langle \frac{\delta \ell}{\delta \overline{\rho}}\,,\,\delta\overline{\rho} \right\rangle \right) \text{d}t \\ &= \int_0^1 \left(\left\langle (1-\alpha_1^2\partial_x^2)u\,,\,\delta u \right\rangle + \left\langle \rho\,,\,\delta\overline{\rho} \right\rangle \right) \text{d}t\,. \end{aligned} \tag{190}$$

Hence, the Euler–Poincaré equations (165) and (166) yield the following modification of the CH2 system (185) with $m = (1-\alpha_1^2\partial_x^2)u$

$$\partial_t m + u\partial_x m + 2m\partial_x u = -\rho\partial_x\overline{\rho} \quad \text{with} \quad \partial_t\rho + \partial_x(\rho u) = 0\,. \tag{191}$$

This modification may seem slight, but it has two important effects. First, very likely, it destroys the complete integrability of the 1D CH2 system (185), although this has not yet been proven. The apparent loss of integrability may seem unfortunate. However, as a sort of compensation for that loss, direct substitution shows that the modified system gains a property not possessed by the original CH2 system. Namely, the modified system admits a finite-dimensional invariant manifold of **singular solutions** in a form that generalizes the peakon solutions of CH to

$$m(t,x) = \sum_{i=1}^{M} P_i(t)\, \delta\Big((x - Q_i(t)\Big) \quad \text{and} \quad \rho(t,x) = \sum_{i=1}^{M} w_i\, \delta\Big((x - Q_i(t)\Big)\,, \tag{192}$$

in which $w_i = \text{cst}$ for $i = 1, \ldots, M$. Moreover, the functions $P_i(t)$ and $Q_i(t)$ satisfy Hamilton's canonical equations,

$$\frac{\mathrm{d}Q_i}{\mathrm{d}t} = \frac{\partial H_M}{\partial P_i} \quad \text{and} \quad \frac{\mathrm{d}P_i}{\mathrm{d}t} = -\,\frac{\partial H_M}{\partial Q_i}\,, \tag{193}$$

with M-particle Hamiltonian,

$$
\begin{aligned}
H_M \;=\; & \frac{1}{2}\sum_{i,j=1}^{M}\Big(P_iP_j\;\mathsf{e}^{-|Q_i-Q_j|/\alpha_1}+w_i\,w_j\,\mathsf{e}^{-|Q_i-Q_j|/\alpha_2}\Big) \\
& +\,\overline{\rho}_0\sum_{i=1}^{M} w_i\,\mathsf{e}^{-|Q_i-Q_j|/\alpha_2}\,. \qquad (194)
\end{aligned}
$$

Just as for the reduced dynamics of the CH equation discussed in Lecture 9, the finite-dimensional invariant manifold of singular solutions (192) of the modified CH2 system obeys canonical Hamiltonian equations. As for the case of CH, this canonical reduction occurs because the singular solutions (192) represent a cotangent-lift momentum map, this time for the left action of the semidirect-product Diff Ⓢ $\mathcal{F}$ on points on the real line. Moreover, as in the passage from the CH equation in one dimension to EPDiff in higher dimensions in Lecture 9, the singular solutions and their momentum-map structure for the modified CH2 system also generalize to arbitrary spatial dimensions. These more technical features of the modified CH2 system are explained in [HTY09].

Emergent singular solutions for a modified CH2

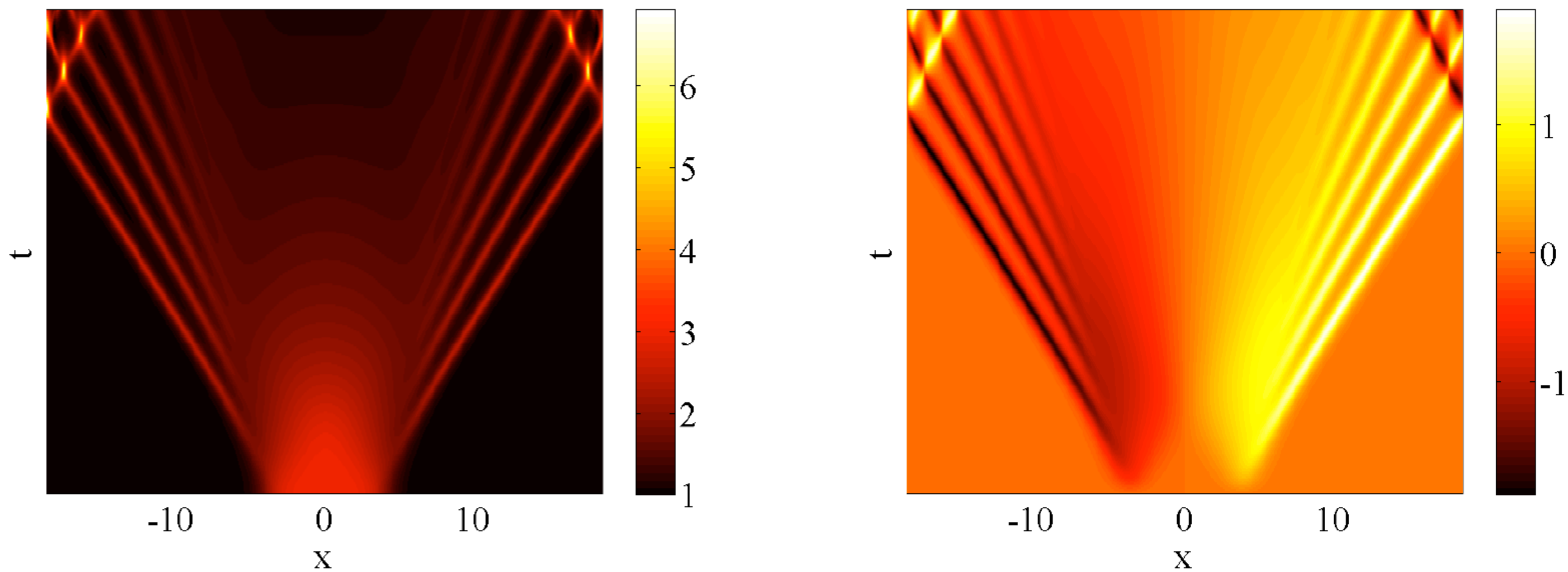


Figure 33: The particular case $Q_i = (1-\alpha_i^2\partial_x^2)$, with $\alpha_i = 0.3, 0.1$. Left figure shows the profile of $\overline{\rho}$, while right shows the velocity u. **Singular solutions emerge spontaneously** and their interaction is seen at late time. (Joint work with Lennon Ó Náraigh, Imperial College London, and Cesare Tronci, EPFL)

The 2-component CH equation and EP(Diff Ⓢ $\mathcal{F}$)

- In 1D, EPDiff recovers the dispersionless CH equation, which has recently been *extended to a system of two coupled 1+1 PDE's (CH2)*

$$\frac{d}{dt}\frac{\delta L}{\delta \mathbf{u}} = -\,\pounds_{\mathbf{u}}\frac{\delta L}{\delta \mathbf{u}} + \rho\,\mathbf{d}\rho\,, \qquad \frac{d}{dt}\rho = -\,\pounds_{\mathbf{u}}\,\rho$$

- Such an extension is again an **integrable geodesic flow on a Lie group**, namely the *semidirect product* Diff Ⓢ $\mathcal{F}$ ($\mathcal{F}$ denotes scalar functions).

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- Such an extension is again an **integrable geodesic flow on a Lie group**, namely the *semidirect product* Diff$\circledS\mathcal{F}$ ($\mathcal{F}$ denotes scalar functions).

- Does such a flow possess singular solutions? Can those be interpreted in terms of momentum maps? What group actions are involved? One approach toward answering these questions is via the study of geodesic motion on semidirect-product Lie groups.

Euler-Poincaré equations on Diff$\circledS\mathcal{F}$

- EP theory yields the **reduced equations on $\mathfrak{X}(\mathbb{R}^n)\circledS\mathcal{F}(\mathbb{R}^n)$**

$$\frac{d}{dt}\frac{\delta L}{\delta \mathbf{u}} = -\,\pounds_{\mathbf{u}}\frac{\delta L}{\delta \mathbf{u}} + \frac{\delta L}{\delta \lambda}\,\mathbf{d}\lambda\,, \qquad \frac{d}{dt}\frac{\delta L}{\delta \lambda} = -\,\pounds_{\mathbf{u}}\frac{\delta L}{\delta \lambda}$$

and geodesic motion is recovered for a *purely quadratic Lagrangian*

$$L(\mathbf{u},\lambda) = \frac{1}{2}\int \mathbf{u}\ Q_1\mathbf{u}\ \mathbf{d}^n\mathbf{x} + \frac{1}{2}\int \lambda\ Q_2\,\lambda\,\mathbf{d}^n\mathbf{x} = \frac{1}{2}\,\|(\mathbf{u},\lambda)\|^2$$

where Q_i are positive-definite, symmetric operators defining a norm for appropriate boundary conditions (say homogeneous or periodic).

Euler-Poincaré equations on Diff Ⓢ $\mathcal{F}$

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- In 1D, the particular choice *$Q_1 = (1-\alpha^2\partial_x^2)$ and $Q_2 = 1$ yields the integrable 2-component CH equations* $\Big(L = 1/2\,\|u\|^2_{H^1} + 1/2\,\|\lambda\|^2_{L^2}\Big)$.

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- In 1D, the particular choice $Q_1 = (1-\alpha^2\partial_x^2)$ *and* $Q_2 = 1$ *yields the integrable 2-component CH equations* $\Big(L = 1/2\,\|u\|^2_{H^1} + 1/2\,\|\lambda\|^2_{L^2}\Big)$.

- However, the choice for integrability ($Q_2 = 1$) **allows singular solutions in momentum (peakons), but prevents singular solutions in density**.

Emergent singular solutions for EP(Diff$\circledS\mathcal{F}$)

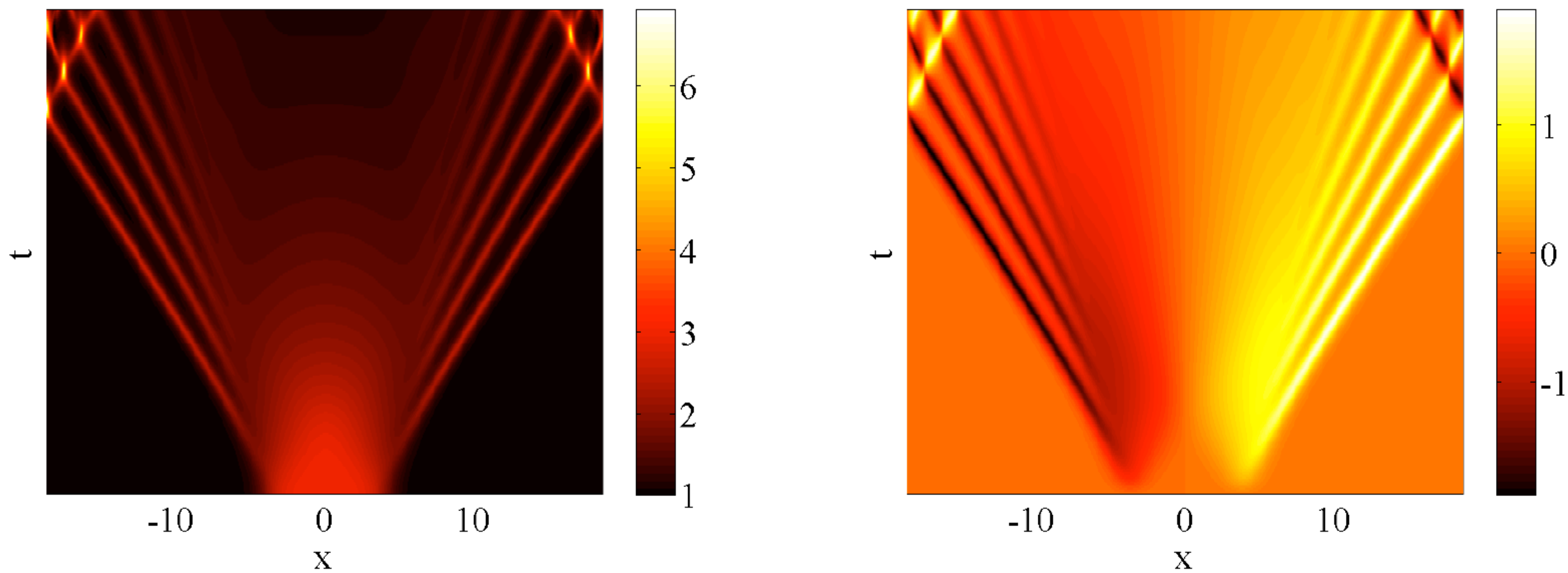

Figure 34: The particular case $Q_i = (1 - \alpha_i^2 \partial_x^2)$, with $\alpha_i = 0.3, 0.1$. Left figure shows the profile of λ, while right shows the velocity u. **Singular solutions emerge spontaneously** and their interaction is seen at late time. (Joint work with Lennon Ó Náraigh, Imperial College London, and Cesare Tronci, EPFL)

Lie-Poisson formulation and singular solutions

- Legendre transform $(\mathbf{m}, \rho) = \delta L / \delta(\mathbf{u}, \lambda)$ yields the LP equations

$$\partial_t \mathbf{m} + \pounds_{\mathbf{u}} \mathbf{m} = \rho \nabla \lambda \,, \qquad \rho_t + \pounds_{\mathbf{u}} \rho = 0$$

with $\mathbf{u} = G_1 * \mathbf{m}$, $\lambda = G_2 * \rho$ and G_i is the Green's function of Q_i.

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- Direct substitution proves the existence of **singular solutions**

$$\Big(\mathbf{m},\rho\Big)=\sum_{i=1}^{N}\int\Big(\mathbf{P}_i(s,t),w_i(s)\Big)\,\delta(\mathbf{x}-\mathbf{Q}_i(s,t))\;\mathrm{d}^k s$$

Theorem 8 *These singular solutions again represent a momentum map*

$$\mathbf{J}:\bigtimes_{i=1}^{N}\Big(T^*\mathrm{Emb}(S,\mathbb{R}^n)\,\times\,\mathrm{Den}(S)\Big)\;\rightarrow\;\mathfrak{X}^*(\mathbb{R}^n)\,\circledS\,\mathrm{Den}(\mathbb{R}^n)\,.$$

Proof strategy. *Canonical Poisson structure on T^*Emb × Den.*

Proof: The *underlying symmetry group* is given by **compositions of diffeomorphisms with fiber translations**

$$\Big(\mathbf{Q}^{(t)},\, \mathbf{P}^{(t)},\, w^{(t)}\Big) = \Big(\eta_t\big(\mathbf{Q}^{(0)}\big),\, \mathbf{P}^{(0)}\cdot T\eta_t^{-1}\big(\mathbf{Q}^{(0)}\big) - \mathsf{d}\Big(w\,\pi_0\big(\mathbf{Q}^{(0)}\big)\Big),\, w^{(0)}\Big)$$
$$= \tau_{-\,\mathsf{d}(w\,\pi_0)}\circ\eta_t\,\left(\mathbf{Q}^{(0)},\, \mathbf{P}^{(0)},\, w^{(0)}\right),$$

Corollary 3 *The momentum map* $\mathbf{J}$ *is equivariant and thus is Poisson.*

The following quadratic Hamiltonian on $\times_{i=1}^{N}\,(T^*\text{Emb}\times\text{Den})$

$$H_N = \frac{1}{2}\sum_{i,j}^{N}\iint \mathbf{P}_i(s,t)\cdot\mathbf{P}_j(s',t)\; G_1\Big(\mathbf{Q}^i(s,t)-\mathbf{Q}^j(s',t)\Big)\;\mathsf{d}^k s\,\mathsf{d}^k s'$$
$$+\frac{1}{2}\sum_{i,j}^{N}\iint w_i(s)\,w_j(s')\,G_2\Big(\mathbf{Q}^i(s,t)-\mathbf{Q}^j(s',t)\Big)\;\mathsf{d}^k s\,\mathsf{d}^k s'\,.$$

governs the **collective dynamics**, with fixed weights $\dot{w}_i = 0$.

Collective Hamiltonian and Kaluza-Klein (KK)

- **Remark 34 (Single-particle Hamiltonian)** *In 1D, one obtains the following single particle Hamiltonian* $H_{N=1}$

$$H_1 = \frac{1}{2}\left(\mathbf{P}^2 + w^2\right) ,$$

This is the **KK Hamiltonian** *on* $T^*(\mathbb{R}^n \times \mathbb{R})$ *for* **free electric charge** w*. Charge conservation follows from the* *cyclic variable* θ *in* $H_1(\mathbf{Q}, \mathbf{P}, \theta, w)$*.*

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- This suggests that the collective Hamiltonian H_N could be treated as a **KK Hamiltonian on the configuration space**

$$Q_{KK} = \times_{i=1}^{N}\Big(\text{Emb}(S, \mathbb{R}^n) \times \mathcal{F}(S)\Big)$$

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- The **KK construction generates the right action momentum map**

$$J_S(\mathbf{Q}, \mathbf{P}, \theta, w) \;=\; \mathbf{P}\cdot \mathsf{d}\mathbf{Q} + w\, \mathsf{d}\theta \;:\; T^*Q_{KK} \to \mathfrak{X}^*(S)$$

which is *conserved*, due to symmetry under Diff(S) reparameterization (This is a version of the *Kelvin circulation theorem*.)

Extension to Yang-Mills dynamics

- The KK formulation generalizes to **non-Abelian gauge theories**

- Such a description applies to certain fluid theories (e.g. liquid crystals) involving extra degrees of freedom (gauge charge, e.g. orientation).

Extension to Yang-Mills dynamics

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- The basic theory is "*chromo-hydrodynamics*", modeling Yang-Mills quark-gluon plasmas. (Gibbons, H, Kupershmidt, 1982)

- Chromo-hydrodynamics is Lie-Poisson on the dual of the Lie algebra

$$\mathfrak{X}(\mathbb{R}^3) \circledS \Big(\mathfrak{g}(\mathbb{R}^3) \oplus \mathrm{Den}(\mathbb{R}^3)\Big)$$

  where $\mathfrak{g}(\mathbb{R}^3) := \big\{\chi \,|\, \chi : \mathbb{R}^3 \to \mathfrak{g}\big\}$ and $\mathfrak{g} = T_e G$ is the Lie algebra of G.

Main message(s)

- **EP(Diff$\circledS\mathfrak{g}$)**

Metamorphosis (Shape dynamics)

$\downarrow$

Kaluza-Klein (KK) formulation

$\downarrow$

Hamiltonian formulation (Lie-Poisson with cocycles)

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Metamorphosis (Shape dynamics)

$\downarrow$

Kaluza-Klein (KK) formulation

$\downarrow$

Hamiltonian formulation (Lie-Poisson with cocycles)

- **Geometry of EP(Diff$\circledS\mathcal{F}$)**

CH2 equation $+$ H^1 norm $\longrightarrow$ *Emergent singularities*

Singular solution momap $+$ Kaluza-Klein $\longrightarrow$ **Conservation laws**

Outlook

- Pairwise interactions of the singular solutions for EP(Diff$\circledS\mathcal{F}$) in 1D have been studied in joint work with L. Ó Náraigh and C. Tronci.

- Interaction of singular solutions in higher dimensions needs more study.

- The interactions of gauge-charged singular solutions for EP(Diff$\circledS\mathfrak{g}$) deserve special attention. These will interact by orientation-dependent collisions of particles carrying angular momentum.

- Can integrability be extended to gauge-charged geodesic solutions?

- EP applications to optimal control problems look especially promising.

Exercise 29 *As in the falling cat problem [Mon93], there is an interpretation of the first equation in (175) as defining the zero-momentum connection form for the horizontal subspace of the quotient space* $G \circledS N/G$*. Compute the curvature of the zero-momentum connection.*

Exercise 30 *Compute the metamorphosis equations* (156) *for the Euclidean metric on the semidirect-product Lie group* $SE(3)$*.*

Exercise 31 *Verify that the conditions* $\psi_{xxt} = \psi_{txx}$ *and* $\mathrm{d}\lambda/\mathrm{d}t = 0$ *together imply the CH2 system (185).*

Exercise 32 *Identify the Lie algebra on whose dual the Lie–Poisson Hamiltonian bracket for the CH2 system in (185) is defined in Exercise 26.*

Exercise 33 *Use Euler–Poincaré theory to derive the higher-dimensional version of the modified CH2 system (191).*

Solutions to selected exercises

Solution 1 *Solution to Exercise 26 One passes from Euler–Poincaré equations (167) on the Lagrangian side to Lie–Poisson Hamiltonian equations via the* **Legendre transformation***, see, e.g., [HMR98]. In our case, we start with the reduced Lagrangian* $\ell(u,\nu,n)$ *in eqn (145) and perform a Legendre transformation in the variables* u *and* ν *only, by writing the Hamiltonian,*

$$h(\mu,\sigma,n) = \Big\langle \mu,\, u \Big\rangle + \Big\langle \sigma,\, \nu \Big\rangle - \ell(u,\nu,n)\,. \tag{195}$$

Variation of the Hamiltonian yields

$$\begin{aligned} \delta h(\mu,\sigma,n) &= \left\langle \frac{\delta h}{\delta \mu}\,,\,\delta\mu \right\rangle + \left\langle \frac{\delta h}{\delta \sigma}\,,\,\delta\sigma \right\rangle + \left\langle \frac{\delta h}{\delta n}\,,\,\delta n \right\rangle \\ &= \Big\langle u,\,\delta\mu \Big\rangle + \Big\langle \nu,\,\delta\sigma \Big\rangle - \left\langle \frac{\delta \ell}{\delta n}\,,\,\delta n \right\rangle \\ &\quad + \left\langle \mu - \frac{\delta \ell}{\delta u}\,,\,\delta u \right\rangle + \left\langle \sigma - \frac{\delta \ell}{\delta \nu}\,,\,\delta \nu \right\rangle . \end{aligned} \tag{196}$$

The last two coefficients vanish under the Legendre transformation, so

$$\mu = \frac{\delta \ell}{\delta u}\,, \quad \sigma = \frac{\delta \ell}{\delta \nu}\,, \tag{197}$$

which recovers the definitions of the momentum variables (μ, σ) in terms of derivatives of the Lagrangian with respect to the velocities (u, ν). One then computes the variational derivatives of the Hamiltonian h as

$$\frac{\delta h}{\delta \mu} = u\,, \quad \frac{\delta h}{\delta \sigma} = \nu\,, \quad \frac{\delta h}{\delta n} = -\,\frac{\delta \ell}{\delta n}\,. \tag{198}$$

Consequently, the Euler–Poincaré equations (167) for metamorphosis in the Eulerian description imply the following equations, for the Legendre-transformed variables, (μ, σ, n), written as a matrix operation, symbolically as

$$\frac{\partial}{\partial t}\begin{bmatrix} \mu \\ \sigma \\ n \end{bmatrix} = -\begin{bmatrix} \mathrm{ad}^*_\Box\,\mu & \sigma \diamond \Box & -\,\Box \diamond n \\ \Box * \sigma & 0 & 1 \\ -\,\pounds_\Box\, n & -1 & 0 \end{bmatrix}\begin{bmatrix} \delta h/\delta \mu \\ \delta h/\delta \sigma \\ \delta h/\delta n \end{bmatrix} =: \mathcal{B}\begin{bmatrix} \delta h/\delta \mu \\ \delta h/\delta \sigma \\ \delta h/\delta n \end{bmatrix}, \tag{199}$$

with boxes $\Box$ indicating where the substitutions occur. The Poisson bracket *defined by the L^2 skew-symmetric Hamiltonian matrix $\mathcal{B}$ is given by*

$$\{f,h\} = \int \begin{bmatrix} \delta f/\delta\mu \\ \delta f/\delta\sigma \\ \delta f/\delta n \end{bmatrix}^T \mathcal{B} \begin{bmatrix} \delta h/\delta\mu \\ \delta h/\delta\sigma \\ \delta h/\delta n \end{bmatrix} \mathrm{d}x \,. \qquad (200)$$

The pair (σ, n) satisfies canonical Poisson-bracket relations. The other parts of the Poisson bracket are linear in the variables (μ, σ, n). This linearity is the signature of a Lie–Poisson bracket. The Lie algebra actions ensure that the Jacobi identity is satisfied. A similar Lie–Poisson bracket was found for complex fluids in [Hol02].

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