$\mathcal{PT}$-symmetry and complex Calogero systems

Paulo Assis

UK-Japan Winter School

Manchester, 7-10 January 2010

University of Kent

Outline

The occurrence of real spectra

In physical theories spectra are expected to be real

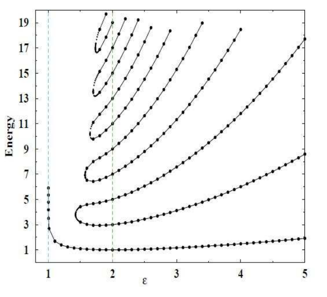
Complex eigenvalues in Quantum Mechanics are usually interpreted as belonging to dissipative (open) systems

- *Ising quantum spin chain in imagninay field corresponds to Yang-Lee model,*
  G.Von Gehlen, J.Phys. A24 (1991) 5371.
- *Solitons in Affine Toda models,*
  T.Hollowood, Nucl.Phys. B384 (1992) 523.
- *Complex Liouville theory related to Hermitian XXZ-quantum spin chain,*
  L.Faddeev and O.Tirkkonen, Nucl.Phys. B453 (1995) 647.

The occurrence of real spectra

Without solving the problem, when are the energies real ?

$$H = \hat{p}^2 - (\imath \hat{x})^\varepsilon \qquad 1 < \varepsilon \in \mathbb{R}$$



Boundary conditions: vanish asymptotically on curves in complex plane

Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$-symmetry, C.M.Bender and S.Boettcher, Phys.Rev.Lett. 80 (1998) 5243.

Identifying Hamiltonians with real spectra

Difficult to predict if the eigenvalues are real beforehand

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$

$\mathcal{PT}$ anti-linear: $\quad x \longrightarrow -x, \quad p \longrightarrow p, \quad \imath \longrightarrow -\imath,$

anti-linear symmetry :

$$[H, A] = 0$$

unbroken anti-linear symmetry :

$$A|\psi_n\rangle = |\psi_n\rangle$$

$$E_n|\psi_n\rangle = H|\psi_n\rangle = HA|\psi_n\rangle = AH|\psi_n\rangle = E_n^* A|\psi_n\rangle = E_n^*|\psi_n\rangle.$$

Normal form of antiunitary operators,
E. P. Wigner, J. Math. Phys. **1** (1960) 409.

Identifying Hamiltonians with real spectra

Difficult to predict if the eigenvalues are real beforehand

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$

$\mathcal{PT}$ anti-linear: $\quad x \longrightarrow -x, \quad p \longrightarrow p, \quad \imath \longrightarrow -\imath,$

anti-linear symmetry :

$$[H, A] = 0$$

unbroken anti-linear symmetry :

$$A|\psi_n\rangle = |\psi_n\rangle$$

$$E_n|\psi_n\rangle = H|\psi_n\rangle = HA|\psi_n\rangle = AH|\psi_n\rangle = E_n^* A|\psi_n\rangle = E_n^*|\psi_n\rangle.$$

Normal form of antiunitary operators,
E. P. Wigner, J. Math. Phys. **1** (1960) 409.

Identifying Hamiltonians with real spectra

Difficult to predict if the eigenvalues are real beforehand

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$

$\mathcal{PT}$ anti-linear: $\quad x \longrightarrow -x, \quad p \longrightarrow p, \quad \imath \longrightarrow -\imath,$

anti-linear symmetry :

$$[H, A] = 0$$

unbroken anti-linear symmetry :

$$A|\psi_n\rangle = |\psi_n\rangle$$

$$E_n|\psi_n\rangle = H|\psi_n\rangle = HA|\psi_n\rangle = AH|\psi_n\rangle = E_n^* A|\psi_n\rangle = E_n^*|\psi_n\rangle.$$

Normal form of antiunitary operators,
E. P. Wigner, J. Math. Phys. **1** (1960) 409.

Identifying Hamiltonians with real spectra

Difficult to predict if the eigenvalues are real beforehand

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$

$$\mathcal{PT} \text{ anti-linear:} \quad x \longrightarrow -x, \quad p \longrightarrow p, \quad \imath \longrightarrow -\imath,$$

anti-linear symmetry :

$$[H, A] = 0$$

unbroken anti-linear symmetry :

$$A|\psi_n\rangle = |\psi_n\rangle$$

$$E_n|\psi_n\rangle = H|\psi_n\rangle = HA|\psi_n\rangle = AH|\psi_n\rangle = E_n^* A|\psi_n\rangle = E_n^*|\psi_n\rangle.$$

Normal form of antiunitary operators,
E. P. Wigner, J. Math. Phys. **1** (1960) 409.

Identifying Hamiltonians with real spectra

Difficult to predict if the eigenvalues are real beforehand

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$

$\mathcal{PT}$ anti-linear: $\quad x \longrightarrow -x, \quad p \longrightarrow p, \quad \imath \longrightarrow -\imath,$

anti-linear symmetry :

$$[H, A] = 0$$

unbroken anti-linear symmetry :

$$A|\psi_n\rangle = |\psi_n\rangle$$

$$E_n|\psi_n\rangle = H|\psi_n\rangle = HA|\psi_n\rangle = AH|\psi_n\rangle = E_n^* A|\psi_n\rangle = E_n^*|\psi_n\rangle.$$

Normal form of antiunitary operators,
E. P. Wigner, J. Math. Phys. **1** (1960) 409.

Identifying Hamiltonians with real spectra

Difficult to predict if the eigenvalues are real beforehand

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$

$\mathcal{PT}$ anti-linear: $\quad x \longrightarrow -x, \quad p \longrightarrow p, \quad \imath \longrightarrow -\imath,$

anti-linear symmetry :

$$[H, A] = 0$$

unbroken anti-linear symmetry :

$$A|\psi_n\rangle = |\psi_n\rangle$$

$$E_n|\psi_n\rangle = H|\psi_n\rangle = HA|\psi_n\rangle = AH|\psi_n\rangle = E_n^* A|\psi_n\rangle = E_n^*|\psi_n\rangle.$$

Normal form of antiunitary operators,
E. P. Wigner, J. Math. Phys. **1** (1960) 409.

Identifying Hamiltonians with real spectra

Difficult to predict if the eigenvalues are real beforehand

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$

$\mathcal{PT}$ anti-linear: $\quad x \longrightarrow -x, \quad p \longrightarrow p, \quad \imath \longrightarrow -\imath,$

anti-linear symmetry :

$$[H, A] = 0$$

unbroken anti-linear symmetry :

$$A|\psi_n\rangle = |\psi_n\rangle$$

$$E_n|\psi_n\rangle = H|\psi_n\rangle = HA|\psi_n\rangle = AH|\psi_n\rangle = E_n^* A|\psi_n\rangle = E_n^*|\psi_n\rangle.$$

Normal form of antiunitary operators,
E. P. Wigner, J. Math. Phys. **1** (1960) 409.

Redefinition of contour: $\epsilon = 4$

$$H^{\epsilon=4} = p_z^2 - \kappa z^4$$

- Contour $$z(x) = -2\imath\sqrt{1+\imath x}$$

$$H^{\epsilon=4} = p_x^2 + \frac{1}{2}p_x + 16\kappa x^2 - 16\kappa + \imath\left(xp^2 - 32\kappa x\right)$$

- Equivalent to

$$h^{\epsilon=4} = \frac{1}{64\kappa}p^4 + \frac{1}{2}p + 16\kappa x^2 \qquad \text{on real line}$$

An equivalent Hermitian Hamiltonian for the $-x^4$ potential,
H. Jones and J. Mateo, Phys. Rev. D **73** (2006) 085002.

Relating Hermitian and non-Hermitian operators

left- and right-eigenvectors are different

$$\langle\phi_n|H = E_n\langle\phi_n| \qquad H|\varphi_n\rangle = E_n|\varphi_n\rangle \qquad H^\dagger = H$$

- Bi-orthogonality

$$\langle\phi_m|\varphi_n\rangle = \delta_{mn} \qquad \text{and} \qquad \sum_n |\varphi_n\rangle\langle\phi_n| = \mathbb{1}$$

- Dyson map : isospectral $\longrightarrow$ $h|\psi_n\rangle = E_n|\psi_n\rangle$

$$h = \eta\, H\, \eta^{-1} = h^\dagger \qquad |\phi_n\rangle = \rho|\varphi_n\rangle = \eta^\dagger|\psi_n\rangle$$

- Change of metric: $\rho = \eta^\dagger\eta$ (unitary evolution)
- Redefinition of observables: $H(\hat{x},\hat{p}) = h(\hat{X},\hat{P})$

Quasi-Hermitian Operators in Quantum Mechanics,
F. G. Scholtz, H. B. Geyer, and F. Hahne, Ann. Phys. **213** (1992) 74.

Darboux transformations

$$H_1 = -\frac{d^2}{dx^2} + V_1(x) = \mathcal{A}^\dagger \mathcal{A}$$

Decomposing

$$\mathcal{A} = \frac{d}{dx} + W(x) \qquad \text{and} \qquad \mathcal{A}^\dagger = -\frac{d}{dx} + W(x)$$

Considering one of the eigenfunctions of H_1 as the vacuum of $\mathcal{A}$

$$\mathcal{A}\psi_0^{(1)}(x) = 0 \qquad \text{so that} \qquad W(x) = -\frac{\psi_0'^{(1)}(x)}{\psi_0^{(1)}(x)}$$

we can construct a partner with the intertwining property

$$H_2 = -\frac{d^2}{dx^2} + V_2(x) = \mathcal{A}\,\mathcal{A}^\dagger$$

Schrödinger operators with complex potential but real spectrum,
F. Cannata, G. Junker and J. Trost, Phys. Lett. A **246** (1998) 219.

ODEs and Integrable Lattice Models

$$\left[-\frac{d^2}{dx^2} + x^{2M} + \alpha x^{M-1} + \frac{l(l+1)}{x^2} - E\right] y(x) = 0$$

$$C^{(\pm)}(E) \equiv W[y_{-1}, y_1](\pm\alpha) \qquad D^{(\pm)}(E) \equiv W[y, x^{l+1}](\pm\alpha)$$

$$C^{(+)}(E)D^{(+)}(E) = \omega^{-(2l+1+\alpha)/2} D^{(-)}(\omega^{-2}E) + \omega^{(2l+1+\alpha)/2} D^{(-)}(\omega^2 E)$$

define the zeros $E = E_k^{(\pm)}$ of C(E) (T-Q relations).

Bethe equations $$\prod_{n=1}^{\infty} \left(\frac{E_n^{(-)} + \omega^2 E}{E_n^{(-)} + \omega^{-2} E} \right) = \omega^{-(2l+1+\alpha)}$$

Spectral equivalences, Bethe ansatz equations, and reality properties, P. Dorey, C. Dunning and R. Tateo, J. Phys. A **34** (2001) 5679.

$\mathcal{PT}$ in classical theories

- Different interesting methods to establish reality of spectra in Quantum Mechanics
- Redefinition of Hilbert space is needed to make sense of non-Hermitian Hamiltonians
- $\mathcal{PT}$-symmetry stands out as a very convenient guiding principle for physical systems
- classical $\mathcal{PT}$-symmetric theories described by complex equations which nevertheless correspond to real energies

Classical $\mathcal{PT}$ symmetric models

Generate new complex systems potentially interesting from a physical point of view
$\Rightarrow$ deform known models in a $\mathcal{PT}$ symmetric way

Many possibilities to deform a PDE:
replacing ordinary space derivatives by a $\mathcal{PT}$-invariant form

$$\partial_x f(x) \to -\imath(\imath f_x)^\varepsilon \equiv f_{x;\varepsilon} \qquad \varepsilon \in \mathbb{N}$$

- $\partial_x^2 f(x) \to f_{x;\varepsilon} \circ f_{x;\varepsilon}$: does not preserve order of PDE
- $\partial_x^n f(x) \to \partial_x^{n-1} f_{x;\varepsilon} = \imath^{\varepsilon-1}\partial_x^{n-1}(f_x)^\varepsilon \equiv f_{nx;\varepsilon}$

PT-symmetric extension of the KdV equation,
C. M. Bender et al, J. Phys. **A**40 (2007) F153.
PT-Symmetric deformations of the KdV equation,
A. Fring, J. Phys. **A**40 (2007) 4215.

Complex deformations of KdV equation

KdV $\qquad u_t + uu_x + u_{xxx} = 0$

- First deformation:

$$u_t - \imath u(\imath u_x)^{\varepsilon} + u_{xxx} = 0$$

  $\varepsilon = 2$ two conserved charges: energy and momentum
  $\varepsilon = 2$ observation of solitary wave -like solutions

- Second deformation:

$$u_t + uu_x + \varepsilon(\varepsilon - 1)(\imath u_x)^{\varepsilon-2} u_{xx}^2 + \varepsilon(\imath u_x)^{\varepsilon-1} u_{xxx} = 0$$

  - three conserved charges; more easily constructed
  - constitutes a Hamiltonian system

$\Longrightarrow$ Highly nonlinear systems
Well behaved solutions

$\mathcal{PT}$-symmetric deformation of Burgers equation

Burgers $\quad u_t + uu_x = \kappa u_{xx}$

$$\longrightarrow \quad u_t + uu_{x;\varepsilon} = \kappa u_{xx;\mu} \quad \text{with} \quad \kappa \in \mathbb{R}, \quad \varepsilon, \mu \in \mathbb{Z}$$

$$u(z) = \sum_{m=0}^{\infty} \lambda_m (z - z_0)^{m+\theta} \qquad \theta = \frac{\varepsilon - \mu - 1}{\varepsilon - \mu + 1} \in \mathbb{Z}_- \quad \Longrightarrow \varepsilon = \mu, \quad \theta = -1$$

- Solve equation order by order

$$\begin{aligned} &\text{at order } -(2\varepsilon + 1): && \lambda_0 + 2\imath\varepsilon\kappa\phi_x = 0, \\ &\text{at order } -2\varepsilon: && \phi_t \delta_{\varepsilon,1} + \lambda_1 \phi_x - \imath\kappa\varepsilon\phi_{xx} = 0, \\ &\text{at order } -(2\varepsilon - 1): && \partial_x(\phi_t \delta_{\varepsilon,1} + \lambda_1 \phi_x - \imath\kappa\varepsilon\phi_{xx}) = 0, \end{aligned}$$

- Convergent series constructed
- Necessary condition for integrability (WTC Painlevé test, roughly)

The quantum Calogero problem (brief review)

One-dimensional problem of three particles interacting in pairs according an $\frac{1}{r^2}$ potential
(possibly with a quadratic confining quadratic well)

$$\left[-\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}+\sum_{i\neq j}^{3}\frac{g}{(x_i-x_j)^2}+\sum_{i\neq j}^{3}\omega^2(x_i-x_j)^2\right]\psi=E\psi$$

change of coordinates $\Rightarrow$ separation of variables

(centre of mass and polar Jacobi coordinates)

$$\begin{aligned}
R &= \frac{1}{3}(x_1+x_2+x_3),\\
r &= \frac{1}{\sqrt{3}}\sqrt{(x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2},\\
\phi &= \arctan\left[\frac{\sqrt{3}(x_1-x_2)}{(x_1-x_3)+(x_2-x_3)}\right]
\end{aligned}$$

The quantum Calogero problem (brief review)

$$x_{1,2} = R + \frac{r\cos\phi}{\sqrt{6}} \pm \frac{r\sin\phi}{\sqrt{2}} \qquad \text{and} \qquad x_3 = R - \sqrt{\frac{2}{3}}\, r\cos\phi.$$

$$\left[-\frac{1}{3}\frac{d^2}{dR^2} - \frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\left(\frac{d^2}{d\phi^2} - \frac{9g}{2\sin^2 3\phi}\right) - E\right]\psi(R, r, \phi) = 0,$$

Angular constant of motion

Radial constant of motion

Centre of mass constant of motion (absorbed as energy shift)

For simplicity $\omega = 0$ (Laguerre $\to$ Bessel)

Classical problem

$$\frac{1}{2}m\dot{r}^2 + \frac{B^2}{r^2} = E \qquad \text{and} \qquad \frac{1}{2}mr^4\dot{\phi}^2 + \frac{9g}{2\sin^2 3\phi} = B^2.$$

Integrability for classical Calogero problem

- Classical particle system

$$H_C = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{N} \frac{g}{(x_i - x_j)^2} \qquad \Longrightarrow \qquad \ddot{x}_i = \sum_{j\neq i}^{N} \frac{2g}{(x_i - x_j)^3}$$

- Lax pair (Moser) $\qquad N \times N$ matrices

$$L_{ij} = p_i \delta_{ij} + \frac{\imath\sqrt{g}}{x_i - x_j}(1 - \delta_{ij}),$$

$$M_{ij} = \sum_{k\neq i}^{N} \frac{\imath\sqrt{g}}{(x_i - x_k)^2}\delta_{ij} - \frac{\imath\sqrt{g}}{(x_i - x_j)^2}(1 - \delta_{ij}),$$

$$\frac{dL}{dt} + [M, L] = 0 \quad \Leftrightarrow \quad \text{Calogero equations of motion}$$

$$L(t) = U(t)L(0)U(t)^{-1} \quad \Rightarrow \quad I_N \equiv \frac{1}{N}\mathrm{tr}L^N : \text{conserved}$$

Classical solutions

- 2 particles

$$x_{1,2}(t) = 2R(t) \pm \sqrt{\frac{g}{E} + 4E(t-t_0)^2},$$

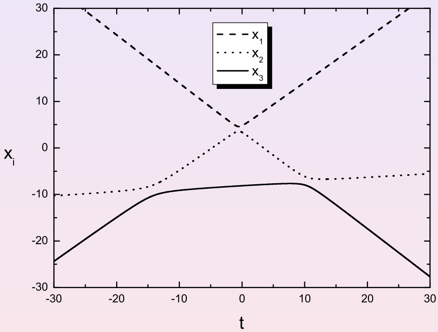
- 3 particles

$$\begin{aligned} x_{1,2}(t) &= R(t) + \frac{1}{\sqrt{6}} r(t)\cos\phi(t) \pm \frac{1}{\sqrt{2}} r(t)\sin\phi(t), \\ x_3(t) &= R(t) - \frac{2}{\sqrt{6}} r(t)\cos\phi(t), \end{aligned}$$

where

$$\begin{aligned} R(t) &= R_0 + V_0\, t, \\ r(t) &= \sqrt{\frac{B^2}{E} + 2E(t-t_0)^2}, \\ \phi(t) &= \frac{1}{3}\cos^{-1}\left\{\varphi_0 \sin\left[\sin^{-1}(\varphi_0\cos 3\phi_0) - 3\tan^{-1}\left(\frac{\sqrt{2}E}{B}(t-t_0)\right)\right]\right\}. \end{aligned}$$

$N = 3$ classical Calogero particles scattering



3 particles trajectory

Calogero particles as poles of nonlinear waves

- Burgers $\quad u_t + \left(\alpha u_{xx} + \beta u^2\right)_x = 0$
- Boussinesq $\quad u_{tt} + \left(\alpha u_{xx} + \beta u^2 - \gamma u\right)_{xx} = 0$
- "Multi-pole" solution

$$u(x,t) = -6\frac{\alpha}{\beta}\sum_{k=1}^{N}\frac{1}{(x - x_k(t))^2}$$

Constraints

$$\dot{x}_k(t) = -12\alpha\sum_{j\neq k}^{N}(x_k(t) - x_j(t))^{-2}, \qquad 0 = \sum_{j\neq k}^{N}(x_k(t) - x_j(t))^{-3}$$

and

$$\ddot{x}_k(t) = -24\alpha\sum_{j\neq k}^{N}(x_k(t) - x_j(t))^{-3}, \qquad \dot{x}_k(t)^2 = 12\alpha\sum_{j\neq k}^{N}(x_k(t) - x_j(t))^{-2} + \gamma$$

Compatibility of constraints with time evolution

- What constraints are compatible with the Hamiltonian flow?
  Airault, McKean, Moser: *Given a multi-particle Hamiltonian*

  $H(x_1, \ldots, x_N, p_1, \ldots, p_N)$ *with flow* $\dot{x}_i = \frac{\partial H}{\partial p_i}$ *and* $\dot{p}_i = -\frac{\partial H}{\partial x_i}$
  together with conserved charges I_n *in involution with* H*,*
  i.e. vanishing Poisson brackets $\{H, I_n\} = 0$ *, then the locus of* $grad(I_n) = 0$ *is invariant with respect to time evolution.*

$$
\begin{aligned}
\frac{d}{dt}\mathrm{grad} I &= \{\mathrm{grad} I, H\} = \mathrm{grad}\{I, H\} - \{I, \mathrm{grad} H\} = \\
&= 0 - \left(\frac{\partial I}{\partial x}\frac{\partial \mathrm{grad} H}{\partial p} - \frac{\partial I}{\partial p}\frac{\partial \mathrm{grad} H}{\partial x}\right) = 0
\end{aligned}
$$

$\Longrightarrow$ We restrict the flow to the locus of $\mathrm{grad}(I_n) = 0$
(provided it is not empty)

Calogero charges

- $I_n = \frac{1}{n}\mathrm{tr}(L^n)$

$$
\begin{aligned}
I_1 &= \sum_{i=1}^{N} p_i \\
I_2 &= \frac{1}{2}\sum_{i=1}^{N} p_i^2 + g\sum_{i\neq j}^{N}\frac{1}{(x_i - x_j)^2} \\
I_3 &= \frac{1}{3}\sum_{i=1}^{N} p_i^3 + g\sum_{i\neq j}^{N}\frac{p_i + p_j}{(x_i - x_j)^2}
\end{aligned}
$$

- I_3-flow restricted to the locus grad$(I_2) = 0$
  $\Rightarrow$ Burgers multi-poles solution
- I_2-flow subject to the constraint grad$(I_3 - \gamma I_1) = 0$
  $\Rightarrow$ Boussinesq multi-poles solution $(g = -12\alpha)$
- Calogero behaviour for poles in Boussinesq solution

Constrained motion

- Is the locus of interest is empty or not?
- $N = 3$
  change of variables

$$\begin{aligned} x_{1,2}(t) &= A_0(t) + A_1(t) \pm A_2(t), \\ x_3(t) &= A_0(t) + \lambda A_1(t), \end{aligned}$$

$$\begin{aligned} A_2(t) &= \frac{\sqrt{-g - 4\gamma(\lambda-1)^2 A_1(t)^2}}{2\sqrt{3\gamma}}, \\ \dot{A}_0(t) &= \sqrt{\gamma} + \frac{3g\sqrt{\gamma}(2+\lambda)}{(\lambda-1)[g + 16\gamma(\lambda-1)^2 A_1(t)^2]}, \\ \dot{A}_1(t) &= \frac{9g\sqrt{\gamma}}{(1-\lambda)[g + 16\gamma(\lambda-1)^2 A_1(t)^2]}, \end{aligned}$$

Complex motion of Boussinesq singularities

3 second order differential equations of motion (+6)
3 first order constraining equations (-3)
1 conserved quantities used: momentum (-1)

$$
\begin{aligned}
x_{1,2}(t) &= c_0 + \sqrt{\gamma} t + \frac{1}{12}\left(\frac{g}{\xi(t)} - \frac{\xi(t)}{\gamma}\right) \pm \frac{\imath}{4\sqrt{3}}\left(\frac{g}{\xi(t)} + \frac{\xi(t)}{\gamma}\right) \\
x_3(t) &= c_0 + \sqrt{\gamma} t - \frac{1}{6}\left(\frac{g}{\xi(t)} - \frac{\xi(t)}{\gamma}\right)
\end{aligned}
$$

with the abbreviation

$$
\xi(t) = \left[-54\gamma^2(\sqrt{\gamma} g t + c_1) + \sqrt{g^3\gamma^3 + [54\gamma^2(\sqrt{\gamma} g t + c_1)]^2}\right]^{\frac{1}{3}}
$$

2 constants of integration

$\mathcal{PT}$-symmetric constrained Calogero

Choosing $c_0, c_1 \in \imath\mathbb{R}$

$$\mathcal{T} : \left(\frac{g}{\xi(t)} \pm \frac{\xi(t)}{\gamma} \right) \longrightarrow \pm \left(\frac{g}{\xi(t)} \pm \frac{\xi(t)}{\gamma} \right)$$

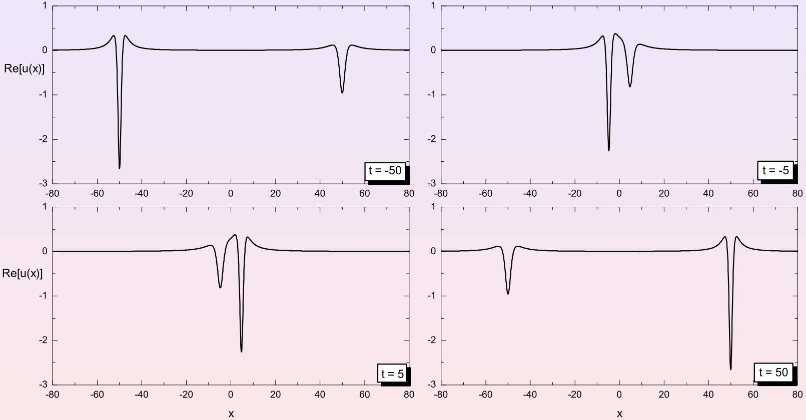
If $\gamma > 0$, then $\mathcal{PT} : x_i \longrightarrow -x_i \quad \leftrightarrow \quad H_C = \frac{N\gamma}{2}$

$\Rightarrow$ $\mathcal{PT}$-symmetry may arise more naturally from field theories without ad-hoc deformations

The Boussinesq solution

$$u(x,t) = -\frac{6\alpha}{\beta} \frac{1}{\left(\varphi - \frac{1}{6} \left(\frac{g}{\xi(t)} - \frac{\xi(t)}{\gamma} \right) \right)^2} + \frac{216\alpha}{\beta} \gamma^2 \xi(t)^2 \left[\frac{g^2\gamma^2 - 12g\gamma^2\varphi\xi(t) - 4\gamma(18\gamma\varphi^2 - g)\xi(t)^2 + 12\gamma\varphi\xi(t)^3 + \xi(t)^4}{(g^2\gamma^2 + 6g\gamma^2\varphi\xi(t) + \gamma(36\gamma\varphi^2 + g)\xi(t)^2 - 6\gamma\varphi\xi(t)^3 + \xi(t)^4)^2} \right]$$

Constraint Boussinesq solution



Wave profile evolution for Boussinesq 3-poles solution.

Calogero deformations: $\mathcal{PT}$-symmetric Weyl reflections

$$H_C = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i\neq j}^{N} V(q_i - q_j) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{\alpha\in\Delta} V(\alpha\cdot q)$$

$$H_C(q,p) \to H_{\mathcal{PT}}(\tilde{q},\tilde{p}) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{\tilde{\alpha}\in\tilde{\Delta}} \frac{g}{(\tilde{\alpha}\cdot q)^2}$$

$$\alpha_i \to \tilde{\alpha}_i = R(\varepsilon)\alpha_i + \imath I(\varepsilon)\sum_{j\neq i} \varsigma_j \lambda_j$$

$$\begin{aligned}
\tilde{q}_1 &= R(\varepsilon)q_1 - \imath\zeta_{\mathbf{g}} I(\varepsilon)(q_2 - q_3)\\
\tilde{q}_2 &= R(\varepsilon)q_2 - \imath\zeta_{\mathbf{g}} I(\varepsilon)(q_3 - q_1)\\
\tilde{q}_3 &= R(\varepsilon)q_3 - \imath\zeta_{\mathbf{g}} I(\varepsilon)(q_1 - q_2)
\end{aligned}$$

$\mathcal{PT}$-symmetric deformations of Calogero models,
A. Fring and M. Znojil, J. Phys. A **40** (2008) 194010.

- This constitutes a non-equivalent deformation

A less obvious connection

The equation

$$u_t + u_x + u^2 = 0$$

is solved by the ansatz

$$u(x,t) = \sum_{i=1}^{N} \frac{1 - \dot{z}_i(t)}{x - z_i(t)}$$

if

$$\ddot{z}_i(t) = 2\sum_{j \neq i}^{N} \frac{(1 - \dot{z}_i(t))(1 - \dot{z}_j(t))}{z_i(t) - z_j(t)}$$

Not conservative

Instead of solving this system, note that

$$u(x,t) = \frac{f(x-t)}{1 + tf(x-t)}$$

The new poles

- Assuming a multi-pole expansion for the arbitrary function

$$f(x) = \sum_{i=1}^{N} \frac{a_i}{\alpha_i - x}, \qquad \text{with } \alpha_i, a_i \in \mathbb{C}$$

- Determine the poles of original field u(x,t)
- $N = 3$

$$\begin{aligned} z_1(t) &= t - \frac{a(t)}{3} + s_+(t) + s_-(t) \\ z_{2,3}(t) &= t - \frac{a(t)}{3} - \frac{1}{2}\left[s_+(t) + s_-(t)\right] \pm \imath \frac{\sqrt{3}}{2}\left[s_+(t) - s_-(t)\right] \end{aligned}$$

where we abbreviated

$$\begin{aligned} s_\pm(t) &= \left[r(t) \pm \sqrt{r^2(t) + q^3(t)}\right]^{1/3} \\ r(t) &= \frac{9a(t)b(t) - 27c(t) - 2a^3(t)}{54}, \quad q(t) = \frac{3b(t) - a^2(t)}{9} \end{aligned}$$

Equivalence with Boussinesq poles

$$\begin{aligned} a(t) &= -a_1 - \alpha_2 - \alpha_3 - t(a_1 + a_2 + a_3) \\ b(t) &= \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 + t[a_1\bar{\alpha}_{23} + a_2\bar{\alpha}_{31} + a_3\bar{\alpha}_{21}] \\ c(t) &= -t(a_1\alpha_2\alpha_3 + a_2\alpha_3\alpha_1 + a_3\alpha_1\alpha_2) - \alpha_1\alpha_2\alpha_3 \end{aligned}$$

A subclass of these solutions is equivalent to Boussinesq poles

$$a_i = -\frac{g}{2}\prod_{j\neq i}(\alpha_i - \alpha_j)^{-2}, \quad g = 4\sum_{\substack{i=1 \\ i<j}}^{3} \alpha_i\alpha_j - \alpha_i^2$$

$$c_0 = \frac{1}{3}\sum_{i=1}^{3}\alpha_i, \quad c_1 = \frac{2}{27}\prod_{\substack{1\leq j<k\leq 3 \\ j,k\neq l}}(\alpha_j + \alpha_k - 2\alpha_l), \quad \gamma = 1$$

$\Rightarrow$ Identical singularity structure for different nonlinear wave eqtns

$\Rightarrow$ Possible to identify constrained (compatible) Hamiltonian flow

Conclusions

- $\mathcal{PT}$-symmetry useful in quantum mechanics
- Identifying potentially interesting deformations of integrable systems by using ideas of $\mathcal{PT}$-symmetry
- Complex particle systems arising from real valued nonlinear partial differential equations
- Possibility to associate $\mathcal{PT}$-symmetry to more natural complex extensions

Work in collaboration with Andreas Fring.

Thank you