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Random complex dynamics and singular functions on the complex plane

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The contents of this talk are included in my preprint:
H. Sumi, *Random complex dynamics and semigroups of holomorphic maps*, preprint 2008, available from http://arxiv.org/abs/0812.4483 or my webpage: http://www.math.sci.osaka-u.ac.jp/~sumi/ .

Some preprints of mine are available from the above webpage.

1 Introduction

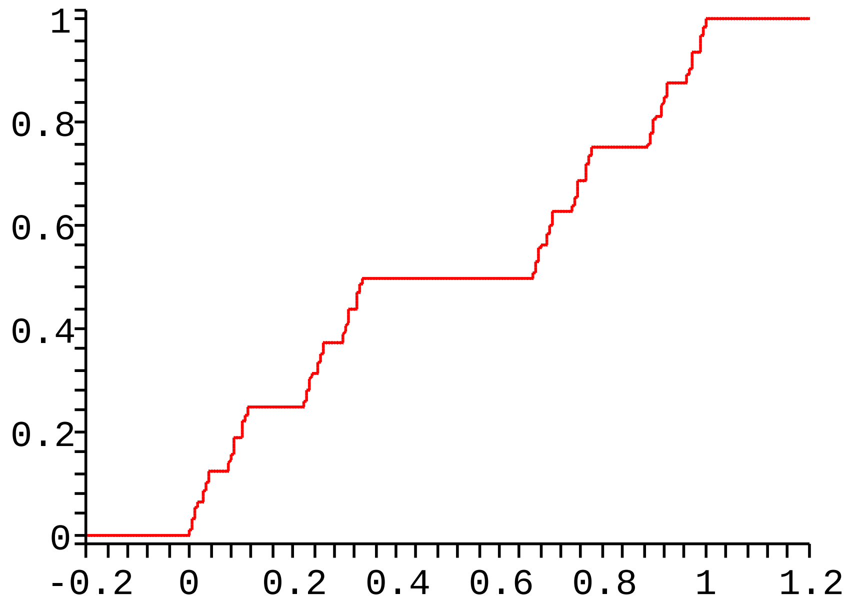
First, we consider the random dynamics on $\mathbb{R}$.

- Let $h_1(x) = 3x$ and $h_2(x) = 3(x-1)+1$ $(x \in \mathbb{R})$.

- We take an initial value $x \in \mathbb{R}$, and at every step, we choose the map h_1 with probability $1/2$ and h_2 with probability $1/2$, and map the point under the chosen map h_j.

- Let $T_{+\infty}(x)$ be the probability of tending to $+\infty$ starting with the initial value $x \in \mathbb{R}$.

Then,....

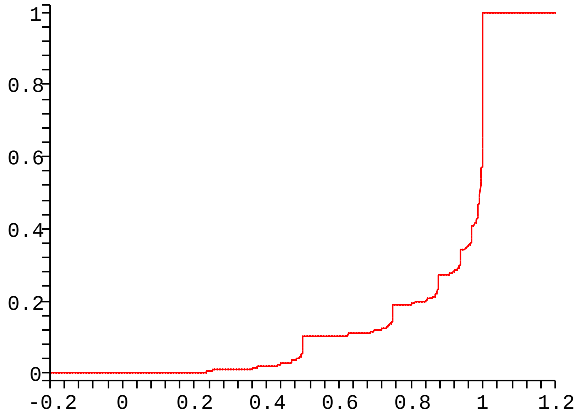


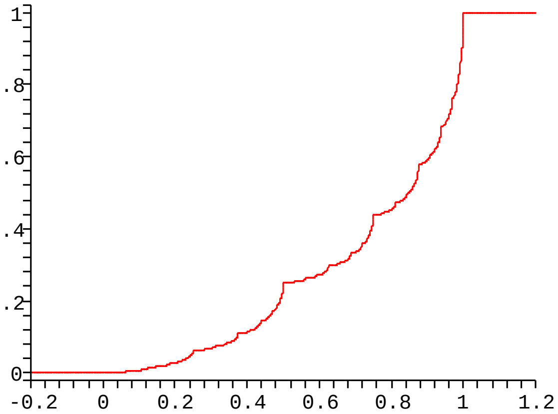
$T_{+\infty}$ is continuous on $\mathbb{R}$, varies only on the Cantor middle third set (which is a thin fractal set), and monotone.
$T_{+\infty}$ is called **the devil's staircase**. This is a typical example of singular functions.

Consider the same thing for the system:
$h_1(x) := 2x$ with probability p
$h_2(x) := 2(x-1)+1$ with probability $1-p$,
where $0 < p < 1$.
Let $T_{+\infty}(x,p)$ be the probability of tending to $+\infty$ starting with the initial value $x \in \mathbb{R}$.

Then,.......

The graph of $x \mapsto T_{+\infty}(x, p)$.
(From left) $p = 0.1,\ p = 0.25$.





The function $x \mapsto T_{+\infty,}(x, p)$ restricted to $[0, 1]$ is called **Lebesgue's singular function** with parameter p.

In this talk, we consider a similar story on the complex plane.

2 Preliminaries

Definition 2.1.

- We denote by $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \cong S^2$ the Riemann sphere and denote by d the spherical distance on $\hat{\mathbb{C}}$.

- We set
  Rat:$=\{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h$ is a non-const. rational map$\}$
  endowed with the distance η defined by
  $\eta(f, g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$.

- We set
  $\mathcal{P} := \{g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid g$ is a polynomial map$, \deg(g) \geq 2\}$
  endowed with the relative topology from Rat.

- Note that Rat and $\mathcal{P}$ are semigroups where the semigroup operation is functional composition.

- A subsemigroup G of Rat is called a **rational semigroup**.

- A subsemigroup G of $\mathcal{P}$ is called a **polynomial semigroup**.

Definition 2.2. Let G be a rational semigroup.

- We set
  $F(G) :=$
  $\{z \in \hat{\mathbb{C}} \mid \exists$ nbd U of z s.t. G is equicontinuous on $U\}$,

  where we say that G is equicontinuous on U if

  $\forall x \in U, \forall \epsilon > 0, \exists \delta > 0$ s.t.
  $d(x,y) < \delta, y \in U \Rightarrow \forall g \in G, d(g(x), g(y)) < \epsilon.$

  This $F(G)$ is called the **Fatou set** of G.

- We set $J(G) := \hat{\mathbb{C}} \setminus F(G)$.
  This is called the **Julia set** of G.

Lemma 2.3. *Let G be a rational semigroup. Then $F(G)$ is open and $J(G)$ is compact. Moreover, for each $h \in G$,*

$$h(F(G)) \subset F(G) \quad \textit{and} \quad h^{-1}(J(G)) \subset J(G).$$

However, the equality $h^{-1}(J(G)) = J(G)$ does not hold in general.

Remark 2.4. The fact we do not have $h^{-1}(J(G)) = J(G)$ is the difficulty in this theory. However, we 'utilize' this fact for the study of the random complex dynamics.

Definition 2.5.

- When a semigroup G is generated by $\{g_1, \ldots, g_m\}$, we write $G = \langle g_1, \ldots, g_m \rangle$.

- For an $h \in \text{Rat}$, we set $J(h) := J(\langle h \rangle)$.

Definition 2.6. For a topological space X, we denote by $\mathfrak{M}_1(X)$ the space of all Borel probability measures on X endowed with the weak topology.

Remark 2.7. If X is a compact metric space, then $\mathfrak{M}_1(X)$ is a compact metric space.

From now on, we take a $\tau \in \mathfrak{M}_1(\text{Rat})$ and we consider the (i.i.d.) random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in \text{Rat}$ according to τ.

Definition 2.8. Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$.

1. We endow $(\mathrm{Rat})^{\mathbb{N}} = \{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots) \mid \forall j, \gamma_j \in \mathrm{Rat}\}$ with the product topology.

2. We set $\tilde{\tau} := \otimes_{j=1}^{\infty} \tau \in \mathfrak{M}_1((\mathrm{Rat})^{\mathbb{N}})$.

3. We denote by $\operatorname{supp} \tau$ the topological support of τ (hence $\operatorname{supp} \tau$ is a closed subset of Rat).

4. Let G_τ be the rational semigroup generated by $\operatorname{supp} \tau$.

5. We set $C(\hat{\mathbb{C}}) := \{\varphi : \hat{\mathbb{C}} \to \mathbb{C} \mid \varphi$ is conti.$\}$ endowed with the sup. norm $\| \ \|_\infty$.

6. Let $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ be the operator defined by:

$$M_\tau(\varphi)(z) := \int_{\mathsf{Rat}} \varphi(g(z))\ d\tau(g),$$

where $\varphi \in C(\hat{\mathbb{C}}), z \in \hat{\mathbb{C}}$.

7. We set

$$C(\hat{\mathbb{C}})^* := \{\rho : C(\hat{\mathbb{C}}) \to \mathbb{C} \mid \rho \text{ is linear and continuous}\}$$

endowed with the weak topology.

8. Let $M_\tau^* : C(\hat{\mathbb{C}})^* \to C(\hat{\mathbb{C}})^*$ be the dual of M_τ.
That is, $M_\tau^*(\rho)(\varphi) := \rho(M_\tau(\varphi))$ for each $\rho \in C(\hat{\mathbb{C}})^*$ and for each $\varphi \in C(\hat{\mathbb{C}})$.

Note that $M_\tau^*(\mathfrak{M}_1(\hat{\mathbb{C}})) \subset \mathfrak{M}_1(\hat{\mathbb{C}})$.

Remark: Let $\Phi : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})$ be the map defined by $\Phi(z) := \delta_z$, where δ_z denotes the Dirac measure at z.

Note that $\Phi : \hat{\mathbb{C}} \to \mathfrak{M}_1(\hat{\mathbb{C}})$ is a topological embedding.

For an $h \in \text{Rat}$, if we set $\tau = \delta_h$, then we have the following commutative diagram:

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{\quad h \quad} & \hat{\mathbb{C}} \\ \Phi \downarrow & & \downarrow \Phi \\ \mathfrak{M}_1(\hat{\mathbb{C}}) & \xrightarrow{\quad M_\tau^* \quad} & \mathfrak{M}_1(\hat{\mathbb{C}}). \end{array}$$

9. We set

$$F_{meas}(\tau) := \{\mu \in \mathfrak{M}_1(\hat{\mathbb{C}}) \mid \exists \text{ nbd } B \text{ of } \mu \text{ in } \mathfrak{M}_1(\hat{\mathbb{C}}) \text{ s.t.} \{(M_\tau^*)^n|_B : B \to \mathfrak{M}_1(\hat{\mathbb{C}})\}_{n\in\mathbb{N}} \text{ is equicontinuous on } B\}.$$

10. We set $J_{meas}(\tau) := \mathfrak{M}_1(\hat{\mathbb{C}}) \setminus F_{meas}(\tau)$.

The following is the key to investigating the random complex dynamics.

Definition 2.9. Let G be a rational semigroup. We set

$$J_{\mathrm{ker}}(G) := \bigcap_{h \in G} h^{-1}(J(G)).$$

This is called the **kernel Julia set** of G.

Remark 2.10. $J_{\mathrm{ker}}(G)$ is a compact subset of $J(G)$. Moreover, for each $h \in G$, $h(J_{\mathrm{ker}}(G)) \subset J_{\mathrm{ker}}(G)$.

Lemma 2.11. *Let* Γ *be a compact subset of* $\mathcal{P}$. *If the interior of* Γ *with respect to the topology of* $\mathcal{P}$ *is not empty, then the polynomial semigroup* G *generated by* Γ *satisfies that* $J_{\ker}(G) = \emptyset$.

The above lemma implies that from a point of view, for most $\tau \in \mathfrak{M}_1(\mathcal{P})$ with compact support, we have $J_{\ker}(G_\tau) = \emptyset$.

Question 2.12. What happens if $J_{\ker}(G_\tau) = \emptyset$?

3 Results

Theorem 3.1 (Theorem A, Cooperation Principle).
Let $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ *be such that* $\mathrm{supp}\,\tau$ *is compact.*
Suppose that $J_{\mathrm{ker}}(G_\tau) = \emptyset$.
Then,

$$F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}}).$$

In other words, if all the maps in $\mathrm{supp}\,\tau$ *cooperate, then*

"*the chaos of the averaged system disappears*"

even if $J(G_\tau) \neq \emptyset$.

Remark: *If* $h \in \mathrm{Rat}$ *with* $\deg(h) \geq 2$, *then* $J_{meas}(\delta_h) \neq \emptyset$.

Notation: $\forall \tau \in \mathfrak{M}_1(\text{Rat})$, let $\mathcal{U}_\tau$ be the space of all finite linear spans of unitary eigenvectors of $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$. Let $\mathcal{B}_{0,\tau} := \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \to 0 \text{ as } n \to \infty\}$.

Theorem 3.2 (Theorem B).
Let $\tau \in \mathfrak{M}_1(\text{Rat})$ be such that $\text{supp}\,\tau$ *is compact.*
Suppose that $J_{\ker}(G_\tau) = \emptyset$ and $J(G_\tau) \neq \emptyset$.
Then, there exists a direct sum decomposition

$$C(\hat{\mathbb{C}}) = \mathcal{U}_\tau \oplus \mathcal{B}_{0,\tau}.$$

Moreover, $\dim_{\mathbb{C}} \mathcal{U}_\tau < \infty$.
Furthermore, for each $\varphi \in \mathcal{U}_\tau$ and for each connected component U of $F(G_\tau)$, $\varphi|_U$ is constant.

Definition 3.3. Let $\tau \in \mathfrak{M}_1(\mathcal{P})$.
For any $z \in \hat{\mathbb{C}}$, we set

$$T_{\infty,\tau}(z) := \tilde{\tau}(\{\gamma \in \mathcal{P}^{\mathbb{N}} \mid \gamma_n \circ \cdots \circ \gamma_1(z) \to \infty \text{ as } n \to \infty\}),$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots)$.

$T_{\infty,\tau}(z)$ is the probability of tending to $\infty \in \hat{\mathbb{C}}$
starting with the initial value $z \in \hat{\mathbb{C}}$
with respect to the random dynamics on $\hat{\mathbb{C}}$ such that
at every step we choose a map $h \in \mathcal{P}$ according to τ.

By the result $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$ in Theorem 3.1, we obtain the following Theorem 3.4.

Theorem 3.4. *Let* $\tau \in \mathfrak{M}_1(\mathcal{P})$ *be such that* $\operatorname{supp} \tau$ *is compact. Suppose that* $J_{\ker}(G_\tau) = \emptyset$.
Then, $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0,1]$ *is continuous on the whole* $\hat{\mathbb{C}}$.
Moreover, for each connected component U *of* $F(G_\tau)$,
$T_{\infty,\tau}|_U$ *is constant. Furthermore,* $M_\tau(T_{\infty,\tau}) = T_{\infty,\tau}$.

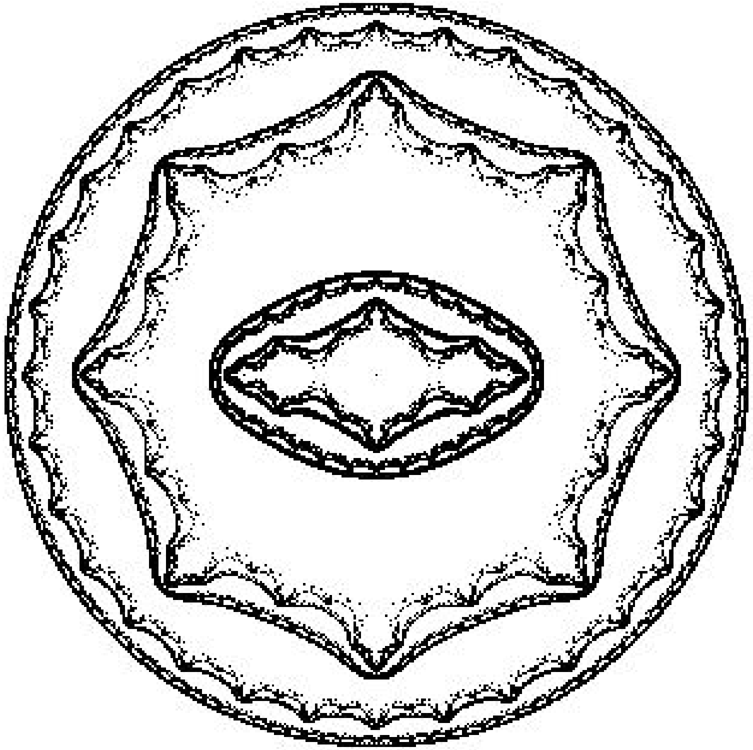
Remark 3.5. If $h \in \mathcal{P}, \tau = \delta_h$, then $T_{\infty,\tau}(\hat{\mathbb{C}}) = \{0, 1\}$, and at every point of $J(h)$ $(\neq \emptyset)$, $T_{\infty,\tau}$ is not continuous.

Remark 3.6. From Theorem 3.4 it follows that if $J_{\ker}(G_\tau) = \emptyset$, then $T_{\infty,\tau}$ is continuous on $\hat{\mathbb{C}}$ and the set of varying points is included in $J(G_\tau)$. Such a function $T_{\infty,\tau}$ is called

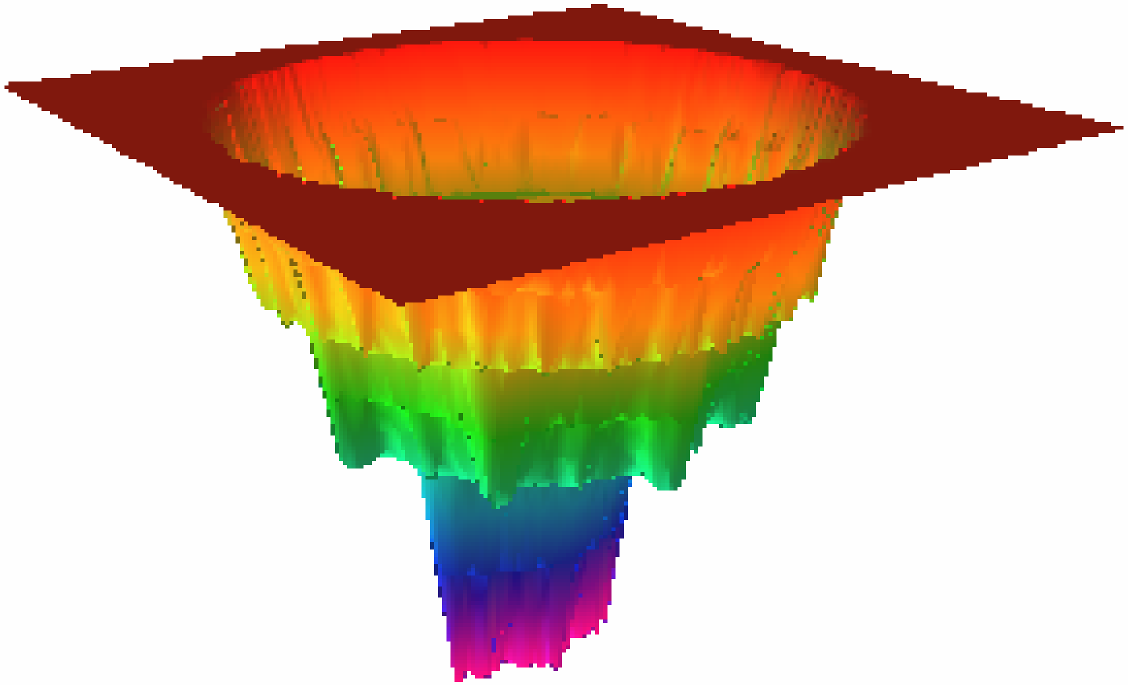
a devil's coliseum

provided that $T_{\infty,\tau} \not\equiv 1$. In fact, $T_{\infty,\tau}$ is a complex analogue of the devil's staircase.

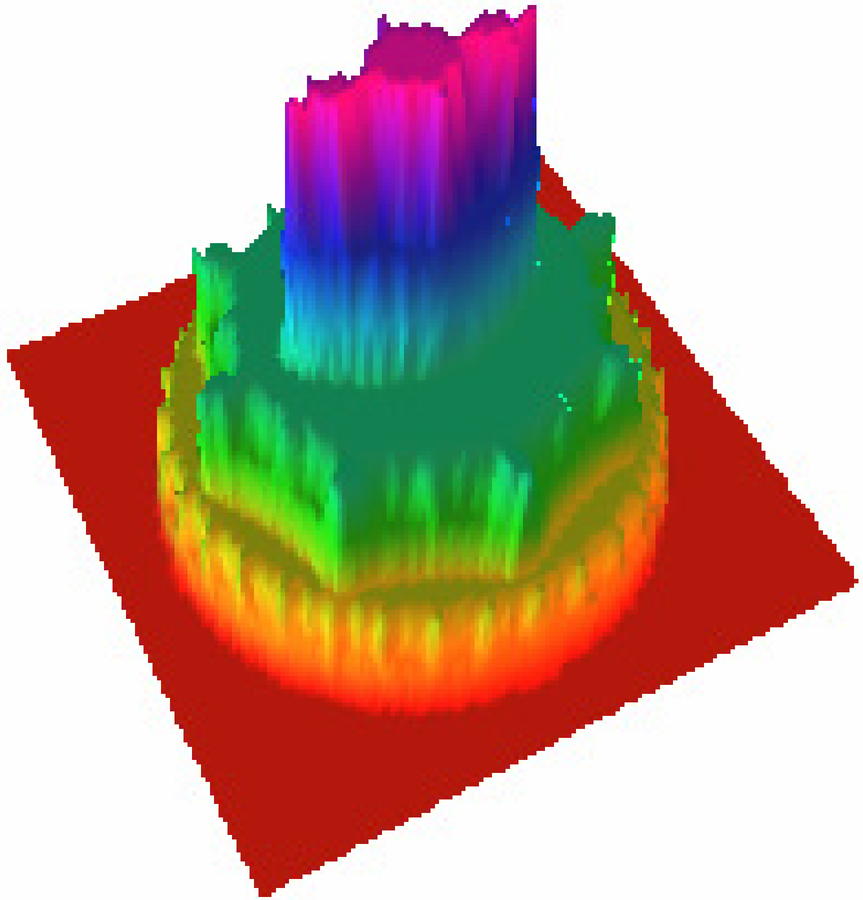
$g_1(z) := z^2 - 1,\ g_2(z) := \frac{z^2}{4},\ h_1 := g_1^2,\ h_2 := g_2^2$. $G := \langle h_1, h_2 \rangle$. $G \in \mathcal{G}_{dis}$.
The figure of $J(G)$. $\sharp \mathrm{Con}(J(G)) > \aleph_0$.



$g_1(z) := z^2 - 1,\ g_2(z) := \frac{z^2}{4},\ h_1 := g_1^2,\ h_2 := g_2^2,\ \tau := \frac{1}{2}\delta_{h_1} + \frac{1}{2}\delta_{h_2}$.
The graph of $z \mapsto T_{\tau,\infty}(z)$.
(Devil's Coliseum (Complex analogue of devil's staircase).)



The graph of $z \mapsto 1 - T_{\tau,\infty}(z)$.



We consider the non-differentiability of non-constant elements $\varphi \in \mathcal{U}_\tau$ at the Julia set $J(G_\tau)$.

Theorem 3.7 (Theorem C).

- *Let* $h_1, h_2 \in \mathcal{P}$ *and let* $G = \langle h_1, h_2 \rangle$.

- *Let* $0 < p_1, p_2 < 1$ *with* $p_1 + p_2 = 1$ *and we set* $\tau := \sum_{i=1}^{2} p_i \delta_{h_i} \in \mathfrak{M}_1(\mathcal{P})$.

- *Let*

$$P(G) := \overline{\bigcup_{h\in G} \{ \textit{ all critical values of } h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}} \ (\subset \hat{\mathbb{C}}).$$

- *We assume that*

(a) G *is hyperbolic* (i.e. $P(G) \subset F(G)$),

(b) $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$, *and*

(c) $\exists z \in \mathbb{C}$ s.t. $\bigcup_{h\in G}\{h(z)\}$ *is bounded in* $\mathbb{C}$.

Then, we have all of the following statements 1,...,4.

1. $J_{\ker}(G) = \emptyset$.

2. $T_{\infty,\tau} \in \mathcal{U}_\tau$ *and* $T_{\infty,\tau}$ *is non-constant.*

3. $\dim_H(J(G)) < 2$, *where* $\dim_H$ *denotes the Hausdorff dimension with respect to the Euclidian distance.*

4. $\exists \mu \in \mathfrak{M}_1(J(G))$ *satisfying all of the following.*

- $\mathrm{supp}\,\mu = J(G)$,
- *for each* $z \in J(G)$, $\mu(\{z\}) = 0$, *and*
- $\exists A \subset J(G)$ *with* $\mu(A) = 1$ s.t.
  $\forall z \in A$, $\forall$*non-const.* $\varphi \in \mathcal{U}_\tau$,

  pointwise Hölder exponent of φ at z

  $$:= \inf\{\alpha \in \mathbb{R} \mid \overline{\lim}_{y \to z} \frac{|\varphi(y) - \varphi(z)|}{|y - z|^\alpha} = \infty\}$$

  $$= \frac{\text{entropy of } (p_1, p_2)}{\text{“averaged Lyapunov exponent”}} < 1$$

  and φ *is not differentiable at* z.

In particular, $\exists A$: *uncountable dense subset of* $J(G)$ *s.t.* $\forall z \in A$, $\forall$*non-const.* $\varphi \in \mathcal{U}_\tau$, φ *is not differentiable at* z.

Remark 3.8. In the proof of statement 4 of the previous theorem, we use

- Birkhoff's ergodic theorem (**ergodic theory**),
- Koebe distortion theorem (**function theory**), and
- Green's function and calculation of Lyapunov exponent (**potential theory**).

4 Example

Proposition 4.1. *Let* $h_1 \in \mathcal{P}$ *be hyperbolic.*

- *Suppose that* $K(h_1)$ *is connected and* $\text{int}K(h_1) \neq \emptyset$, *where* $K(h_1) := \{z \in \mathbb{C} \mid \{h_1^n(z)\}_{n\in\mathbb{N}}\text{is bounded}\}$.
- *Let* $b \in \text{int}K(h_1)$.
- *Let* $d \in \mathbb{N}$ *with* $d \geq 2$ *be* s.t. $(\deg(h_1), d) \neq (2, 2)$.

Then $\exists c > 0$ s.t. $\forall a \in \mathbb{C}$ *with* $0 < |a| < c$,
setting $h_2(z) = a(z - b)^d + b$,
$\{h_1, h_2\}$ *satisfies the assumption of Theorem C, i.e.,*

(a) $G = \langle h_1, h_2 \rangle$ *is hyperbolic,*
(b) $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$, *and*
(c) $\exists z \in \mathbb{C}$ s.t. $\bigcup_{h\in G}\{h(z)\}$ *is bounded in* $\mathbb{C}$.

5 Summary

- We simultaneously develop the theory of random complex dynamics and that of the dynamics of semigroups of holomorphic maps.

- Both fields are related to each other very deeply.

- While we study these fields, singular functions on the complex plane (devil's coliseums) appear.

Supplement: we give a precise definition of μ and give a detail in statement of 4 in Theorem C.

- *Let* $\Gamma = \{h_1, h_2\}$ *and for each* $(\gamma, y) \in \Gamma^{\mathbb{N}} \times \mathbb{C}$, *we set*

$$\mathcal{G}_\gamma(y) := \lim_{n\to\infty} \frac{1}{\deg(\gamma_n \circ \cdots \circ \gamma_1)} \log^+ |\gamma_n \circ \cdots \circ \gamma_1(y)|,$$

where $\log^+(a) := \max\{\log a, 0\}$ *for each* $a > 0$.

- *For each* $\gamma \in \Gamma^{\mathbb{N}}$,
let $\mu_\gamma := dd^c \mathcal{G}_\gamma \in \mathfrak{M}_1(J(G))$, *where* $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$.
We set $\mu := \int_{\Gamma^{\mathbb{N}}} \mu_\gamma \, d\tilde{\tau}(\gamma) \in \mathfrak{M}_1(J(G))$.

- *For each* $\gamma \in \Gamma^{\mathbb{N}}$,
let $\Omega(\gamma) := \sum_c \mathcal{G}_\gamma(c)$, *where* c *runs over all critical points of* γ_1 *in* $\mathbb{C}$.

4. • supp $\mu = J(G)$,

• *for each* $z \in J(G)$, $\mu(\{z\}) = 0$, *and*

• $\exists A \subset J(G)$ *with* $\mu(A) = 1$ s.t. $\forall z \in A$, $\forall$*non-constant* $\varphi \in \mathcal{U}_\tau$,

pointwise Hölder exponent of φ at z

$$:= \inf\{\alpha \in \mathbb{R} \mid \overline{\lim}_{y \to z} \frac{|\varphi(y) - \varphi(z)|}{|y - z|^\alpha} = \infty\}$$

$$= \frac{-(\sum_{i=1}^2 p_i \log p_i)}{\sum_{i=1}^2 p_i \log(\deg(h_i)) + \int_{\Gamma^{\mathbb{N}}} \Omega(\gamma)\ d\tilde{\tau}(\gamma)} < 1$$

and φ *is not differentiable at* z.

In particular, $\exists A$*: uncountable dense subset of* $J(G)$ *s.t.* $\forall z \in A$, $\forall$*non-constant* $\varphi \in \mathcal{U}_\tau$, φ *is not differentiable at* z.