THE GEOMETRIC STRUCTURE OF THE COMPLEX FLUID E QUATIONS

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PLAN OF THE PRESENTATION

- **Example: Ericksen-Leslie equations**

- **Example: Eringen equations**

- **The general theory**

PUNCHLINE: All these equations are obtained by Euler-Poincaré and Lie-Poisson reduction from material representation. These reduction procedures need to be extended to include affine terms and the groups have a relatively complicated internal structure adapted to complex fluids.

EXAMPLE: ERICKSEN-LESLIE EQUATIONS

Liquid crystal state: a distinct phase of matter observed between the crystalline (solid) and isotropic (liquid) states. Three main types of liquid crystal states, depending upon the amount of order:

Nematic liquid crystal phase: characterized by rod-like molecules, no positional order, but tend to point in the same direction.

Cholesteric (or *chiral nematic*) liquid crystal phase: molecules resemble helical springs, which may have opposite chiralities. Molecules exhibit a privileged direction, which is the axis of the helices.

Smectic liquid crystals are essentially different form both nematics and cholesterics: they have one more degree of orientational order. Smectics generally form layers within which there is a loss of positional order, while orientational order is still preserved.

Three main theories:

Director theory due to Oseen, Frank, Zöcher, Ericksen and Leslie

Micropolar and *microstretch theories*, due to Eringen, which take into account the microinertia of the particles and which is applicable, for example, to *liquid crystal polymers*

Ordered micropolar approach, due to Lhuillier and Rey, which combines the director theory with of the micropolar models.

In all that follows $\mathcal{D} \subset \mathbb{R}^3$ and all boundary conditions are ignored: in all integration by parts the boundary terms vanish. We fix a volume form μ on $\mathcal{D}$.

EXAMPLE: DIRECTOR THEORY (nematics, cholesterics)

Assumption: only the direction and not the sense of the molecules matter. The preferred orientation of the molecules around a point is described by a unit vector $\mathbf{n} : \mathcal{D} \to S^2$, called the *director*, and $\mathbf{n}$ and $-\mathbf{n}$ are assumed to be equivalent.

Ericksen-Leslie equations in a domain $\mathcal{D}$, constraint $\|\mathbf{n}\| = 1$, are:

$$\begin{cases} \rho\left(\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u}\right) = \operatorname{grad}\frac{\partial F}{\partial \rho^{-1}} - \partial_j\left(\rho\frac{\partial F}{\partial \mathbf{n}_{,j}}\cdot\nabla\mathbf{n}\right), \\ \rho J\frac{D^2}{dt^2}\mathbf{n} - 2q\mathbf{n} + \mathbf{h} = 0, \\ \frac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \qquad \frac{D}{dt} := \frac{\partial}{\partial t} + \nabla_{\mathbf{u}} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla \end{cases}$$

$\mathbf{u}$ *Eulerian velocity*, ρ *mass density*, $\mathbf{n} : \mathcal{D} \to \mathbb{R}^3$ *director* ($\mathbf{n}$ equivalent to -$\mathbf{n}$), J *microinertia constant*, and $F(\mathbf{n}, \mathbf{n}_{,i})$ is the *free energy*. The axiom of objectivity requires that

$$F(\rho^{-1}, A^{-1}\mathbf{n}, A^{-1}\nabla\mathbf{n}A) = F(\rho^{-1}, \mathbf{n}, \nabla\mathbf{n}),$$

for all $A \in O(3)$ for nematics, or for all $A \in SO(3)$ for cholesterics.

$$\mathbf{h} := \rho\frac{\partial F}{\partial \mathbf{n}} - \partial_i\left(\rho\frac{\partial F}{\partial \mathbf{n}_{,i}}\right)$$

is called the *molecular field*. q is unknown and determined by

$$2q := \mathbf{n}\cdot\mathbf{h} - \rho J\left\|\frac{D\mathbf{n}}{dt}\right\|^2$$

This is seen in the following way.

Take the dot product with $\mathbf{n}$ of the second equation to get

$$2q = \rho J \mathbf{n} \cdot \frac{D^2}{dt^2}\mathbf{n} + \mathbf{n} \cdot \mathbf{h} = \mathbf{n} \cdot \mathbf{h} - \rho J \left\| \frac{D\mathbf{n}}{dt} \right\|^2$$

since $\|\mathbf{n}\|^2 = 1$ implies $\mathbf{n} \cdot \frac{D\mathbf{n}}{dt} = 0$ and hence, taking one more material derivative gives

$$\mathbf{n} \cdot \frac{D^2}{dt^2}\mathbf{n} = - \left\| \frac{D\mathbf{n}}{dt} \right\|^2 .$$

Think of the function q in the Ericksen-Leslie equation the way one regards the pressure in ideal incompressible homogeneous fluid dynamics, namely, the q is an unknown function determined by the imposed constraint $\|\mathbf{n}\| = 1$.

WHAT IS THE STRUCTURE OF THESE EQUATIONS?

Let $(\mathbf{u}, \rho, \mathbf{n})$ *be a solution of the Ericksen-Leslie equations such that* $\|\mathbf{n}\| = 1$ *and define*

$$\boldsymbol{\nu} := \mathbf{n} \times \frac{D}{dt}\mathbf{n} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3), \qquad \frac{D}{dt} := \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad \textit{material derivative.}$$

Then $(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n})$ *is a solution of the equations*

$$(motion) \quad \begin{cases} \rho \left(\dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} \right) = \operatorname{grad} \dfrac{\partial F}{\partial \rho^{-1}} - \partial_i \left(\rho \dfrac{\partial F}{\partial \mathbf{n}_{,i}} \cdot \nabla \mathbf{n} \right), \\ \rho J \dfrac{D}{dt}\boldsymbol{\nu} = \mathbf{h} \times \mathbf{n}, \end{cases}$$

$$(advection) \quad \begin{cases} \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \dfrac{D}{dt}\mathbf{n} = \boldsymbol{\nu} \times \mathbf{n}, \end{cases}$$

Evolution of ρ, $\mathbf{n}$ *(where* $\eta \in \operatorname{Diff}(\mathcal{D})$, $\chi \in \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ *) is*

$$\rho = J(\eta^{-1})(\rho_0 \circ \eta^{-1}) \ \textit{ and } \ \mathbf{n} = (\chi \mathbf{n}_0) \circ \eta^{-1}.$$

These equations are Euler-Poincaré/Lie-Poisson for the group

$$\left(\mathrm{Diff}(\mathcal{D})\,\circledS\,\mathcal{F}(\mathcal{D},\mathrm{SO}(3))\right)\circledS\left(\mathcal{F}(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathbb{R}^3)\right).$$

EXPLANATION:

- $\mathrm{Diff}(\mathcal{D})$ acts on $\mathcal{F}(\mathcal{D},\mathrm{SO}(3))$ via the *right* action

$$(\eta,\chi)\in\mathrm{Diff}(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathrm{SO}(3))\mapsto\chi\circ\eta\in\mathcal{F}(\mathcal{D},\mathrm{SO}(3)).$$

Therefore, the group multiplication in $\mathrm{Diff}(\mathcal{D})\,\circledS\,\mathcal{F}(\mathcal{D},\mathrm{SO}(3))$ is

$$(\eta,\chi)(\varphi,\psi)=(\eta\circ\varphi,(\chi\circ\varphi)\psi).$$

- The bracket of $\mathfrak{X}(\mathcal{D})\,\circledS\,\mathcal{F}(\mathcal{D},\mathfrak{so}(3))$ is

$$\mathrm{ad}_{(\mathbf{u},\nu)}(\mathbf{v},\zeta)=(\mathrm{ad}_{\mathbf{u}}\mathbf{v},\mathrm{ad}_\nu\zeta+\mathbf{d}\nu\cdot\mathbf{v}-\mathbf{d}\zeta\cdot\mathbf{u}),$$

where $\mathrm{ad}_{\mathbf{u}}\mathbf{v}=-[\mathbf{u},\mathbf{v}]$, $\mathrm{ad}_\nu\zeta\in\mathcal{F}(\mathcal{D},\mathfrak{so}(3))$ is given by $\mathrm{ad}_\nu\zeta(x):=\mathrm{ad}_{\nu(x)}\zeta(x)$, and $\mathbf{d}\nu\cdot\mathbf{v}\in\mathcal{F}(\mathcal{D},\mathfrak{so}(3))$ is given by $\mathbf{d}\nu\cdot\mathbf{v}(x):=\mathbf{d}\nu(x)(\mathbf{v}(x))$.

- $(\eta,\chi)\in\mathrm{Diff}(\mathcal{D})\,\circledS\,\mathcal{F}(\mathcal{D},\mathrm{SO}(3))$ acts *linearly* and on the *right* on the advected quantities $(\rho,\mathbf{n})\in\mathcal{F}(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathbb{R}^3)$, by

$$(\rho,\mathbf{n})\mapsto\left(J(\eta)(\rho\circ\eta),\chi^{-1}(\mathbf{n}\circ\eta)\right).$$

- The associated infintesimal action and diamond operations are

$$\mathbf{n}\mathbf{u} = \nabla\mathbf{n}\cdot\mathbf{u}, \quad \mathbf{n}\boldsymbol{\nu} = \mathbf{n}\times\boldsymbol{\nu}, \quad \mathbf{m}\diamond_1\mathbf{n} = -\nabla\mathbf{n}^T\cdot\mathbf{m} \quad \text{and} \quad \mathbf{m}\diamond_2\mathbf{n} = \mathbf{n}\times\mathbf{m},$$

where $\boldsymbol{\nu}, \mathbf{m}, \mathbf{n} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3)$.

- EP equations for $\left(\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))\right) \circledS \left(\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathbb{R}^3)\right)$:

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \mathbf{u}} = -\pounds_{\mathbf{u}}\dfrac{\delta l}{\delta \mathbf{u}} - \operatorname{div}\mathbf{u}\dfrac{\delta l}{\delta \mathbf{u}} - \dfrac{\delta l}{\delta \boldsymbol{\nu}}\cdot\mathbf{d}\boldsymbol{\nu} + \rho\,\mathbf{d}\dfrac{\delta l}{\delta \rho} - \left(\nabla\mathbf{n}^T\cdot\dfrac{\delta l}{\delta \mathbf{n}}\right)^\flat, \\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \boldsymbol{\nu}} = \boldsymbol{\nu}\times\dfrac{\delta l}{\delta \boldsymbol{\nu}} - \operatorname{div}\left(\dfrac{\delta l}{\delta \boldsymbol{\nu}}\mathbf{u}\right) + \mathbf{n}\times\dfrac{\delta l}{\delta \mathbf{n}}, \end{cases}$$

- The advection equations are:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \\ \dfrac{\partial}{\partial t}\mathbf{n} + \nabla\mathbf{n}\cdot\mathbf{u} + \mathbf{n}\times\boldsymbol{\nu} = 0. \end{cases}$$

- Reduced Lagrangian for nematic and cholesteric liquid crystals:

$$l(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n}) := \frac{1}{2}\int_{\mathcal{D}}\rho\|\mathbf{u}\|^2\mu + \frac{1}{2}\int_{\mathcal{D}}\rho J\|\boldsymbol{\nu}\|^2\mu - \int_{\mathcal{D}}\rho F(\rho^{-1}, \mathbf{n}, \nabla\mathbf{n})\mu.$$

- The functional derivatives of the Lagrangian l are:

$$\mathbf{m} := \frac{\delta l}{\delta \mathbf{u}} = \rho \mathbf{u}^\flat, \quad \boldsymbol{\kappa} := \frac{\delta l}{\delta \boldsymbol{\nu}} = \rho J \boldsymbol{\nu},$$

$$\frac{\delta l}{\delta \rho} = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{1}{2} J \|\boldsymbol{\nu}\|^2 - F + \frac{1}{\rho}\frac{\partial F}{\partial \rho^{-1}}, \quad \frac{\delta l}{\delta \mathbf{n}} = -\rho \frac{\partial F}{\partial \mathbf{n}} + \partial_i \left(\rho \frac{\partial F}{\partial \mathbf{n}_{,i}} \right) = -\mathbf{h}.$$

- By the Legendre transformation, the Hamiltonian is:

$$h(\mathbf{m}, \boldsymbol{\kappa}, \rho, \mathbf{n}) := \frac{1}{2} \int_{\mathcal{D}} \frac{1}{\rho} \|\mathbf{m}\|^2 \mu + \frac{1}{2J} \int_{\mathcal{D}} \frac{1}{\rho} \|\boldsymbol{\kappa}\|^2 \mu + \int_{\mathcal{D}} \rho F(\rho^{-1}, \mathbf{n}, \nabla \mathbf{n}) \mu.$$

- The Poisson bracket for liquid crystals is given by:

$$\begin{aligned}
\{f,g\}(\mathbf{m}, \rho, \boldsymbol{\kappa}, \mathbf{n}) &= \int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}} \right] \mu \\
&\quad + \int_{\mathcal{D}} \boldsymbol{\kappa} \cdot \left(\frac{\delta f}{\delta \boldsymbol{\kappa}} \times \frac{\delta g}{\delta \boldsymbol{\kappa}} + \mathbf{d}\frac{\delta f}{\delta \boldsymbol{\kappa}} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \boldsymbol{\kappa}} \cdot \frac{\delta f}{\delta \mathbf{m}} \right) \mu \\
&\quad + \int_{\mathcal{D}} \rho \left(\mathbf{d}\frac{\delta f}{\delta \rho} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \rho} \cdot \frac{\delta f}{\delta \mathbf{m}} \right) \mu \\
&\quad + \int_{\mathcal{D}} \left[\left(\mathbf{n} \times \frac{\delta f}{\delta \boldsymbol{\kappa}} + \nabla \mathbf{n} \cdot \frac{\delta f}{\delta \mathbf{m}} \right) \frac{\delta g}{\delta \mathbf{n}} - \left(\mathbf{n} \times \frac{\delta g}{\delta \boldsymbol{\kappa}} + \nabla \mathbf{n} \cdot \frac{\delta g}{\delta \mathbf{m}} \right) \frac{\delta f}{\delta \mathbf{n}} \right] \mu.
\end{aligned}$$

- The Kelvin circulation theorem for liquid crystals reads:

$$\frac{d}{dt}\oint_{c_t} \mathbf{u}^\flat = \oint_{c_t} \frac{1}{\rho}\nabla \mathbf{n}^T\cdot\mathbf{h} \quad \text{where} \quad \mathbf{h} = \rho\frac{\partial F}{\partial \mathbf{n}} - \partial_i\left(\rho\frac{\partial F}{\partial \mathbf{n}_{,i}}\right).$$

Now do the converse: show that the EP equations imply the Ericksen-Leslie equations. For this one needs to show first that if $\boldsymbol{\nu}$ and $\mathbf{n}$ are solutions of the EP equations then:

(i) $\|\mathbf{n}_0\| = 1$ *implies* $\|\mathbf{n}\| = 1$ *for all time.*

(ii) $\frac{D}{dt}(\mathbf{n}\cdot\boldsymbol{\nu}) = 0$. *Therefore,* $\mathbf{n}_0\cdot\boldsymbol{\nu}_0 = 0$ *implies* $\mathbf{n}\cdot\boldsymbol{\nu} = 0$ *for all time.*

(iii) *Suppose that* $\mathbf{n}_0\cdot\boldsymbol{\nu}_0 = 0$ *and* $\|\mathbf{n}_0\| = 1$. *Then*

$$\frac{D}{dt}\mathbf{n} = \boldsymbol{\nu}\times\mathbf{n} \quad \textit{becomes} \quad \boldsymbol{\nu} = \mathbf{n}\times\frac{D}{dt}\mathbf{n}$$

and

$$\rho J\frac{D}{dt}\boldsymbol{\nu} = \mathbf{h}\times\mathbf{n} \quad \textit{becomes} \quad \rho J\frac{D^2}{dt^2}\mathbf{n} - 2q\mathbf{n} + \mathbf{h} = 0.$$

If $(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n})$ is a solution of the Euler-Poincaré equations with initial conditions $\mathbf{n}_0$ and $\boldsymbol{\nu}_0$ satisfying $\|\mathbf{n}_0\| = 1$ and $\mathbf{n}_0 \cdot \boldsymbol{\nu}_0 = 0$, then $(\mathbf{u}, \rho, \mathbf{n})$ is a solution of the Ericksen-Leslie equations.

The q does not appear in the Euler-Poincaré formulation relative to the variables $(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n})$, since in this case, the constraint $\|\mathbf{n}\| = 1$ is automatically satisfied.

Consequence of this theorem: the Ericksen-Leslie equations are obtained by Lagrangian reduction. Right-invariant Lagrangian

$$L_{(\rho_0, \mathbf{n}_0)} : T\left[\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))\right] \to \mathbb{R}$$

induced by the Lagrangian l (make it right invariant after freezing the parameters $(\rho_0, \mathbf{n}_0)$). Assume that $\|\mathbf{n}_0\| = 1$ and $\boldsymbol{\nu}_0 \cdot \mathbf{n}_0 = 0$. A curve $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ is a solution of the Euler-Lagrange equations for $L_{(\rho_0, \mathbf{n}_0)}$, with initial condition $\mathbf{u}_0$, $\boldsymbol{\nu}_0$ iff

$$(\mathbf{u}, \nu) := (\dot{\eta} \circ \eta^{-1}, \dot{\chi}\chi^{-1} \circ \eta^{-1})$$

is a solution of the Ericksen-Leslie equations, where

$$\rho = J(\eta^{-1})(\rho_0 \circ \eta^{-1}) \quad \text{and} \quad \mathbf{n} = (\chi \mathbf{n}_0) \circ \eta^{-1}.$$

The curve $\eta \in \mathrm{Diff}(\mathcal{D})$ describes the *Lagrangian motion of the fluid* or *macromotion* and the curve $\chi \in \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ describes the *local molecular orientation relative to a fixed reference frame* or *micromotion*. Standard choice for the initial value of the director is

$$\mathbf{n}_0(x) := (0,0,1), \;\; \text{for all } x \in \mathcal{D}.$$

In this case we obtain

$$\mathbf{n} = \begin{pmatrix} \chi_{13} \\ \chi_{23} \\ \chi_{33} \end{pmatrix} \circ \eta^{-1}.$$

This relation is usually taken as a definition of the director, when the 3-axis is chosen as the reference axis of symmetry.

Standard choice for F is the *Oseen-Zöcher-Frank free energy*:

$$\rho F(\rho^{-1}, \mathbf{n}, \nabla \mathbf{n}) = K_2 \underbrace{(\mathbf{n} \cdot \operatorname{curl} \mathbf{n})}_{\textbf{chirality}} + \frac{1}{2} K_{11} \underbrace{(\operatorname{div} \mathbf{n})^2}_{\textbf{splay}} + \frac{1}{2} K_{22} \underbrace{(\mathbf{n} \cdot \operatorname{curl} \mathbf{n})^2}_{\textbf{twist}}$$
$$+ \frac{1}{2} K_{33} \underbrace{\|\mathbf{n} \times \operatorname{curl} \mathbf{n}\|^2}_{\textbf{bend}},$$

where $K_2 \neq 0$ for cholesterics and $K_2 = 0$ for nematics. The free energy can also contain additional terms due to external electromagnetic fields. The constants K_{11}, K_{22}, K_{33} are respectively associated to the three principal distinct director axis deformations in nematic liquid crystals, namely, splay, twist, and bend.

One-constant approximation: $K_{11} = K_{22} = K_{33} = K$. Free energy is, up to the addition of a divergence,

$$\rho F(\rho^{-1}, \mathbf{n}, \nabla \mathbf{n}) = \frac{1}{2} K \|\nabla \mathbf{n}\|^2.$$

Recall that the molecular field was given by

$$\frac{\delta l}{\delta \mathbf{n}} = -\rho \frac{\partial F}{\partial \mathbf{n}} + \partial_i \left(\rho \frac{\partial F}{\partial \mathbf{n}_{,i}} \right) = -\mathbf{h}.$$

In the case of the Oseen-Zöcher-Frank free energy for nematics (that is, $K_2 = 0$), the vector $\mathbf{h}$ is given by

$$\begin{aligned} \mathbf{h} =& K_{11} \operatorname{grad} \operatorname{div} \mathbf{n} - K_{22}(A \operatorname{curl} \mathbf{n} + \operatorname{curl}(A\mathbf{n})) \\ &+ K_{33}(\mathbf{B} \times \operatorname{curl} \mathbf{n} + \operatorname{curl}(\mathbf{n} \times \mathbf{B})), \end{aligned}$$

where $A := \mathbf{n} \cdot \operatorname{curl} \mathbf{n}$ and $\mathbf{B} := \mathbf{n} \times \operatorname{curl} \mathbf{n}$.

In the case of the one-constant approximation, $\mathbf{h} = -K \Delta \mathbf{n}$.

EXAMPLE: ERINGEN EQUATIONS

This is the micropolar theory of liquid crystals.

Microfluids are fluids whose material points are *small deformable particles*. Examples of microfluids include *liquid crystals, blood, polymer melts, bubbly fluids, suspensions with deformable particles, biological fluids*.

SKETCH OF ERINGEN'S THEORY

A material particle P in the fluid is characterized by its position X and by a vector Ξ attached to P that denotes the orientation and intrinsic deformation of P. Both X and Ξ have their own motions, $X \mapsto x = \eta(X, t)$ and $\Xi \mapsto \xi = \chi(X, \Xi, t)$, called respectively the *macromotion* and *micromotion*.

The material particles are thought of as very small, so a linear approximation in Ξ is permissible for the micromotion:

$$\xi = \chi(X,t)\Xi,$$

where $\chi(X,t) \in \mathrm{GL}(3)^+ := \{A \in \mathrm{GL}(3) \mid \det(A) > 0\}$.

The classical Eringen theory considers only three possible groups in the description of the micromotion of the particles:

$$\mathrm{GL}(3)^+ \supset \mathrm{K}(3) \supset \mathrm{SO}(3),$$

where

$$\mathrm{K}(3) = \left\{A \in \mathrm{GL}(3)^+ \mid \text{there exists } \lambda \in \mathbb{R} \text{ such that } AA^T = \lambda I_3\right\}.$$

These cases correspond to *micromorphic*, *microstretch*, and *micropolar* fluids. The Lie group $K(3)$ is a closed subgroup of $GL(3)^+$ that is associated to rotations and stretch.

The general theory admits other groups describing the micromotion.

Eringen's equations for non-dissipative micropolar liquid crystals:

$$\begin{cases} \rho\frac{D}{dt}\mathbf{u}_l = \partial_l\frac{\partial\Psi}{\partial\rho^{-1}} - \partial_k\left(\rho\frac{\partial\Psi}{\partial\boldsymbol{\gamma}_k^a}\boldsymbol{\gamma}_l^a\right), \quad \rho\sigma_l = \partial_k\left(\rho\frac{\partial\Psi}{\partial\boldsymbol{\gamma}_k^l}\right) - \varepsilon_{lmn}\rho\frac{\partial\Psi}{\partial\boldsymbol{\gamma}_m^a}\boldsymbol{\gamma}_n^a, \\ \frac{D}{dt}\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \qquad \frac{D}{dt}j_{kl} + (\varepsilon_{kpr}j_{lp} + \varepsilon_{lpr}j_{kp})\boldsymbol{\nu}_r = 0, \\ \frac{D}{dt}\boldsymbol{\gamma}_l^a = \partial_l\boldsymbol{\nu}_a + \nu_{ab}\boldsymbol{\gamma}_l^b - \boldsymbol{\gamma}_r^a\partial_l u_r. \end{cases}$$

$\mathbf{u} \in \mathfrak{X}(\mathcal{D})$ Eulerian velocity, $\rho \in \mathcal{F}(\mathcal{D})$ mass density, $\boldsymbol{\nu} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3)$, microrotation rate, where we use the standard isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$, $j_{kl} \in \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3))$ microinertia tensor (symmetric), σ_k spin inertia is defined by

$$\sigma_k := j_{kl}\frac{D}{dt}\boldsymbol{\nu}_l + \varepsilon_{klm}j_{mn}\boldsymbol{\nu}_l\boldsymbol{\nu}_n = \frac{D}{dt}(j_{kl}\boldsymbol{\nu}_l),$$

and $\gamma = (\gamma_i^{ab}) \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$ wryness tensor. This variable is denoted by $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_i^a)$ when it is seen as a form with values in $\mathbb{R}^3$.

$\Psi = \Psi(\rho^{-1}, j, \gamma) : \mathbb{R} \times \mathrm{Sym}(3) \times \mathfrak{gl}(3) \to \mathbb{R}$ is the free energy.

The axiom of objectivity requires that

$$\Psi(\rho^{-1}, A^{-1}jA, A^{-1}\gamma A) = \Psi(\rho^{-1}, j, \gamma),$$

for all $A \in \mathrm{O}(3)$ (for nematics and nonchiral smectics), or for all $A \in \mathrm{SO}(3)$ (for cholesterics and chiral smectics).

These equations are Euler-Poincaré/Lie-Poisson for the group

$$\left[\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))\right] \circledS \left[\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3)) \times \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))\right].$$

EXPLANATION:

- $\mathrm{Diff}(\mathcal{D})$ acts on $\mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ via the *right* action

$$(\chi, \eta) \in \mathcal{F}(\mathcal{D}, \mathrm{SO}(3)) \times \mathrm{Diff}(\mathcal{D}) \times \mapsto \chi \circ \eta \in \mathcal{F}(\mathcal{D}, \mathrm{SO}(3)).$$

Therefore, the group multiplication in $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ is

$$(\eta, \chi)(\varphi, \psi) = (\eta \circ \varphi, (\chi \circ \varphi)\psi).$$

• The bracket of $\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))$ is

$$\mathrm{ad}_{(\mathbf{u},\nu)}(\mathbf{v},\zeta) = (\mathrm{ad}_{\mathbf{u}}\,\mathbf{v}, \mathrm{ad}_{\nu}\,\zeta + \mathbf{d}\nu\cdot\mathbf{v} - \mathbf{d}\zeta\cdot\mathbf{u}),$$

where $\mathrm{ad}_{\mathbf{u}}\,\mathbf{v} = -[\mathbf{u},\mathbf{v}]$, $\mathrm{ad}_{\nu}\,\zeta \in \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))$ is given by $\mathrm{ad}_{\nu}\,\zeta(x) := \mathrm{ad}_{\nu(x)}\,\zeta(x)$, and $\mathbf{d}\nu\cdot\mathbf{v} \in \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))$ is given by $\mathbf{d}\nu\cdot\mathbf{v}(x) := \mathbf{d}\nu(x)(\mathbf{v}(x))$.

• $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ acts on $\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3)) \times \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))$ on the right. Need explicit action on the dual

$$\left[\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3)) \times \mathcal{F}(\mathcal{D}, \mathfrak{so}(3))\right]^* \cong \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3)) \times \Omega^1(\mathcal{D}, \mathfrak{so}(3))$$

via L^2 inner product.

• $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ acts *linearly* and on the *right* on the advected quantities $(\rho, j) \in \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3))$, by

$$(\rho, j) \mapsto \left(J\eta(\rho\circ\eta), \chi^T(j\circ\eta)\chi\right).$$

• $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ acts on $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$ by

$$\gamma \mapsto \chi^{-1}(\eta^*\gamma)\chi + \chi^{-1}T\chi.$$

This is a *right affine* action. Note that γ transforms as a connection.

- The reduced Lagrangian

$$l : \left[\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathbb{R}^3)\right] \circledS \left[\mathcal{F}(\mathcal{D}) \oplus \mathcal{F}(\mathcal{D}, \mathsf{Sym}(3)) \oplus \Omega^1(\mathcal{D}, \mathfrak{so}(3))\right] \to \mathbb{R}$$

is given by

$$l(\mathbf{u}, \boldsymbol{\nu}, \rho, j, \gamma) = \frac{1}{2}\int_{\mathcal{D}} \rho \|\mathbf{u}\|^2 \mu + \frac{1}{2}\int_{\mathcal{D}} \rho\,(j\boldsymbol{\nu} \cdot \boldsymbol{\nu})\,\mu - \int_{\mathcal{D}} \rho \Psi(\rho^{-1}, j, \gamma)\mu.$$

The affine Euler-Poincaré equations for l are:

$$\begin{cases} \rho\left(\dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u}\right) = \mathsf{grad}\,\dfrac{\partial \Psi}{\partial \rho^{-1}} - \partial_k\left(\rho \dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}^a_k}\boldsymbol{\gamma}^a\right), \\ j\dfrac{D}{dt}\boldsymbol{\nu} - (j\boldsymbol{\nu}) \times \boldsymbol{\nu} = -\dfrac{1}{\rho}\,\mathsf{div}\left(\rho \dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}}\right) + \boldsymbol{\gamma}^a \times \dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}^a}, \\ \dfrac{\partial}{\partial t}\rho + \mathsf{div}(\rho \mathbf{u}) = 0, \qquad \dfrac{D}{dt}j + [j, \widehat{\boldsymbol{\nu}}] = 0, \\ \dfrac{\partial}{\partial t}\gamma + \pounds_{\mathbf{u}}\gamma + \mathbf{d}^{\gamma}\nu = 0, \end{cases}$$

which are the Eringen equations after the change of variables $\gamma \mapsto -\gamma$. Here $\mathbf{d}^{\gamma}\boldsymbol{\nu}(\mathbf{v}) := \mathbf{d}\boldsymbol{\nu}(\mathbf{v}) + [\gamma(\mathbf{v}), \boldsymbol{\nu}]$.

$L_{(\rho_0, j_0, \gamma_0)} : T\left[\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))\right] \to \mathbb{R}$ induced by the Lagrangian l by right translation and freezing the parameters . A curve $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ is a solution of the Euler-Lagrange equations associated to $L_{(\rho_0, j_0, \gamma_0)}$ if and only if the curve

$$(\mathbf{u}, \boldsymbol{\nu}) := (\dot{\eta} \circ \eta^{-1}, \dot{\chi}\chi^{-1} \circ \eta^{-1}) \in \mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$$

is a solution of the previous equations with initial conditions (ρ_0, j_0, γ_0). The evolution of the mass density ρ, the microinertia j, and the wryness tensor γ is given by

$$\rho = J(\eta^{-1})(\rho_0 \circ \eta^{-1}), \quad j = \left(\chi j_0 \chi^{-1}\right) \circ \eta^{-1}, \quad \gamma = \eta_* \left(\chi \gamma_0 \chi^{-1} + \chi T \chi^{-1}\right).$$

If the initial value γ_0 is zero, then the evolution of γ is given by

$$\gamma = \eta_* \left(\chi T \chi^{-1}\right).$$

This relation is usually taken as a definition of γ when using the Eringen equations without the last one. This is often the case in the literature.

PROBLEM: Eringen defines a smectic liquid crystal in the micropolar theory by the condition $\mathrm{Tr}(\gamma) = \gamma^1_1 + \gamma^2_2 + \gamma^3_3 = 0$. But this is *not* preserved by the evolution $\gamma = \eta_* \left(\chi\gamma_0\chi^{-1} + \chi T \chi^{-1}\right)$, in general. This is consistent with: the equation

$$\frac{\partial \boldsymbol{\gamma}}{\partial t} + \boldsymbol{\pounds}_{\mathbf{u}}\boldsymbol{\gamma} + \mathbf{d}\boldsymbol{\nu} + \boldsymbol{\gamma} \times \boldsymbol{\nu} = 0.$$

does not show that if the initial condition for $\boldsymbol{\gamma}$ has trace zero then $\mathrm{Tr}\,\boldsymbol{\gamma} = 0$ for all time.

So we believe that Eringen's definition of smectic is incorrect. Here is a proposal. Find a function F that is invariant under the action

$$\gamma \mapsto \chi^{-1}(\eta^* \gamma)\chi + \chi^{-1} T \chi.$$

In fact, the η plays no role so we need an $\mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$-invariant function under the action

$$\mathbf{v} \mapsto \chi^{-1}\mathbf{v} + \chi^{-1} T \chi,$$

where $\mathbf{v} : \mathcal{D} \to \mathbb{R}^3$, $\chi : \mathcal{D} \to \mathrm{SO}(3)$.

The affine Lie-Poisson bracket is in this case equal to:

$$\{f,g\}(\mathbf{m},\boldsymbol{\kappa},\rho,j) = \int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right] \mu$$
$$+ \int_{\mathcal{D}} \kappa \cdot \left(\mathsf{ad}_{\frac{\delta f}{\delta \kappa}} \frac{\delta g}{\delta \kappa} + \mathbf{d}\frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \kappa} \cdot \frac{\delta f}{\delta \mathbf{m}} \right) \mu$$
$$+ \int_{\mathcal{D}} \rho \left(\mathbf{d}\left(\frac{\delta f}{\delta \rho}\right) \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\left(\frac{\delta g}{\delta \rho}\right) \frac{\delta f}{\delta \mathbf{m}} \right) \mu$$
$$+ \int_{\mathcal{D}} j \cdot \left(\mathsf{div}\left(\frac{\delta f}{\delta j}\frac{\delta g}{\delta \mathbf{m}}\right) + \left[\frac{\delta f}{\delta j}, \frac{\delta g}{\delta \kappa}\right] - \mathsf{div}\left(\frac{\delta g}{\delta j}\frac{\delta f}{\delta \mathbf{m}}\right) - \left[\frac{\delta g}{\delta j}, \frac{\delta f}{\delta \kappa}\right] \right) \mu$$
$$+ \int_{\mathcal{D}} \left[\left(\mathbf{d}^{\gamma}\frac{\delta f}{\delta \kappa} + \boldsymbol{\pounds}_{\frac{\delta f}{\delta \mathbf{m}}}\gamma \right) \cdot \frac{\delta g}{\delta \gamma} - \left(\mathbf{d}^{\gamma}\frac{\delta g}{\delta \kappa} + \boldsymbol{\pounds}_{\frac{\delta g}{\delta \mathbf{m}}}\gamma \right) \cdot \frac{\delta f}{\delta \gamma} \right] \mu$$

(second to last term brackets are commutator bracket of matrices).
The circulation theorems are:

$$\frac{d}{dt}\oint_{c_t} \mathbf{u}^{\flat} = \oint_{c_t} \frac{\partial \Psi}{\partial i} \cdot \mathbf{d}i + \frac{\partial \Psi}{\partial \gamma} \cdot \mathbf{i}\_\, \mathbf{d}\gamma - \frac{1}{\rho}\,\mathsf{div}\left(\rho \frac{\partial \Psi}{\partial \gamma}\right) \cdot \gamma$$
$$\frac{d}{dt}\oint_{c_t} \boldsymbol{\gamma} = \oint_{c_t} \boldsymbol{\nu} \times \boldsymbol{\gamma}$$

One can show that the ordered micropolar theory of Lhuillier-Rey is a direct generalization of the Ericksen-Leslie director theory. So one needs to compare the Lhuillier-Rey theory to the Eringen theory.

PROBLEM: How does one pass from ordered micropolar (or Ericksen-Leslie) theory to Eringen theory? Eringen says that it is given by $\gamma = \nabla \mathbf{n} \times \mathbf{n}$ and $j := J(I_3 - \mathbf{n} \otimes \mathbf{n})$. If so, then transformation laws should be preserved.

a.) If $\mathbf{n} \mapsto \chi^{-1}(\mathbf{n} \circ \eta)$ is the transformation law for $\mathbf{n}$, which is imposed by Lhuillier-Rey (and also Ericksen-Leslie) theory, then j transforms as $j \mapsto \chi^T (j \circ \eta)\chi$, which is correct. However, γ does not transform as $\gamma \mapsto \chi^{-1}(\eta^* \gamma)\chi + \chi^{-1} T\chi$.

b.) One can find, by a brutal computation, what the Eringen equations should be under this transformation, if $(\mathbf{u}, \boldsymbol{\nu}, \rho, j, \mathbf{n})$ are solutions of the Lhuillier-Rey equations. The resulting system is almost the Eringen system: there are two bad factors of j/J.

THE GENERAL THEORY

$\rho : G \to \mathrm{Aut}(V)$ denotes a *right* Lie group representation. Form the semidirect product $S = G \circledS V$ whose group multiplication is

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + \rho_{g_2}(v_1)).$$

The Lie alebra $\mathfrak{s} = \mathfrak{g} \circledS V$ of S has bracket

$$\mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) = [(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1),$$

where $v\xi$ denotes the induced action of $\mathfrak{g}$ on V, that is,

$$v\xi := \left.\frac{d}{dt}\right|_{t=0} \rho_{\exp(t\xi)}(v) \in V.$$

If $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^*$ we have

$$\mathrm{ad}^*_{(\xi, v)}(\mu, a) = (\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi),$$

where $a\xi \in V^*$ and $v \diamond a \in \mathfrak{g}^*$ are given by

$$a\xi := \left.\frac{d}{dt}\right|_{t=0} \rho^*_{\exp(-t\xi)}(a) \quad \text{and} \quad \langle v \diamond a, \xi \rangle_{\mathfrak{g}} := -\langle a\xi, v \rangle_V,$$

$\langle \cdot, \cdot \rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbb{R}$ are the duality parings.

Lagrangian semidirect product theory

- $L : TG \times V^* \to \mathbb{R}$ which is right G-invariant.

- So, if $a_0 \in V^*$, define the Lagrangian $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of G_{a_0} on G, where $G_{a_0} := \{g \in G \mid \rho_g^* a_0 = a_0\}$.

- Right G-invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(T_g R_{g^{-1}}(v_g), \rho_g^*(a_0)) = L(v_g, a_0).$$

- For a curve $g(t) \in G$, let $\xi(t) := TR_{g(t)^{-1}}(\dot{g}(t))$ and define the curve $a(t)$ as the unique solution of the following linear differential equation with time dependent coefficients

$$\dot{a}(t) = -a(t)\xi(t),$$

with initial condition $a(0) = a_0$. Solution is $a(t) = \rho^*_{g(t)}(a_0)$.

i *With a_0 held fixed, Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0,$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii *$g(t)$ satisfies the Euler-Lagrange equations for L_{a_0} on G.*

iii *The constrained variational principle*

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0,$$

holds on $\mathfrak{g} \times V^*$, upon using variations $(\delta\xi, \delta a)$ of the form

$$\delta\xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta a = -a\eta,$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.

iv *The Euler-Poincaré equations hold on $\mathfrak{g} \times V^*$:*

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} = -\operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a.$$

Hamiltonian semidirect product theory

- $H : T^*G \times V^* \to \mathbb{R}$ which is right G-invariant.

- So, if $a_0 \in V^*$, define the Hamiltonian $H_{a_0} : TG \to \mathbb{R}$ by $H_{a_0}(\alpha_g) := H(\alpha_g, a_0)$. Then H_{a_0} is right invariant under the lift to TG of the right action of G_{a_0} on G.

- Right G-invariance of H permits us to define $h : \mathfrak{g}^* \times V^* \to \mathbb{R}$ by

$$h(T_e^* R_g(\alpha_g), \rho_g^*(a_0)) = H(\alpha_g, a_0).$$

For $\alpha(t) \in T^*_{g(t)}G$ and $\mu(t) := T^* R_{g(t)}(\alpha(t)) \in \mathfrak{g}^*$, the following are equivalent:

i *$\alpha(t)$ satisfies Hamilton's equations for H_{a_0} on T^*G.*

ii *The Lie-Poisson equation holds on $\mathfrak{s}^*$:*

$$\frac{\partial}{\partial t}(\mu, a) = -\,\mathrm{ad}^*_{\left(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a}\right)}(\mu, a) = -\left(\mathrm{ad}^*_{\frac{\delta h}{\delta \mu}}\mu + \frac{\delta h}{\delta a} \diamond a, a\frac{\delta h}{\delta \mu}\right), \quad a(0) = a_0$$

where $\mathfrak{s}$ is the semidirect product Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$. The associated Poisson bracket is the Lie-Poisson bracket on the semidirect product Lie algebra $\mathfrak{s}^*$, that is,

$$\{f, g\}(\mu, a) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle + \left\langle a, \frac{\delta f}{\delta a}\frac{\delta g}{\delta \mu} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mu}\right\rangle.$$

As on the Lagrangian side, the evolution of the advected quantities is given by $a(t) = \rho^*_{g(t)}(a_0)$.

Legendre transformation: $h(\mu, a) := \langle \mu, \xi \rangle - l(\xi, a)$, where $\mu = \frac{\delta l}{\delta \xi}$. If it is invertible, since

$$\frac{\delta h}{\delta \mu} = \xi \quad \text{and} \quad \frac{\delta h}{\delta a} = -\frac{\delta l}{\delta a},$$

the Lie-Poisson equations for h are equivalent to the Euler-Poincaré equations for l together with the advection equation $\dot{a} + a\xi = 0$.

Affine Lagrangian semidirect product theory

Let $c \in \mathcal{F}(G, V^*)$ be a right one-cocycle, that is, it verifies the property $c(fg) = \rho^*_{g^{-1}}(c(f)) + c(g)$ for all $f, g \in V^*$. This implies that $c(e) = 0$ and $c(g^{-1}) = -\rho^*_g(c(g))$. Instead of the contragredient representation $\rho^*_{g^{-1}}$ of G on V^* form the affine *right* representation

$$\theta_g(a) = \rho^*_{g^{-1}}(a) + c(g).$$

Note that

$$\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(t\xi)}(a) = a\xi + \mathbf{d}c(\xi).$$

and

$$\langle a\xi + \mathbf{d}c(\xi), v\rangle_V = \langle \mathbf{d}c^T(v) - v \diamond a, \xi\rangle_{\mathfrak{g}},$$

where $\mathbf{d}c : \mathfrak{g} \to V^*$ is defined by $\mathbf{d}c(\xi) := T_e c(\xi)$, and $\mathbf{d}c^T : V \to \mathfrak{g}^*$ is defined by

$$\langle \mathbf{d}c^T(v), \xi\rangle_{\mathfrak{g}} := \langle \mathbf{d}c(\xi), v\rangle_V.$$

- $L : TG \times V^* \to \mathbb{R}$ right G-invariant under the affine action $(v_h, a) \mapsto (T_h R_g(v_h), \theta_g(a)) = (T_h R_g(v_h), \rho^*_{g^{-1}}(a) + c(g))$.

- So, if $a_0 \in V^*$, define $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of $G^c_{a_0}$ on G, where $G^c_{a_0} := \{g \in G \mid \theta_g(a_0) = a_0\}$.

- Right G-invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(T_g R_{g^{-1}}(v_g), \theta_{g^{-1}}(a_0)) = L(v_g, a_0).$$

- For a curve $g(t) \in G$, let $\xi(t) := TR_{g(t)^{-1}}(\dot{g}(t))$ and define the curve $a(t)$ as the unique solution of the following affine differential equation with time dependent coefficients

$$\dot{a}(t) = -a(t)\xi(t) - \mathbf{d}c(\xi(t)),$$

with initial condition $a(0) = a_0$. The solution can be written as $a(t) = \theta_{g(t)^{-1}}(a_0)$.

i With a_0 held fixed, Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t)) dt = 0,$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii $g(t)$ satisfies the Euler-Lagrange equations for L_{a_0} on G.

iii The constrained variational principle

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t)) dt = 0,$$

holds on $\mathfrak{g} \times V^*$, upon using variations of the form

$$\delta\xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta a = -a\eta - \mathbf{d}c(\eta),$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.

iv The affine Euler-Poincaré equations hold on $\mathfrak{g} \times V^*$:

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} = -\,\mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a - \mathbf{d}c^T\left(\frac{\delta l}{\delta a}\right).$$

Lagrangian Approach to Continuum Theories of Perfect Complex Fluids

Two key observations:

1. Enlarge the configuration manifold $\mathrm{Diff}(\mathcal{D})$ to a bigger group G that contains variables in the Lie group $\mathcal{O}$ of order parameters.

2. The usual advection equations (for the mass density, the entropy, the magnetic field, etc) need to be augmented by a new advected quantity on which the group G acts by an *affine representation*.

$\mathcal{O}$ the *order parameter Lie group*, $\mathcal{F}(\mathcal{D},\mathcal{O}) := \{\chi : \mathcal{D} \to \mathcal{O} \text{ smooth}\}$

Basic idea for complex fluids: enlarge the "particle relabeling group" $\mathrm{Diff}(\mathcal{D})$ to the semidirect product $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D},\mathcal{O})$.

Diff$(\mathcal{D})$ acts on $\mathcal{F}(\mathcal{D},\mathcal{O})$ via the *right* action

$$(\eta,\chi)\in \mathrm{Diff}(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathcal{O})\mapsto \chi\circ\eta\in\mathcal{F}(\mathcal{D},\mathcal{O}).$$

Therefore, the group multiplication is given by

$$(\eta,\chi)(\varphi,\psi)=(\eta\circ\varphi,(\chi\circ\varphi)\psi).$$

Fix a volume form μ on $\mathcal{D}$, so identify densities with functions, one-form densitities with one-forms, etc. But the dual actions will be of course different once these identifications are used.

The Lie algebra $\mathfrak{g}$ of the semidirect product group is

$$\mathfrak{g}=\mathfrak{X}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathfrak{o}),$$

and the Lie bracket is computed to be

$$[(\mathbf{u},\nu),(\mathbf{v},\zeta)]=\mathrm{ad}_{(\mathbf{u},\nu)}(\mathbf{v},\zeta)=(\mathrm{ad}_{\mathbf{u}}\mathbf{v},\mathrm{ad}_\nu\zeta+\mathbf{d}\nu\cdot\mathbf{v}-\mathbf{d}\zeta\cdot\mathbf{u}),$$

where $\mathrm{ad}_{\mathbf{u}}\mathbf{v}=-[\mathbf{u},\mathbf{v}]$, $\mathrm{ad}_\nu\zeta\in\mathcal{F}(\mathcal{D},\mathfrak{o})$ is given by $\mathrm{ad}_\nu\zeta(x):=\mathrm{ad}_{\nu(x)}\zeta(x)$, and $\mathbf{d}\nu\cdot\mathbf{v}\in\mathcal{F}(\mathcal{D},\mathfrak{o})$ is given by $\mathbf{d}\nu\cdot\mathbf{v}(x):=\mathbf{d}\nu(x)(\mathbf{v}(x))$.

$$\mathfrak{g}^* = \Omega^1(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$$

through the pairing

$$\langle(\mathbf{m}, \kappa), (\mathbf{u}, \nu)\rangle = \int_{\mathcal{D}} (\mathbf{m} \cdot \mathbf{u} + \kappa \cdot \nu)\, \mu.$$

The dual map to $\mathrm{ad}_{(\mathbf{u},\nu)}$ is

$$\mathrm{ad}^*_{(\mathbf{u},\nu)}(\mathbf{m}, \kappa) = \Big(\pounds_{\mathbf{u}}\mathbf{m} + (\mathrm{div}\,\mathbf{u})\mathbf{m} + \kappa \cdot \mathbf{d}\nu, \mathrm{ad}^*_\nu \kappa + \mathrm{div}(\mathbf{u}\kappa)\Big).$$

Explanation of the symbols:

- $\kappa \cdot \mathbf{d}\nu \in \Omega^1(\mathcal{D})$ denotes the one-form defined by

$$\kappa \cdot \mathbf{d}\nu(v_x) := \kappa(x)(\mathbf{d}\nu(v_x))$$

- $\mathrm{ad}^*_\nu \kappa \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ denotes the $\mathfrak{o}^*$-valued mapping defined by

$$\mathrm{ad}^*_\nu \kappa(x) := \mathrm{ad}^*_{\nu(x)}(\kappa(x)).$$

- $\mathbf{u}\kappa$ is the 1-contravariant tensor field with values in $\mathfrak{o}^*$ defined by

$$\mathbf{u}\kappa(\alpha_x) := \alpha_x(\mathbf{u}(x))\kappa(x) \in \mathfrak{o}^*.$$

So $\mathbf{u}\kappa$ is a generalization of the notion of a vector field. $\mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ denotes the space of all $\mathfrak{o}^*$-valued 1-contravariant tensor fields.

• $\operatorname{div}(\mathbf{u})$ denotes the divergence of the vector field $\mathbf{u}$ with respect to the fixed volume form μ. Recall that it is defined by the condition

$$(\operatorname{div}\mathbf{u})\mu = \boldsymbol{\pounds}_{\mathbf{u}}\mu.$$

This operator can be naturally extended to the space $\mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ as follows. For $w \in \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ we write $w = w_a \varepsilon^a$ where (ε^a) is a basis of $\mathfrak{o}^*$ and $w_a \in \mathfrak{X}(\mathcal{D})$. We define $\operatorname{div} : \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*) \to \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ by

$$\operatorname{div} w := (\operatorname{div} w_a)\varepsilon^a.$$

Note that if $w = \mathbf{u}\kappa$ we have

$$\operatorname{div}(\mathbf{u}\kappa) = \mathbf{d}\kappa \cdot \mathbf{u} + (\operatorname{div}\mathbf{u})\kappa.$$

Split the space of advected quantities in two: usual ones and new ones that involve affine actions and cocycles.

Affine representation space: $V_1^* \oplus V_2^*$, V_i^* are subspaces of the space of all tensor fields on $\mathcal{D}$, possibly with values in a vector space.

• V_1^* is only acted upon by the component $\mathrm{Diff}(\mathcal{D})$ of G.

• The action of G on V_2^* is affine, with the restriction that the affine term only depends on the second component $\mathcal{F}(\mathcal{D}, \mathcal{O})$ of G.

• Right affine representation of $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$ on $V_1^* \oplus V_2^*$:

$$(a, \gamma) \in V_1^* \oplus V_2^* \mapsto (a\eta, \gamma(\eta, \chi) + C(\chi)) \in V_1^* \oplus V_2^*,$$

where $\gamma(\eta, \chi)$ denotes the representation of $(\eta, \chi) \in G$ on $\gamma \in V_2^*$, and $C \in \mathcal{F}(\mathcal{F}(\mathcal{D}, \mathcal{O}), V_2^*)$ satisfies the identity

$$C((\chi \circ \varphi)\psi) = C(\chi)(\varphi, \psi) + C(\psi).$$

The representation ρ and the affine term c in the general theory are

$$\rho^*_{(\eta,\chi)^{-1}}(a, \gamma) = (a\eta, \gamma(\eta, \chi)) \quad \text{and} \quad c(\eta, \chi) = (0, C(\chi)).$$

- The infinitesimal action of $(\mathbf{u},\nu) \in \mathfrak{g}$ on $\gamma \in V_2^*$ is:

$$\gamma(\mathbf{u},\nu) = \gamma\mathbf{u} + \gamma\nu.$$

- The diamond operation: for $(v,w) \in V_1 \oplus V_2$ we have

$$(v,w) \diamond (a,\gamma) = (v \diamond a + w \diamond_1 \gamma, w \diamond_2 \gamma),$$

where $\diamond_1$ and $\diamond_2$ are associated to the induced representations of the first and second component of G on V_2^*. On the right hand side, $\diamond$ is associated to the representation of Diff($\mathcal{D}$) on V_1^*. Usually, V_1^* is naturally the dual of some space V_1 of tensor fields on $\mathcal{D}$. For example the (p,q) tensor fields are naturally in duality with the (q,p) tensor fields. For $a \in V_1^*$ and $v \in V_1$, the duality pairing is given by

$$\langle a, v \rangle = \int_{\mathcal{D}} (a \cdot v)\mu,$$

where $\cdot$ denotes the contraction of tensor fields.

- The affine cocycle is $c(\eta,\chi) = (0, C(\chi))$. Hence

$$\mathbf{d}c^T(v,w) = (0, \mathbf{d}C^T(w)).$$

• For a Lagrangian $l(\mathbf{u}, \nu, a, \gamma)$

$$l : \Big[\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o})\Big] \circledS \Big[V_1^* \oplus V_2^*\Big] \to \mathbb{R},$$

the affine Euler-Poincaré equations are

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \mathbf{u}} = -\pounds_{\mathbf{u}}\dfrac{\delta l}{\delta \mathbf{u}} - (\operatorname{div}\mathbf{u})\dfrac{\delta l}{\delta \mathbf{u}} - \dfrac{\delta l}{\delta \nu}\cdot \mathbf{d}\nu + \dfrac{\delta l}{\delta a}\diamond a + \dfrac{\delta l}{\delta \gamma}\diamond_1 \gamma \\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \nu} = -\operatorname{ad}^*_\nu \dfrac{\delta l}{\delta \nu} - \operatorname{div}\left(\mathbf{u}\dfrac{\delta l}{\delta \nu}\right) + \dfrac{\delta l}{\delta \gamma}\diamond_2 \gamma - \mathbf{d}C^T\left(\dfrac{\delta l}{\delta \gamma}\right), \end{cases}$$

and the advection equations are

$$\begin{cases} \dot{a} + a\mathbf{u} = 0 \\ \dot{\gamma} + \gamma\mathbf{u} + \gamma\nu + \mathbf{d}C(\nu) = 0. \end{cases}$$

Complex Fluids Example

Choice: $V_2 = \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$, $V_2^* := \Omega^1(\mathcal{D}, \mathfrak{o})$. V_1 is some space of tensor fields on $\mathcal{D}$.

Affine representation:

$$(a, \gamma) \mapsto (a\eta, \mathrm{Ad}_{\chi^{-1}}\, \eta^* \gamma + \chi^{-1} T\chi),$$

where $\mathrm{Ad}_{\chi^{-1}}\, \eta^* \gamma + \chi^{-1} T\chi$ is the $\mathfrak{o}$-valued one-form given by

$$\left(\mathrm{Ad}_{\chi^{-1}}\, \eta^* \gamma + \chi^{-1} T\chi\right)(v_x) := \mathrm{Ad}_{\chi(x)^{-1}}(\eta^* \gamma(v_x)) + \chi(x)^{-1} T_x \chi(v_x),$$

for $v_x \in T_x\mathcal{D}$. One can check that $\gamma(\eta, \chi) := \mathrm{Ad}_{\chi^{-1}}\, \eta^* \gamma$ is a right representation of G on V_2^* and that $C(\chi) = \chi^{-1} T\chi$ verifies the cocycle condition.

This formula corresponds to the action of the automorphism group of the trivial principal bundle $\mathcal{O} \times \mathcal{D}$, on the space connections.

For this example we have

$$\gamma \mathbf{u} = \boldsymbol{\pounds}_{\mathbf{u}} \gamma, \quad \gamma \nu = -\operatorname{ad}_\nu \gamma \quad \text{and} \quad \mathbf{d}C(\nu) = \mathbf{d}\nu,$$

where $\operatorname{ad}_\nu \gamma \in \Omega^1(M, \mathfrak{o})$ and $\mathbf{d}\nu \in \Omega^1(\mathcal{D}, \mathfrak{o})$ are the one-forms

$$(\operatorname{ad}_\nu \gamma)(v_x) := \operatorname{ad}_{\nu(x)}(\gamma(v_x)) = [\nu(x), \gamma(v_x)], \quad \mathbf{d}\nu(v_x) := T_x \nu(v_x) \in \mathfrak{o}.$$

A direct computation shows that

$$\begin{aligned} w \diamond_1 \gamma &= (\operatorname{div} w) \cdot \gamma - w \cdot \mathbf{i}_{\_} \mathbf{d}\gamma \in \Omega^1(\mathcal{D}), \\ w \diamond_2 \gamma &= -\operatorname{Tr}(\operatorname{ad}^*_\gamma w) \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*), \\ \mathbf{d}C^T(w) &= -\operatorname{div} w \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*), \end{aligned}$$

where Tr denotes the trace of the $\mathfrak{o}^*$-valued $(1,1)$ tensor

$$\operatorname{ad}^*_\gamma w : T^*\mathcal{D} \times T\mathcal{D} \to \mathfrak{o}^*, \quad (\alpha_x, v_x) \mapsto \operatorname{ad}^*_{\gamma(v_x)}(w(\alpha_x)).$$

In coordinates we have $\operatorname{Tr}(\operatorname{ad}^*_\gamma w) = \operatorname{ad}^*_{\gamma_i} w^i$.

The affine Euler-Poincaré equations become in this case

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \mathbf{u}} = -\pounds_{\mathbf{u}}\dfrac{\delta l}{\delta \mathbf{u}} - (\operatorname{div}\mathbf{u})\dfrac{\delta l}{\delta \mathbf{u}} - \dfrac{\delta l}{\delta \nu}\cdot \mathbf{d}\nu + \dfrac{\delta l}{\delta a}\diamond a + \left(\operatorname{div}\dfrac{\delta l}{\delta \gamma}\right)\cdot\gamma - \dfrac{\delta l}{\delta \gamma}\cdot \mathbf{i}\_\mathbf{d}\gamma \\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \nu} = -\operatorname{ad}^*_{\nu}\dfrac{\delta l}{\delta \nu} + \operatorname{div}\left(\dfrac{\delta l}{\delta \gamma} - \mathbf{u}\dfrac{\delta l}{\delta \nu}\right) - \operatorname{Tr}\left(\operatorname{ad}^*_{\gamma}\dfrac{\delta l}{\delta \gamma}\right), \end{cases}$$

and the advection equations are

$$\begin{cases} \dot{a} + a\mathbf{u} = 0 \\ \dot{\gamma} + \pounds_{\mathbf{u}}\gamma - \operatorname{ad}_{\nu}\gamma + \mathbf{d}\nu = 0. \end{cases}$$

These are, up to sign conventions, the equations for complex fluids given by Holm[2002].

Write these equations more geometrically; γ defines a connection:

$$(v_x, \xi_{\mathsf{h}}) \in T_x\mathcal{D} \times T_{\mathsf{h}}\mathcal{O} \mapsto \operatorname{Ad}_{\mathsf{h}^{-1}}(\gamma(x)(v_x) + TR_{\mathsf{h}^{-1}}(\xi_{\mathsf{h}})) \in \mathfrak{o}.$$

Covariant differential is denoted by $\mathbf{d}^{\gamma}$. For a function $\nu \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$

$$\mathbf{d}^{\gamma}\nu(\mathbf{v}) := \mathbf{d}\nu(\mathbf{v}) + [\gamma(\mathbf{v}), \nu].$$

The covariant divergence of $w \in \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ is the function

$$\mathrm{div}^\gamma w := \mathrm{div}\, w - \mathrm{Tr}(\mathrm{ad}^*_\gamma w) \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*),$$

defined as minus the adjoint of the covariant differential, that is,

$$\int_\mathcal{D} (\mathbf{d}^\gamma \nu \cdot w)\, \mu = -\int_\mathcal{D} (\nu \cdot \mathrm{div}^\gamma w)\, \mu$$

for all $\nu \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$.

Note that the Lie derivative of $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{o})$ can be written as

$$\begin{aligned} \pounds_\mathbf{u} \gamma(\mathbf{v}) &= \mathbf{d}(\gamma(\mathbf{u}))(\mathbf{v}) + \mathbf{i}_\mathbf{u} \mathbf{d}\gamma(\mathbf{v}) \\ &= \mathbf{d}^\gamma(\gamma(\mathbf{u}))(\mathbf{v}) - [\gamma(\mathbf{v}), \gamma(\mathbf{u})] + \mathbf{d}\gamma^\gamma(\mathbf{u}, \mathbf{v}) - [\gamma(\mathbf{u}), \gamma(\mathbf{v})] \\ &= \mathbf{d}^\gamma(\gamma(\mathbf{u}))(\mathbf{v}) + \mathbf{i}_\mathbf{u} B(\mathbf{v}), \end{aligned}$$

where

$$B := \mathbf{d}^\gamma \gamma = \mathbf{d}\gamma + [\gamma, \gamma],$$

is the *curvature* of the connection induced by γ.

Note also that, using covariant differentiation, we have

$$w \diamond_1 \gamma = (\mathrm{div}\, w) \cdot \gamma - w \cdot \mathbf{i}_\_ \mathbf{d}\gamma = (\mathrm{div}^\gamma w) \cdot \gamma - w \cdot \mathbf{i}_\_ B.$$

Therefore, in terms of $\mathbf{d}^\gamma, \mathrm{div}^\gamma$, and $B = \mathbf{d}^\gamma \gamma$, the equations read

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \mathbf{u}} = -\pounds_{\mathbf{u}}\dfrac{\delta l}{\delta \mathbf{u}} - (\mathrm{div}\,\mathbf{u})\dfrac{\delta l}{\delta \mathbf{u}} - \dfrac{\delta l}{\delta \nu}\cdot \mathbf{d}\nu + \dfrac{\delta l}{\delta a}\diamond a + \left(\mathrm{div}^\gamma \dfrac{\delta l}{\delta \gamma}\right)\cdot \gamma - \dfrac{\delta l}{\delta \gamma}\cdot \mathbf{i}_{\_} B \\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \nu} = -\,\mathrm{ad}^*_\nu \dfrac{\delta l}{\delta \nu} - \mathrm{div}\left(\mathbf{u}\dfrac{\delta l}{\delta \nu}\right) + \mathrm{div}^\gamma \dfrac{\delta l}{\delta \gamma}, \end{cases}$$

and

$$\begin{cases} \dot{a} + a\mathbf{u} = 0 \\ \dot{\gamma} + \mathbf{d}^\gamma(\gamma(\mathbf{u})) + \mathbf{i}_{\mathbf{u}} B + \mathbf{d}^\gamma \nu = 0. \end{cases}$$

The Curvature Representation

Want to reformulate the reduction process and the equations of motion in terms of $(\mathbf{u}, \nu, a, B)$, instead of $(\mathbf{u}, \nu, a, \gamma)$, where $B = \mathbf{d}^\gamma \gamma = \mathbf{d}\gamma + [\gamma, \gamma] \in \Omega^2(\mathcal{D}, \mathfrak{o})$ is the *curvature* of γ. With this choice of variables the action of G becomes linear instead of affine. We shall also assume that the Lagrangian L, and hence also l, depend on γ only through B. We shall use therefore standard Euler-Poincaré reduction for semidirect products.

If $\gamma' = \mathrm{Ad}_{\chi^{-1}} \eta^* \gamma + \chi^{-1} T\chi$ then $\mathbf{d}^{\gamma'}\gamma' = \mathrm{Ad}_{\chi^{-1}} \eta^* \mathbf{d}^{\gamma}\gamma$. Thus the representation of $\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$ on $V_1^* \oplus \Omega^2(\mathcal{D}, \mathfrak{o})$ is given by

$$(a, B) \mapsto (a\eta, \mathrm{Ad}_{\chi^{-1}} \eta^* B).$$

The associated infinitesimal action of $(\mathbf{u}, \nu) \in \mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o})$ is

$$(a, B)(\mathbf{u}, \nu) = (a\mathbf{u}, B(\mathbf{u}, \nu)) = (a\mathbf{u}, \boldsymbol{\pounds}_{\mathbf{u}} B - \mathrm{ad}_{\nu} B).$$

Duality: contraction and integration with respect to the fixed volume form μ, so the space $\Omega_k(\mathcal{D}, \mathfrak{o}^*)$ of k-contravariant skew symmetric tensor fields with values in $\mathfrak{o}^*$ is dual to $\Omega^k(\mathcal{D}, \mathfrak{o})$. Define the divergence operators, $\mathrm{div}, \mathrm{div}^{\gamma} : \Omega_k(\mathcal{D}, \mathfrak{o}^*) \to \Omega_{k-1}(\mathcal{D}, \mathfrak{o}^*)$, to be minus the adjoint of the exterior derivatives $\mathbf{d}$ and $\mathbf{d}^{\gamma}$, respectively. For example, div^{γ} is defined on $\Omega_k(\mathcal{D}, \mathfrak{o}^*)$ by

$$\int_{\mathcal{D}} (\mathbf{d}^{\gamma}\alpha \cdot \omega)\, \mu = -\int_{\mathcal{D}} (\alpha \cdot \mathrm{div}^{\gamma} \omega)\, \mu,$$

where $\alpha \in \Omega^{k-1}(\mathcal{D}, \mathfrak{o})$ and $\omega \in \Omega_k(\mathcal{D}, \mathfrak{o}^*)$ Note that we have used the notations $\Omega_1(\mathcal{D}, \mathfrak{o}^*) = \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ and $\Omega_0(\mathcal{D}, \mathfrak{o}^*) = \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$.

If $(v, b) \in V_1 \oplus \Omega_2(\mathcal{D}, \mathfrak{o}^*)$ and $(a, B) \in V_1^* \oplus \Omega^2(\mathcal{D}, \mathfrak{o})$, then

$$(v, b) \diamond (a, B) = (v \diamond a + b \diamond_1 B, b \diamond_2 B),$$

where

$$b \diamond_1 B = (\mathrm{div}\, b) \cdot \mathbf{i}\_ B - b \cdot \mathbf{i}\_ \mathbf{d}B \in \Omega^1(\mathcal{D})$$

$$b \diamond_2 B = -\,\mathrm{Tr}(\mathrm{ad}^*_B\, b) = -\,\mathrm{ad}^*_{B_{ij}}\, b^{ij} \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*).$$

The (usual semidirect product) Euler-Poincaré equation are

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \mathbf{u}} = -\pounds_{\mathbf{u}}\dfrac{\delta l}{\delta \mathbf{u}} - (\mathrm{div}\,\mathbf{u})\dfrac{\delta l}{\delta \mathbf{u}} - \dfrac{\delta l}{\delta \nu} \cdot \mathbf{d}\nu + \dfrac{\delta l}{\delta a} \diamond a \\ \qquad\qquad + \left(\mathrm{div}\, \dfrac{\delta l}{\delta B}\right) \cdot \mathbf{i}\_ B - \dfrac{\delta l}{\delta B} \cdot \mathbf{i}\_ \mathbf{d}B \\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \nu} = -\,\mathrm{ad}^*_{\nu}\, \dfrac{\delta l}{\delta \nu} + \dfrac{\delta l}{\delta B} \diamond_2 B - \mathrm{Tr}\left(\mathrm{ad}^*_B\, \dfrac{\delta l}{\delta B}\right), \end{cases}$$

and the advection equations are

$$\begin{cases} \dot{a} + a\mathbf{u} = 0 \\ \dot{B} + \pounds_{\mathbf{u}} B - \mathrm{ad}_{\nu}\, B = 0. \end{cases}$$

The affine Euler-Poincaré equations imply these standard ones.

Affine Hamiltonian semidirect product theory

$R_g^{T^*}$ is the lift of right translation on G: $R_g^{T^*}(\alpha_f) = T^* R_{g^{-1}}(\alpha_f)$.

Let $C : G \times G \to T^*G$ be a smooth map such that $C_g(f) \in T^*_{fg}G$, for all $f, g \in G$ and define $\Psi_g : T^*G \to T^*G$ by

$$\Psi_g(\alpha_f) := R_g^{T^*}(\alpha_f) + C_g(f),$$

where $C : G \times G \to T^*G$ satisfies $C_g(f) \in T^*_{fg}G$, for all $f, g \in G$.

The following are equivalent.

i Ψ_g is a right action.

ii For all $f, g, h \in G$, the affine term C verifies the property

$$C_{gh}(f) = C_h(fg) + R_h^{T^*}(C_g(f)).$$

iii There exists $\alpha \in \Omega^1(G)$ such that $C_g(f) = \alpha(fg) - R_g^{T^*}(\alpha(f))$.

The one-form α is unique if we assume that $\alpha(e) = 0$, which is what we will do from now on. Let $\Omega^1_0(G) := \{\alpha \in \Omega^1(G) \mid \alpha(e) = 0\}$

Let G act on $(T^*G, \Omega_{\text{can}})$ by the right affine action

$$\Psi_g(\beta_f) := R_g^{T^*}(\beta_f) + C_g(f),$$

where $C \in \mathcal{C}(G)$. Let $\alpha \in \Omega^1_0(G)$ be the one-form associated to Ψ_g.

i *$t_\alpha : \beta_q \in (T^*G, \Omega_{\text{can}}) \mapsto \beta_q - \alpha(q) \in (T^*G, \Omega_{\text{can}} - \pi_G^* \mathbf{d}\alpha)$ symplectic. The action induced by Ψ_g on $(T^*G, \Omega_{\text{can}} - \pi_G^* \mathbf{d}\alpha)$ through t_α is $R_g^{T^*}$.*

ii *Suppose that $\mathbf{d}\alpha$ is G-invariant. Then the action Ψ_g is symplectic relative to the canonical symplectic form Ω_{can}.*

iii *Suppose that there is a smooth map* $\phi : G \to \mathfrak{g}^*$ *that satisfies*

$$\mathbf{i}_{\xi^L}\mathbf{d}\alpha = \mathbf{d}\langle \phi, \xi \rangle$$

for all $\xi \in \mathfrak{g}$*. Then* $\mathbf{J}_\alpha = \mathbf{J}_R \circ t_\alpha - \phi \circ \pi_G$ *is a momentum map for the action* Ψ_g *relative to* Ω_{can}*. We can always choose* ϕ *such that* $\phi(e) = 0$*. Then, the nonequivariance one-cocycle of* $\mathbf{J}_\alpha$ *is* $-\phi$*.*

iv G_μ^ϕ *is the isotropy group of* μ *relative to the affine action* $\mu \mapsto \mathrm{Ad}_g^*(\mu) - \phi(g)$*. The symplectic reduced space* $(\mathbf{J}_\alpha^{-1}(\mu)/G_\mu^\phi, \Omega_\mu)$ *is symplectically diffeomorphic to the affine coadjoint orbit* $(\mathcal{O}_\mu^\phi, \omega_{\mathbf{d}\alpha}^+)$*, the symplectic diffeomorphism being induced by the* G_μ^ϕ*-invariant smooth map* $\alpha_g \in \mathbf{J}_\alpha^{-1}(\mu) \mapsto \Psi_{g^{-1}}(\alpha_g) \in \mathcal{O}_\mu^\sigma$*. Here*

$$\begin{aligned}\omega_{\mathbf{d}\alpha}^+(\lambda)\left(\mathrm{ad}_\xi^*\lambda - \Sigma(\xi,\cdot), \mathrm{ad}_\eta^*\lambda - \Sigma(\eta,\cdot)\right) \\ = \langle \lambda, [\xi,\eta]\rangle - \Sigma(\xi,\eta),\end{aligned}$$

where $\Sigma(\xi,\cdot) := T_e\phi(\xi)$*.*

Now work on the semidirect product $G \circledS V$. Modify the cotangent lift of right translation by adding the term

$$C_{(g,v)}(f,u) := (0_{fg}, v + \rho_g(u), c(g)),$$

where $c \in \mathcal{F}(G, V^*)$ is a group one-cocycle , that is, it verifies $c(fg) = \rho^*_{g^{-1}}(c(f)) + c(g)$. Thus, the affine right action on T^*S is:

$$\Psi_{(g,v)}(\alpha_f, (u,a)) = (R_g^{T^*}(\alpha_f), v + \rho_g(u), \rho^*_{g^{-1}}(a) + c(g))$$

All properties of the preceding theorem hold (long calculations). For example, $\alpha \in \Omega^1_0(S)$ associated to the affine term C is given by

$$\alpha(g,v)(\xi_g, (v,u)) = \langle c(g), u \rangle,$$

for $(\xi_g, (v,u)) \in T_{(g,v)}S$,

$$\mathbf{J}_\alpha(\beta_f, (u,a)) = (T^*L_f(\beta_f) + u \diamond a - \mathbf{d}c^T(u), a)$$

and

$$-\phi(f,u) = (u \diamond c(f) - \mathbf{d}c^T(u), c(f)) \in \mathfrak{s}^*.$$

So, by the theorem

$\mathbf{J}_\alpha^{-1}(\mu, a)/S^\phi_{(\mu,a)}$ is symplectomorphic to

$$\mathcal{O}^\phi_{(\mu,a)} = \left\{ \left(\mathrm{Ad}^*_g \mu + u \diamond (\rho^*_{g^{-1}}(a) + c(g)) - \mathbf{d}c^T(u), \rho^*_{g^{-1}}(a) + c(g) \right) \Big| \, (g, u) \in S \right\}$$

relative to the symplectic form

$$\begin{aligned}\omega^+_{\mathcal{B}}(\lambda, b) \Big(\left(\mathrm{ad}^*_\xi \lambda + u \diamond b - \mathbf{d}c^T(u), b\xi + \mathbf{d}c(\xi) \right), \\ \left(\mathrm{ad}^*_\eta \lambda + w \diamond b - \mathbf{d}c^T(w), b\eta + \mathbf{d}c(\eta) \right) \Big) \\ = \langle \lambda, [\xi, \eta] \rangle + \langle b, u\eta - w\xi \rangle + \langle \mathbf{d}c(\eta), u \rangle - \langle \mathbf{d}c(\xi), w \rangle.\end{aligned}$$

Recall that the affine coadjoint orbits $\mathcal{O}^\phi_{(\mu,a)}$ are the symplectic leaves of the Poisson manifold $\mathfrak{s}^*$ with Poisson bracket

$$\begin{aligned}\{f, g\}_{\mathbf{d}\alpha}(\mu, a) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle + \left\langle a, \frac{\delta f}{\delta a} \frac{\delta g}{\delta \mu} - \frac{\delta g}{\delta a} \frac{\delta f}{\delta \mu} \right\rangle \\ + \left\langle \mathbf{d}c \left(\frac{\delta f}{\delta \mu} \right), \frac{\delta g}{\delta a} \right\rangle - \left\langle \mathbf{d}c \left(\frac{\delta g}{\delta \mu} \right), \frac{\delta f}{\delta a} \right\rangle.\end{aligned}$$

With this geometric background we can state the analogue of the affine Lagrangian semidirect product theorem.

$H : T^*G \times V^* \to \mathbb{R}$ right-invariant under the G-action

$$(\alpha_h, a) \mapsto (R_g^{T^*}(\alpha_h), \theta_g(a)) := (R_g^{T^*}(\alpha_h), \rho^*_{g^{-1}}(a) + c(g)).$$

In particular, the function $H_{a_0} := H|_{T^*G\times\{a_0\}} : T^*G \to \mathbb{R}$ is invariant under the induced action of the isotropy subgroup $G^c_{a_0}$ of a_0 relative to the affine action θ, for any $a_0 \in V^*$. Recall that

$$\theta_g(a) := ag + c(g)$$

for any $g \in G$ and $a \in V^*$.

For $\alpha(t) \in T^*_{g(t)}G$ and $\mu(t) := T^*_e R_{g(t)}(\alpha(t)) \in \mathfrak{g}^*$, the following are equivalent:

i *$\alpha(t)$ satisfies Hamilton's equations for H_{a_0} on T^*G.*

ii *The following affine Lie-Poisson equation holds on $\mathfrak{s}^*$:*

$$\frac{\partial}{\partial t}(\mu, a) = \left(-\operatorname{ad}^*_{\frac{\delta h}{\delta \mu}} \mu - \frac{\delta h}{\delta a} \diamond a + \mathbf{d}c^T\left(\frac{\delta h}{\delta a}\right), -a\frac{\delta h}{\delta \mu} - \mathbf{d}c\left(\frac{\delta h}{\delta \mu}\right)\right),\ a(0) = a_0.$$

The evolution of the advected quantities is given by $a(t) = \theta_{g(t)^{-1}}(a_0)$.

Hamiltonian Approach to Continuum Theories of Perfect Complex Fluids

This is the counterpart of the Lagrangian approach, so the Lie-Poisson space is

$$\left(\left[\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o})\right] \circledS \left[V_1 \oplus V_2\right]\right)^* \cong \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \times V_1^* \times V_2^*.$$

with affine Poisson bracket given by

$$\begin{aligned}\{f, g\}(\mathbf{m}, \kappa, a, \gamma) = & \int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right] \mu \\ & + \int_{\mathcal{D}} \kappa \cdot \left(\mathrm{ad}_{\frac{\delta f}{\delta \kappa}} \frac{\delta g}{\delta \kappa} + \mathbf{d}\frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \kappa} \cdot \frac{\delta f}{\delta \mathbf{m}}\right) \mu \\ & + \int_{\mathcal{D}} a \cdot \left(\frac{\delta f}{\delta a}\frac{\delta g}{\delta \mathbf{m}} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mathbf{m}}\right) \\ & + \int_{\mathcal{D}} \gamma \cdot \left(\frac{\delta f}{\delta \gamma}\frac{\delta g}{\delta \mathbf{m}} + \frac{\delta f}{\delta \gamma}\frac{\delta g}{\delta \kappa} - \frac{\delta g}{\delta \gamma}\frac{\delta f}{\delta \mathbf{m}} - \frac{\delta g}{\delta \gamma}\frac{\delta f}{\delta \kappa}\right) \mu \\ & + \int_{\mathcal{D}} \left(\mathbf{d}C\left(\frac{\delta f}{\delta \kappa}\right) \cdot \frac{\delta g}{\delta \gamma} - \mathbf{d}C\left(\frac{\delta g}{\delta \kappa}\right) \cdot \frac{\delta f}{\delta \gamma}\right) \mu.\end{aligned}$$

For a Hamiltonian $h = h(\mathbf{m}, \kappa, a, \gamma) : \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \times V_1^* \times V_2^* \to \mathbb{R}$, the affine Lie-Poisson equations of become

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{m} = -\pounds_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{m} - \operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\mathbf{m} - \kappa \cdot \mathbf{d}\dfrac{\delta h}{\delta \kappa} - \dfrac{\delta h}{\delta a} \diamond a - \dfrac{\delta h}{\delta \gamma} \diamond_1 \gamma \\ \dfrac{\partial}{\partial t}\kappa = -\operatorname{ad}^*_{\frac{\delta h}{\delta \kappa}} \kappa - \operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\kappa\right) - \dfrac{\delta h}{\delta \gamma} \diamond_2 \gamma + \mathbf{d}C^T\left(\dfrac{\delta h}{\delta \gamma}\right) \\ \dfrac{\partial}{\partial t}a = -a\dfrac{\delta h}{\delta \mathbf{m}} \\ \dfrac{\partial}{\partial t}\gamma = -\gamma\dfrac{\delta h}{\delta \mathbf{m}} - \gamma\dfrac{\delta h}{\delta \kappa} - \mathbf{d}C\left(\dfrac{\delta h}{\delta \kappa}\right). \end{cases}$$

Complex Fluids Example

$V_2 = \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$, $V_2^* := \Omega^1(\mathcal{D}, \mathfrak{o})$ and all formulas were already presented. The affine Lie-Poisson equations become in this case:

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{m} = -\pounds_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{m} - \mathsf{div}\left(\frac{\delta h}{\delta \mathbf{m}}\right)\mathbf{m} - \kappa \cdot \mathbf{d}\frac{\delta h}{\delta \kappa} - \frac{\delta h}{\delta a} \diamond a \\ \qquad\qquad - \left(\mathsf{div}^{\gamma} \frac{\delta h}{\delta \gamma}\right)\gamma + \frac{\delta h}{\delta \gamma} \cdot \mathbf{i}_{\_}\,\mathbf{d}^{\gamma}\gamma \\ \frac{\partial}{\partial t}\kappa = -\,\mathsf{ad}^{*}_{\frac{\delta h}{\delta \kappa}}\,\kappa - \mathsf{div}\left(\frac{\delta h}{\delta \mathbf{m}}\kappa\right) - \mathsf{div}^{\gamma}\,\frac{\delta h}{\delta \gamma} \\ \frac{\partial}{\partial t}a = -a\frac{\delta h}{\delta \mathbf{m}} \\ \frac{\partial}{\partial t}\gamma = -\mathbf{d}^{\gamma}\left(\gamma\left(\frac{\delta h}{\delta \mathbf{m}}\right)\right) - \mathbf{i}_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{d}^{\gamma}\gamma - \mathbf{d}^{\gamma}\frac{\delta h}{\delta \kappa} \end{cases}$$

or, in matrix notation (like in Holm[2002] up to sign conventions)

$$\begin{bmatrix} \dot{m}_i \\ \dot{\kappa}_a \\ \dot{a} \\ \dot{\gamma}_i^a \end{bmatrix} = - \begin{bmatrix} m_k\partial_i + \partial_k m_i & \kappa_b\partial_i & (\Box \diamond a)_i & \partial_j\gamma_i^b - \gamma_{j,i}^b \\ \partial_k\kappa_a & \kappa_c C_{ba}^c & 0 & \delta_a^b\partial_j - C_{ca}^b\gamma_j^c \\ a\Box\partial_k & 0 & 0 & 0 \\ \gamma_k^a\partial_i + \gamma_{i,k}^a & \delta_b^a\partial_i + C_{cb}^a\gamma_i^c & 0 & 0 \end{bmatrix} \begin{bmatrix} (\delta h/\delta m)^k \\ (\delta h/\delta \kappa)^b \\ \delta h/\delta a \\ (\delta h/\delta \gamma)_b^j \end{bmatrix}$$

The associated affine Lie-Poisson bracket is

$$\begin{aligned}\{f,g\}(\mathbf{m},\kappa,a,\gamma) = & \int_{\mathcal{D}} \mathbf{m}\cdot\left[\frac{\delta f}{\delta \mathbf{m}},\frac{\delta g}{\delta \mathbf{m}}\right]\mu \\ & + \int_{\mathcal{D}} \kappa\cdot\left(\mathrm{ad}_{\frac{\delta f}{\delta\kappa}}\frac{\delta g}{\delta\kappa} + \mathbf{d}\frac{\delta f}{\delta\kappa}\cdot\frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta\kappa}\cdot\frac{\delta f}{\delta \mathbf{m}}\right)\mu \\ & + \int_{\mathcal{D}} a\cdot\left(\frac{\delta f}{\delta a}\frac{\delta g}{\delta \mathbf{m}} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mathbf{m}}\right)\mu \\ & + \int_{\mathcal{D}}\left[\left(\mathbf{d}^{\gamma}\frac{\delta f}{\delta\kappa} + \boldsymbol{\pounds}_{\frac{\delta f}{\delta \mathbf{m}}}\gamma\right)\cdot\frac{\delta g}{\delta\gamma} - \left(\mathbf{d}^{\gamma}\frac{\delta g}{\delta\kappa} + \boldsymbol{\pounds}_{\frac{\delta g}{\delta \mathbf{m}}}\gamma\right)\cdot\frac{\delta f}{\delta\gamma}\right]\mu.\end{aligned}$$

Curvature Representation

The affine action on connections becomes a linear action on the curvature and one can therefore reduce. The relevant group is

$$[\mathrm{Diff}(\mathcal{D})\,\circledS\,\mathcal{F}(\mathcal{D},\mathcal{O})]\,\circledS\,\left[V_1^*\oplus\Omega^2(\mathcal{D},\mathfrak{o})\right],$$

where $\mathrm{Diff}(\mathcal{D})\,\circledS\,\mathcal{F}(\mathcal{D},\mathcal{O})$ acts on $\Omega^2(\mathcal{D},\mathfrak{o})$ by the representation

$$B\mapsto \mathrm{Ad}_{\chi^{-1}}\,\eta^* B,$$

and where the space V_1^* is only acted upon by the subgroup $\mathrm{Diff}(\mathcal{D})$.

The Lie-Poisson reduction for semidirect products are

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{m} = -\pounds_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{m} - \operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\mathbf{m} - \kappa \cdot \mathbf{d}\dfrac{\delta h}{\delta \kappa} - \dfrac{\delta h}{\delta a} \diamond a \\ \qquad\qquad - \operatorname{div} \dfrac{\delta h}{\delta B} \cdot \mathbf{i}\_ B + \dfrac{\delta h}{\delta B} \cdot \mathbf{i}\_ \mathbf{d}B \\ \dfrac{\partial}{\partial t}\kappa = -\operatorname{ad}^*_{\frac{\delta h}{\delta \kappa}} \kappa - \operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\kappa\right) + \operatorname{Tr}\left(\operatorname{ad}^*_B \dfrac{\delta h}{\delta B}\right) \\ \dfrac{\partial}{\partial t}a = -a\dfrac{\delta h}{\delta \mathbf{m}} \\ \dfrac{\partial}{\partial t}B = -\pounds_{\frac{\delta h}{\delta \mathbf{m}}} B + \operatorname{ad}_{\frac{\delta h}{\delta \kappa}} B \end{cases}$$

if h depends on the connection only through the curvature.

The Lie-Poisson bracket is in this case:

$$
\begin{aligned}
\{f,g\}(\mathbf{m},\kappa,a,B) = & \int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right] \mu \\
& + \int_{\mathcal{D}} \kappa \cdot \left(\mathrm{ad}_{\frac{\delta f}{\delta \kappa}} \frac{\delta g}{\delta \kappa} + \mathbf{d}\frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \kappa} \cdot \frac{\delta f}{\delta \mathbf{m}} \right) \mu \\
& + \int_{\mathcal{D}} a \cdot \left(\frac{\delta f}{\delta a}\frac{\delta g}{\delta \mathbf{m}} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mathbf{m}} \right) \mu \\
& + \int_{\mathcal{D}} \left[\left(\pounds_{\frac{\delta f}{\delta \mathbf{m}}} B - \mathrm{ad}_{\frac{\delta f}{\delta \kappa}} B \right) \cdot \frac{\delta g}{\delta B} - \left(\pounds_{\frac{\delta g}{\delta \mathbf{m}}} B - \mathrm{ad}_{\frac{\delta g}{\delta \kappa}} B \right) \cdot \frac{\delta f}{\delta B} \right] \mu .
\end{aligned}
$$

The map

$$
(\mathbf{m}, \nu, a, \gamma) \mapsto (\mathbf{m}, \nu, a, \mathbf{d}^{\gamma}\gamma)
$$

is a Poisson map relative to the affine Lie-Poisson bracket and this Lie-Poisson bracket.

THE CIRCULATION THEOREMS

For compressible adiabatic fluids the Kelvin circulation theorem is

$$\frac{d}{dt}\oint_{\gamma_t} \mathbf{u}^\flat = \oint_{\gamma_t} T\mathbf{d}s,$$

where $\gamma_t \subset \mathcal{D}$ is a closed curve which moves with the fluid velocity $\mathbf{u}$, $T = \partial e/\partial s$ is the temperature, and e, s denote respectively the specific internal energy and the specific entropy.

Abstract Lagrangian version: Work under the hypotheses of the affine Euler-Poincaré reduction. Let $\mathcal{C}$ be a manifold on which G acts on the left and suppose we have an equivariant map $\mathcal{K} : \mathcal{C} \times V^* \to \mathfrak{g}^{**}$, that is, for all $g \in G, a \in V^*, c \in \mathcal{C}$, we have

$$\langle \mathcal{K}(gc, \theta_g(a)), \mu \rangle = \langle \mathcal{K}(c, a), \mathrm{Ad}_g^* \mu \rangle,$$

where gc denotes the action of G on $\mathcal{C}$, and θ_g is the affine action of G on V^*.

Define the Kelvin-Noether quantity $I : \mathcal{C} \times \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$I(c, \xi, a) := \left\langle \mathcal{K}(c, a), \frac{\delta l}{\delta \xi}(\xi, a) \right\rangle.$$

Fixing $c_0 \in \mathcal{C}$, let $\xi(t), a(t)$ satisfy the affine Euler-Poincaré equations and define $g(t)$ to be the solution of $\dot{g}(t) = TR_{g(t)}\xi(t)$ and, say, $g(0) = e$. Let $c(t) = g(t)c_0$ and $I(t) := I(c(t), \xi(t), a(t))$. Then

$$\frac{d}{dt} I(t) = \left\langle \mathcal{K}(c(t), a(t)), \frac{\delta l}{\delta a} \diamond a - \mathbf{d}c^T \left(\frac{\delta l}{\delta a} \right) \right\rangle.$$

Abstract Hamiltonian version: Some examples do not admit a Lagrangian formulation. Nevertheless, a Kelvin-Noether theorem is still valid for the Hamiltonian formulation. The Kelvin-Noether quantity is now the mapping $J : \mathcal{C} \times \mathfrak{g}^* \times V^* \to \mathbb{R}$ defined by

$$J(c, \mu, a) := \langle \mathcal{K}(c, a), \mu \rangle.$$

Fixing $c_0 \in \mathcal{C}$, let $\mu(t), a(t)$ satisfy the affine Lie-Poisson equations and define $g(t)$ to be the solution of

$$\dot{g}(t) = TR_{g(t)}\left(\frac{\delta h}{\delta \mu}\right), \quad g(0) = e.$$

Let $c(t) = g(t)c_0$ and $J(t) := J(c(t), \mu(t), a(t))$. Then

$$\frac{d}{dt}J(t) = \left\langle \mathcal{K}(c(t), a(t)), -\frac{\delta h}{\delta a} \diamond a + \mathbf{d}c^T\left(\frac{\delta h}{\delta a}\right)\right\rangle.$$

In the case of dynamics on the group $G = \text{Diff}(\mathcal{D})$, the standard choice for the equivariant map $\mathcal{K}$ is

$$\langle \mathcal{K}(c, a), \mathbf{m}\rangle := \oint_c \frac{1}{\rho}\mathbf{m},$$

where $c \in \mathcal{C} = \text{Emb}(S^1, \mathcal{D})$, the manifold of all embeddings of the circle S^1 in $\mathcal{D}$, $\mathbf{m} \in \Omega^1(\mathcal{D})$, and ρ is advected as $(J\eta)(\rho \circ \eta)$.

Consider the affine Euler-Poincaré equations for complex fluids. Suppose that one of the linear advected variables, say ρ, is advected as $(J\eta)(\rho\circ\eta)$. Then

$$\frac{d}{dt}\oint_{c_t}\frac{1}{\rho}\frac{\delta l}{\delta \mathbf{u}} = \oint_{c_t}\frac{1}{\rho}\left(-\frac{\delta l}{\delta \nu}\cdot\mathbf{d}\nu + \frac{\delta l}{\delta a}\diamond a + \frac{\delta l}{\delta \gamma}\diamond_1 \gamma\right),$$

where c_t is a loop in $\mathcal{D}$ which moves with the fluid velocity $\mathbf{u}$.

Similarly, consider the affine Lie-Poisson equations for complex fluids. Suppose that one of the linear advected variables, say ρ, is advected as $(J\eta)(\rho\circ\eta)$. Then

$$\frac{d}{dt}\oint_{c_t}\frac{1}{\rho}\mathbf{m} = \oint_{c_t}\frac{1}{\rho}\left(-\kappa\cdot\mathbf{d}\frac{\delta h}{\delta \kappa} - \frac{\delta h}{\delta a}\diamond a - \frac{\delta h}{\delta \gamma}\diamond_1 \gamma\right),$$

where c_t is a loop in $\mathcal{D}$ which moves with the fluid velocity $\mathbf{u}$, defined be the equality

$$\mathbf{u} := \frac{\delta h}{\delta \mathbf{m}}.$$

There is also a circulation theorem associated to the variable γ because of the equation

$$\frac{\partial}{\partial t}\gamma + \boldsymbol{\pounds}_{\mathbf{u}}\gamma = -\mathbf{d}\nu + \mathrm{ad}_{\nu}\,\gamma.$$

Let η_t be the flow of the vector field $\mathbf{u}$, let c_0 be a loop in $\mathcal{D}$ and let $c_t := \eta_t \circ c_0$. Then, by change of variables, we have

$$\begin{aligned}\frac{d}{dt}\oint_{c_t}\gamma = \frac{d}{dt}\oint_{c_0}\eta_t^*\gamma &= \oint_{c_0}\eta_t^*(\dot{\gamma} + \boldsymbol{\pounds}_{\mathbf{u}}\gamma) \\ &= \oint_{c_0}\eta_t^*(-\mathbf{d}\nu + \mathrm{ad}_{\nu}\,\gamma) = \oint_{c_t}\mathrm{ad}_{\nu}\,\gamma \in \mathfrak{o},\end{aligned}$$

that is, the γ-circulation law is

$$\frac{d}{dt}\oint_{c_t}\gamma = \oint_{c_t}\mathrm{ad}_{\nu}\,\gamma \in \mathfrak{o}.$$