An Introduction to Dynamical Systems and Fractals 3

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Funded by the Engineering and Physical Sciences Research Council (EPSRC)

and the University of Manchester.

Overview

So far, we covered the first 5 parts of the plan

- Some properties of the Cantor set
- Ways to characterise fractals
- Hyperbolic iterated function systems (IFSs)
- Semi-infinite strings of symbols seen as a metric space
- Topological equivalence with the Cantor set
- Non-hyperbolic IFSs
- Digital forcing/controlling of an inverted pendulum

Now we shall look at a digital control problems from the point of view of iterated function systems

A Control Example

- Consider a compass needle constrained to rotate in a vertical plane, with all its mass concentrated at its north pole.
- External electro-magnets can exert a constant, spatially-uniform, horizontal force on the needle's north pole in the plane of its rotation when the pole is near its unstable equilibrium.
- The magnets can be switched instantaneously so that the force is either to the left or to the right.
- The timing of the switching is based on a τ-periodic clock.
- At each tick of the clock the horizontal force can be switched or not as required, the horizontal force is then maintained at the chosen value until the next time the clock ticks.
- Mathematically, this problem is solved by integrating the equations of motion of the system for τ seconds given the two possible values of the horizontal force.
- The resulting dynamical system is an iterated function system, but one for which the mappings are not contractions.

Equations of motion

- Let us say that the centre of mass of the compass needle is a distance r from its pivot point, that its mass is m, and, therefore, that its moment of inertia is $I = mr^2$.
- θ as the angle of the pendulum from the upwards pointing vertical and J the corresponding canonical momentum
- The hamiltonian for the system is written:

$$H(\theta, J) = \frac{J^2}{2I} + mgr(1 + \cos\theta) \mp \mu(\theta + O(\theta^3))$$

where g is the acceleration due to gravity and μ is fixed magnitude of the applied force.

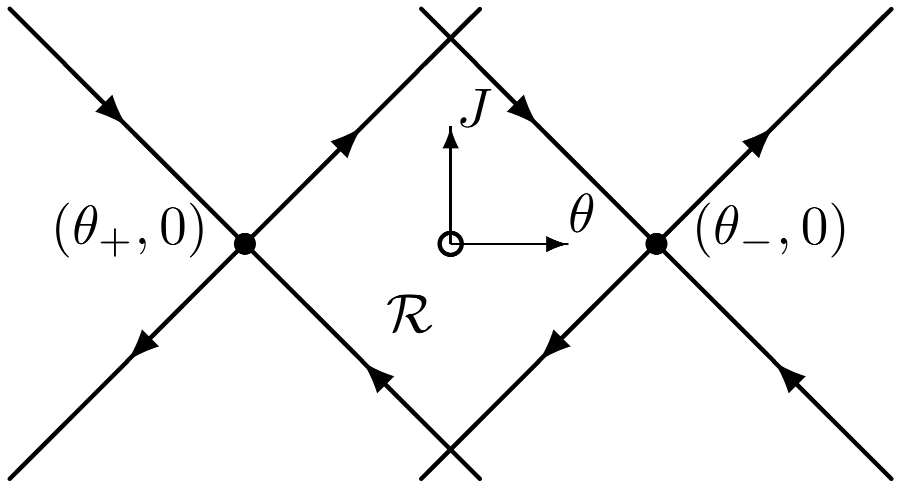
- The equations of motion of the system are therefore

$$\begin{aligned} \dot{\theta} &= J/I \\ \dot{J} &= mgr \sin\theta \pm \mu(1 + O(\theta^2)) \end{aligned}$$

- Note we have left out friction. There are some differences in detail if friction is included.

Linearised equations of motion

- Linearise about the unstable equilibrium of the unperturbed system (assuming $\mu << mgr$)
- For the two perturbations $\pm\mu$ there are new fixed points at $(\theta_\pm, J_\pm) = (\mp\mu/mgr, 0)$



- The eigenvalues are the same for both perturbations: $\lambda_u = \lambda = -\lambda_s$ with $\lambda = \sqrt{g/r}$
- Stable and unstable eigenvectors at $\pm\phi$ from θ axis, where $\tan\phi = -I\lambda$

Time-τ maps

- Use coordinates along the stable and unstable manifolds
- Restrict to the rhombus $\mathcal{R} = \{(x_s, x_u) | -\tilde{\mu} \leq x_s, x_u \leq \tilde{\mu}\}$) where $\tilde{\mu} = \frac{\mu}{2\lambda|\sin\phi|}$
- Integrating for τ seconds we get maps $w_{\pm} : \mathcal{R} \to \mathcal{R}$

$$\begin{aligned} w_+^{(s)}(x_s) &= \Lambda^{-1}x_s - (1-\Lambda^{-1})\tilde{\mu} \\ w_+^{(u)}(x_u) &= \Lambda x_u - (1-\Lambda)\tilde{\mu} \end{aligned}$$

and

$$\begin{aligned} w_-^{(s)}(x_s) &= \Lambda^{-1}x_s + (1-\Lambda^{-1})\tilde{\mu} \\ w_-^{(u)}(x_u) &= \Lambda x_u + (1-\Lambda)\tilde{\mu} \end{aligned}$$

- Where $\Lambda = e^{\lambda\tau} > 1$.
- These constitute the IFS, but they are not contraction mappings.

Non-wandering points: 1

- Is it possible to maintain the pendulum near its inverted fixed point—in $\mathcal{R}$— simply by applying the discrete forces?
- In the new coordinates there is a (trivial) splitting $\mathcal{R} = \mathcal{R}^s \times \mathcal{R}^u$ where $\mathcal{R}^s, \mathcal{R}^u = [-\tilde{\mu}, \tilde{\mu}]$
- Consider two iterated function systems:
  - one for stable motion forward in time, $\{w_+^{(s)}, w_-^{(s)} | w_\pm^{(s)} : \mathcal{R}^s \to \mathcal{R}^s\}$
  - one for unstable motion backward in time, $\{(w_+^{(u)})^{-1}, (w_-^{(u)})^{-1} | (w_\pm^{(u)})^{-1} : \mathcal{R}^u \to \mathcal{R}^u\}$,
- Time reversal symmetry means that both have the form:

$$
\begin{aligned}
w_+^{(s)}(x) \text{ or } (w_+^{(u)}(x))^{-1} &= \Lambda^{-1}x - (1-\Lambda^{-1})\tilde{\mu} \\
w_-^{(s)}(x) \text{ or } (w_-^{(u)}(x))^{-1} &= \Lambda^{-1}x + (1-\Lambda^{-1})\tilde{\mu}
\end{aligned}
$$

- Note that these are not independent, since they must have a common symbol space, Σ

Non-wandering points: 2

- Both IFSs consist of similarities with contraction factor Λ^{-1}.
- They have self-similar invariant sets satisfying:

$$\begin{aligned}\mathcal{W}^s &= w_+^{(s)}(\mathcal{W}^s) \cup w_-^{(s)}(\mathcal{W}^s) \subset \mathcal{R}^s \\ \mathcal{W}^u &= (w_+^{(u)})^{-1}(\mathcal{W}^u) \cup (w_-^{(u)})^{-1}(\mathcal{W}^u) \subset \mathcal{R}^u\end{aligned}$$

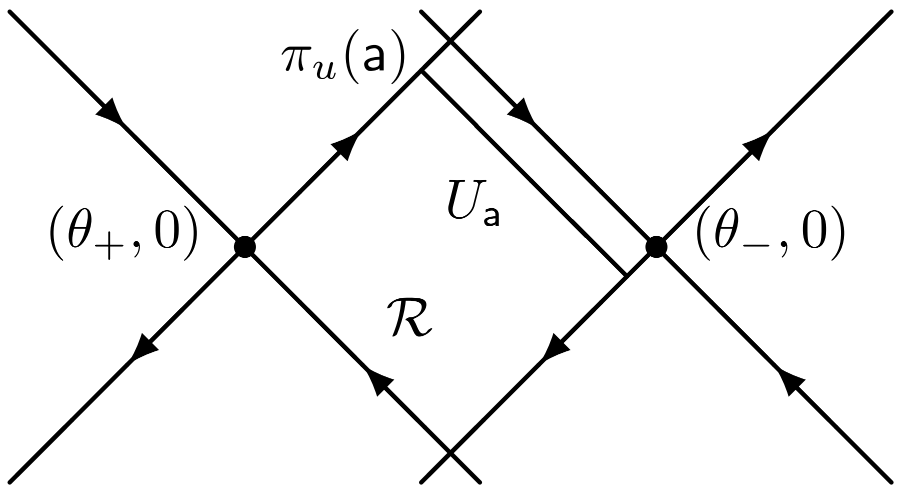
- For each there is a projection $\pi_s : \Sigma \to \mathcal{W}^s$ and $\pi_u : \Sigma \to \mathcal{W}^u$
- Recall, these are defined for any $\mathsf{a} = (a_1, a_2, \ldots) \in \Sigma$ as:

$$\pi(\mathsf{a}) = \lim_{n\to\infty} w_{a_1} \circ w_{a_2} \circ \ldots \circ w_{a_n}(x)$$

- The maps are contractions, therefore these projections are continuous surjections onto their respective invariant sets.

Non-wandering points: 3

- Let us now consider the subset $U_{\mathsf{a}} = \{(x, \pi_u(\mathsf{a}) | x \in \mathcal{R}^s\} \subset \mathcal{R}$
- There is one such set for each point in the invariant set $\mathcal{W}^u$.



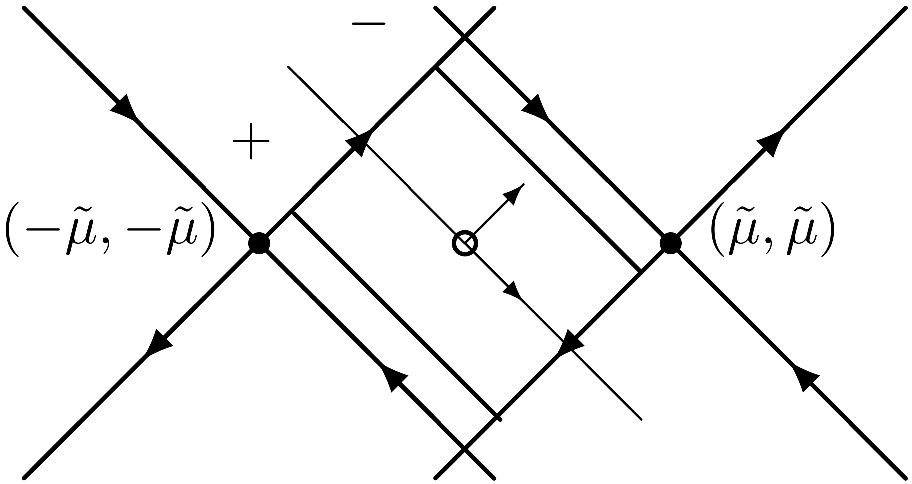
- These sets have the following interesting property

$$
\begin{aligned}
w_{a_1}(U_{\mathsf{a}}) &= \{(w_{a_1}^{(s)}(x), w_{a_1}^{(u)} \circ \pi_u((a_1, a_2, a_3, \ldots)) | x \in \mathcal{R}^s\} \\
&= \{(w_{a_1}^{(s)}(x), w_{a_1}^{(u)} \circ (w_{a_1}^{(u)})^{-1} \circ (w_{a_2}^{(u)})^{-1} \circ (w_{a_3}^{(u)})^{-1} \circ \ldots) | x \in \mathcal{R}^s\} \\
&= \{(w_{a_1}^{(s)}(x), \pi_u \circ \sigma(\mathsf{a}) | x \in \mathcal{R}^s\} \subset U_{\sigma(\mathsf{a})} \subset \mathcal{R}
\end{aligned}
$$

- For the correct sequence of maps, no points of U_{a} leave $\mathcal{R}$

Controllable states and their control

- We call $\mathcal{B} = \bigcup_{a \in \Sigma} U_a$ the set of controllable points in $\mathcal{R}$
- The stucture of $\mathcal{B}$ depends on the value of Λ
  $\Lambda > 2$: $\mathcal{B}$ is the product $[-\tilde{\mu}, \tilde{\mu}]$ with a Cantor set.
  $\Lambda \leq 2$: $\mathcal{B}$ is the whole of $\mathcal{R}$.



- In either case a controllable point can be controlled by seeing which fixed point is nearer
- If $\Lambda > 2$ controllable points sparse: $\dim_H \mathcal{B} = 1 + \log(2)/(\lambda\tau)$
- If $\Lambda \leq 2$ controllable points everywhere $\mathcal{B} = \mathcal{R}$
- If $\Lambda \leq 2$ in the overlap region either control will do.

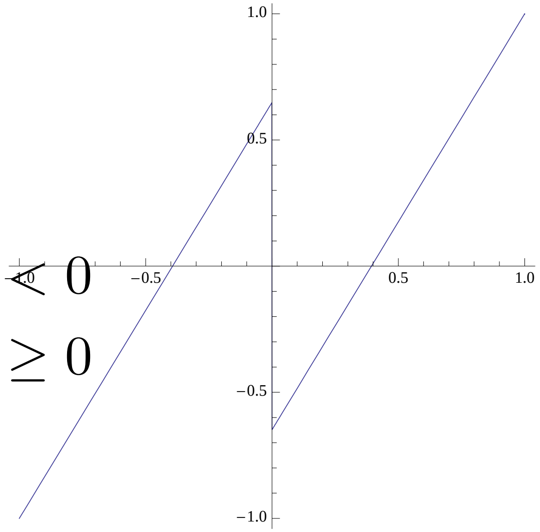
Chaotic example from control theory

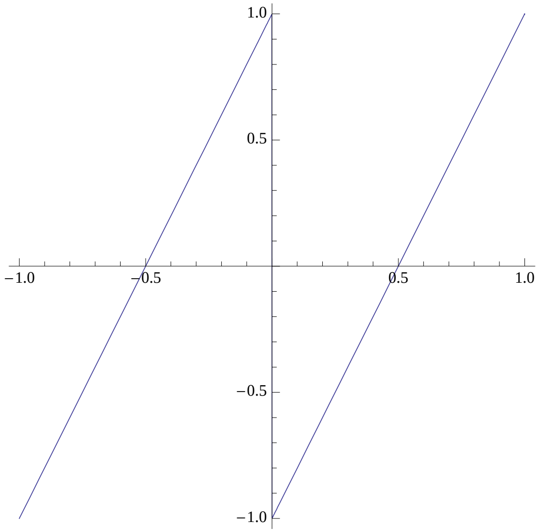
- The control strategy based on expanding away from the nearer fixed point is deterministic.

$$x_{n+1} \quad = \quad \begin{cases} w_+^{(u)}(x_n) = \Lambda x_n - (1-\Lambda)\tilde{\mu} & x_n < 0 \\ w_-^{(u)}(x_n) = \Lambda x_n + (1-\Lambda)\tilde{\mu} & x_n \geq 0 \end{cases}$$

where $\Lambda = e^{\lambda\tau}$

- The map is expanding (Liapunov number $= \Lambda$)
- In addition, if $\Lambda \leq 2$ the interval $[-(\Lambda-1)\tilde{\mu}, (\Lambda-1)\tilde{\mu}]$ contains a a chaotic attractor.
- When $\Lambda > 2$ no attractor, however there is a fractal non-wandering set (Julia set)
- If $\Lambda \leq 2$ there is also a non-deterministic control strategy.





Remarks

- There are several questions that remain with this:
- ? What happens if I have two slightly different affine maps?
- ? It the splitting property robust?
- ? Often hyperbolic IFSs can be regarded as the inverse of expanding maps, what does this say about the control problem?
- ? . . .
- ? The behaviour near the stable fixed point of the pendulum?
- ? What happens globally?

Conclusion

We have discussed the following:

- Some properties of the Cantor set
- Ways to characterise fractals
- Hyperbolic iterated function systems (IFSs)
- Semi-infinite strings of symbols seen as a metric space
- Topological equivalence with the Cantor set
- × Non-hyperbolic IFSs
- Digital forcing/controlling of an inverted pendulum

But really have just scratched the surface.

A very brief bibliography

These references are literally those that I had to hand when preparing these lectures. The list is not remotely a vague approximation of complete, and cannot even be said to contain the most important contributions. I think they are a good start if you are coming new to this area.

1. Basic texts:
   (a) Barnsley M. *Fractals Everywhere*, 1988, Academic Press Inc., San Diego.
   (b) Bressoud D. M. *A Radical Approach to Lebesgue's Theory of Integration*, 2008, Cambridge University Press.
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   (a) Diaconis P., Freedman D. (1999). Iterated random functions, *SIAM Review*, **41**, 45–76.
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