Recurrent of a multi-dimensional diffusion process in a Brownian environment

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1 Introduction

It is well known that multi-dimensional Brownian motion is recurrent if its dimension equals to 1 or 2. And the case where its dimension is larger than 2, then Brownian motion is transient.

Multi-dimensional Brownian motion is constructed from independent d copies of one-dimensional Brownian motion. Instead of Brownian motions, we consider independent d copies of one-dimensional Brox-type diffusion process $X(t)$, which is formally described as follows:

$$\begin{cases} dX(t) &= dB(t) - \frac{1}{2}W'(X(t))dt, \\ X(0) &= 0, \end{cases}$$

where $B(t)$ and $W(x)$ are independent one-dimensional Brownian motions starting at 0 and $W'(x)$ denotes a formal derivative of $W(x)$. $W(x)$ has an influence on a suitable scaling for convergence of $X(t)$, and thus $W(x)$ is regarded as "environment".

Since $W(x)$ is not differentiable in general, the meaning of a solution of the stochastic differential equation above is not clear. Hence for a fixed environment $W(x)$, we deal with diffusion processes X_W with a generator in terms of Feller

$$L_W = \frac{1}{2}e^{W(x)}\frac{d}{dx}\left(e^{-W(x)}\frac{d}{dx}\right).$$

We construct such a diffusion process from a one-dimensional Brownian motion through a time change and a space change, and thereby multi-dimensional diffusion process in a Brownian environment we consider is constructed from d independent suitably scaled Brownian motions.

Brox[1] showed sub-diffusive property of X_W, that is, $(\log t)^{-2}X_W(t)$ converges weakly to a functional of $W(x)$. Since X_W has a quite different scaling for convergence from Brownian motion, we expect different long time behaviour of the multi-dimensional diffusion processes.

2 Main Result

Let $\mathbf{W}$ be a set of d independent one-dimensional Brownian environments. For a fixed $\mathbf{W}$, we consider the following generator, which corresponds to the multi-dimensional diffusion processes $X_{\mathbf{W}}$:

$$L_{\mathbf{W}} = \sum_{k=1}^{d} \frac{1}{2} e^{W_k(x_k)} \frac{\partial}{\partial x_k} \left\{ e^{-W_k(x_k)} \frac{\partial}{\partial x_k} \right\}.$$

Our main theorem is as follows:

Theorem
$X_{\mathbf{W}}$ is recurrent for almost all Brownian environments and any dimension d.

Next we consider a one-dimensional diffusion process in a non-positive reflected Brownian environment. This diffusion process is recurrent for almost all environments, and Tanaka[3] showed that $(\log t)^{-2}\widetilde{X}(t)$ converges weakly, namely its scaling property is same as that of $X(t)$. Let $-|\mathbf{W}|$ be a set of d independent one-dimensional non-positive reflected Brownian environments, and we consider limiting behaviour of multi-dimensional diffusion processes $\widetilde{X}_{-|\mathbf{W}|}$ in the same manner as the case above. Then we have following:

Theorem
$\widetilde{X}_{-|\mathbf{W}|}$ is transient for almost all non-positive reflected Brownian environments and any dimension $d = 2, 3, 4, \ldots$.

By using Ichihara's recurrent or transient test (cf [2]), these are shown in a similar manner to that for Lévy's multi-parameter Brownian environment's case studied by Tanaka[4].

References

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[4] Tanaka, H., 1993. Recurrence of a diffusion process in a multi-dimensional Brownian environment, Proc. Japan Acad. Ser. A Math. Sci. 69, 377–381.