Asymptotic behavior of a diffusion process with a one-sided Brownian potential

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This talk is based on a joint work with K. Kawazu (Yamaguchi University).

We consider a diffusion process with a one-sided Brownian potential starting from the origin and study the asymptotic behavior of the process as time goes to infinity. We begin with describing our model. Denote by $\mathbb{W}$ the space of continuous functions w defined in $\mathbb{R}$ and vanishing identically on $[0, \infty)$. Let P be the Wiener measure on $\mathbb{W}$, namely, let P be the probability measure on $\mathbb{W}$ such that $\{w(-x), x \geq 0, P\}$ is a Brownian motion with time parameter x. Ω denotes the space of real-valued continuous functions defined on $[0, \infty)$. For $\omega \in \Omega$ we write $X(t) = X(t, \omega) = \omega(t) =$ the value of ω at t. For $w \in \mathbb{W}$ and $x_0 \in \mathbb{R}$, let $P_w^{x_0}$ be the probability measure on Ω such that $\{X(t), t \geq 0, P_w^{x_0}\}$ is a diffusion process with generator

$$\mathcal{L}_w = \frac{1}{2} e^{w(x)} \frac{d}{dx} \left(e^{-w(x)} \frac{d}{dx} \right)$$

starting from x_0. We define the probability measure $\mathcal{P}^{x_0}$ on $\mathbb{W} \times \Omega$ by

$$\mathcal{P}^{x_0}(dw d\omega) = P(dw) P_w^{x_0}(d\omega).$$

We regard $\{X(t), t \geq 0, \mathcal{P}^{x_0}\}$ as a process defined on the probability space $(\mathbb{W} \times \Omega, \mathcal{P}^{x_0})$ and call it a diffusion process with a one-sided Brownian potential. This model was introduced by Kawazu, Suzuki and Tanaka([2]). Our aim is to clarify the asymptotic behavior of $\{X(t), t \geq 0, \mathcal{P}^0\}$ as $t \to \infty$.

When $w(x)$ does not vanish identically for $x \geq 0$, or more precisely speaking, when $\{w(x), x \geq 0, P\}$ and $\{w(-x), x \geq 0, P\}$ are independent Brownian motions, the corresponding diffusion process $\{X(t), t \geq 0, \mathcal{P}^{x_0}\}$ was introduced by Brox ([1]) and Schumacher ([3]) as a diffusion analogue of Sinai's

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random walk ([4]). In [1] and [3], it was proved that $\{(\log t)^{-2}X(t), t \geq 0, \mathcal{P}^0\}$ has a nondegenerate limit distribution.

Let us introduce the result in [2]. Let $\mathcal{M}$ be the space of probability laws on Ω and ρ be the Prokhorov metric on $\mathcal{M}$. Let $\{X(t), t \geq 0, \mathcal{P}^0\}$ be a diffusion process with a one-sided Brownian potential. Put

$$X_\lambda(t) = \lambda^{-1/2}X(\lambda t), \quad t \geq 0,$$

for a constant $\lambda > 0$, and denote by $P_\lambda(w)(\in \mathcal{M})$ the probability law of the process $\{X_\lambda(t), t \geq 0, P_w^0\}$. We also denote by $P_N(\in \mathcal{M})$ the probability law of the process vanishing identically, and by $P_R(\in \mathcal{M})$ the probability law of the reflecting Brownian motion on $[0, \infty)$ starting from 0.

Theorem 1 ([2]) *For any* ε *such that* $0 < \varepsilon < \rho(P_N, P_R)/2$

$$\lim_{\lambda\to\infty} P\{\rho(P_\lambda(w), P_N) < \varepsilon\} = \frac{1}{2},$$
$$\lim_{\lambda\to\infty} P\{\rho(P_\lambda(w), P_R) < \varepsilon\} = \frac{1}{2}.$$

In particular, the following hold.

$$\lim_{t\to\infty} \mathcal{P}^0\{-\varepsilon < t^{-1/2}X(t) \leq x\} = \frac{1}{2} + \frac{1}{2}\int_0^x \sqrt{\frac{2}{\pi}}e^{-y^2/2}dy, \quad x > 0, \varepsilon > 0.$$
$$\lim_{t\to\infty} \mathcal{P}^0\left\{0 \leq t^{-1/2}\max_{0\leq s\leq t} X(s) \leq x\right\} = \frac{1}{2} + \frac{1}{2}P_R\left\{\max_{0\leq s\leq 1} X(s) \leq x\right\}, \quad x > 0.$$

Our present results (Theorems 2 and 3 stated below) imply Theorem 1. To state the theorems, we introduce some notation. We put, for $\lambda > 0$ and $\omega \in \Omega$,

$$a_\lambda(t) = a_\lambda(t, \omega) = \int_0^t \mathbf{1}_{(0,\infty)}(X_\lambda(s))ds, \quad t \geq 0,$$

and define

$$a_\lambda^{-1}(t) = \inf\{s > 0 : a_\lambda(s) > t\}, \quad t \geq 0,$$

the right-continuous inverse function of $a_\lambda(t)$. We also put

$$G_\lambda(t) = X_\lambda(a_\lambda^{-1}(t)), \quad t \geq 0.$$

Then $\{G_\lambda(t), t \geq 0, P_w^0\}$ is a reflecting Brownian motion on $[0, \infty)$ starting from 0.

For $w \in \mathbb{W}$ and $a \in \mathbb{R}$, we put

$$\sigma(a) = \sigma(a, w) = \sup\{x < 0 : w(x) = a\},$$

and introduce two subsets A and B of $\mathbb{W}$ by

$$A = \{w \in \mathbb{W} : \sigma(1/2) > \sigma(-1/2)\},$$
$$B = \{w \in \mathbb{W} : \sigma(1/2) < \sigma(-1/2)\},$$

each of which has a half P-measure. For $w \in \mathbb{W}$ and $\lambda > 0$, define $w_\lambda \in \mathbb{W}$ by

$$w_\lambda(x) = \lambda^{-1} w(\lambda^2 x), \quad x \in \mathbb{R}.$$

Then we have

$$\{w_\lambda, P\} \stackrel{\mathrm{d}}{=} \{w, P\},$$

where $\stackrel{\mathrm{d}}{=}$ means the equality in distribution. We also introduce subsets A_λ and B_λ of $\mathbb{W}$ by

$$A_\lambda = \{w \in \mathbb{W} : w_\lambda \in A\},$$
$$B_\lambda = \{w \in \mathbb{W} : w_\lambda \in B\},$$

each of which has a half P-measure by the above property.

In the following theorems, $P\{\cdots|\cdot\}$ denotes the conditional probability. We put $\widetilde{A}_\lambda = A_{\log\lambda}$ and $\widetilde{B}_\lambda = B_{\log\lambda}$.

Theorem 2 *For any $T > 0$ and $\varepsilon > 0$,*

$$\lim_{\lambda\to\infty} P\left\{ P_w^0 \left\{ \sup_{0\le t\le T} |X_\lambda(t) - G_\lambda(t)| < \varepsilon \right\} > 1 - \varepsilon \;\middle|\; \widetilde{A}_\lambda \right\} = 1.$$

For $w \in \mathbb{W}$, we put

$$\zeta = \zeta(w) = \sup\left\{ x < 0 : w(x) - \min_{x\le y\le 0} w(y) = 1 \right\},$$
$$M = M(w) = \begin{cases} \sigma(1/2), & \text{if } w \in A, \\ \zeta(w), & \text{if } w \in B, \end{cases}$$
$$V = V(w) = \min_{x\ge M} w(x).$$

We also define $b = b(w)$ in $(M, 0)$ by $w(b) = V$. Note that b is determined uniquely by w (P-a.s.).

Theorem 3 *For any* $\varepsilon > 0$,

$$\lim_{t\to\infty} P\left\{P_w^0\left\{\left|(\log t)^{-2}X(t) - b(w_{\log t})\right| < \varepsilon\right\} > 1-\varepsilon \mid \widetilde{B}_t\right\} = 1.$$

Concerning the occupation time on $(0,\infty)$ of our process, we have the following.

Theorem 4 *For any* $\varepsilon > 0$,

$$\lim_{t\to\infty} P\left\{P_w^0\left\{\frac{1}{t}\int_0^t \mathbf{1}_{(0,\infty)}(X(s))ds > 1-\varepsilon\right\} > 1-\varepsilon \;\middle|\; \widetilde{A}_t\right\} = 1,$$

$$\lim_{t\to\infty} P\left\{P_w^0\left\{\frac{1}{t}\int_0^t \mathbf{1}_{(0,\infty)}(X(s))ds < \varepsilon\right\} > 1-\varepsilon \;\middle|\; \widetilde{B}_t\right\} = 1.$$

Corollary 5 *The probability distribution of* $t^{-1}\int_0^t \mathbf{1}_{(0,\infty)}(X(s))ds$ *under* $\mathcal{P}^0$ *converges to* $(1/2)\delta_0 + (1/2)\delta_1$ *as* $t \to \infty$.

To state the result on the maximum process of $X(t)$, we put

$$H(w) = \max_{M \le x \le 0} w(x).$$

Note that $H(w) = 1/2$ if $w \in A$ and $0 < H(w) < 1/2$ if $w \in B$.

Theorem 6 *For any* $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathcal{P}^0\left\{\left|\frac{\log \max_{0\le s\le t} X(s)}{\log t} - H(w_{\log t})\right| > \varepsilon\right\} = 0.$$

Our present results together with those of the previous paper ([2]) immediately imply the following theorem.

Theorem 7 *Let* $\widetilde{X}_t$ *be any one of the left column in the following. In each case the distribution of* $\widetilde{X}_t$ *under* $\mathcal{P}^0$ *tends to a limit distribution as* $t \to \infty$, *which is described as follows.*

$\widetilde{X}_t$	*limit distribution*	*support*
$t^{-1/2}X(t)$	μ_{I}	$[0,\infty)$
$(\log t)^{-2}X(t)$	μ_{II}	$(-\infty,0)\cup\{\infty\}$
$t^{-1/2}\max\limits_{0\le s\le t}X(s)$	μ_{III}	$[0,\infty)$
$\dfrac{\log \max\limits_{0\le s\le t}X(s)}{\log t}$	μ_{IV}	$(0,\frac{1}{2}]$
$(\log t)^{-2}\min\limits_{0\le s\le t}X(s)$	μ_{V}	$(-\infty,0)$

$$\mu_{\mathrm{I}}(dx)=\frac{1}{2}\sqrt{\frac{2}{\pi}}e^{-x^2/2}dx+\frac{1}{2}\delta_0(dx).$$
$$\mu_{\mathrm{II}}(dx)=P\{(b\in dx)\cap B\}+\frac{1}{2}\delta_\infty(dx).$$
$$\mu_{\mathrm{III}}(dx)=\frac{1}{2}P_R\left\{\max_{0\le s\le 1}X(s)\in dx\right\}+\frac{1}{2}\delta_0(dx).$$
$$\mu_{\mathrm{IV}}(dx)=P\{H\in dx\}.$$
$$\mu_{\mathrm{V}}(dx)=P\{M\in dx\}.$$

Moreover, the Laplace transforms of the distributions of b, H *and* M *appearing in the definition of* $\mu_{\mathrm{II}},\mu_{\mathrm{IV}}$ *and* μ_{V} *are as follows. For* $\xi>0$,

$$\begin{aligned}
E[e^{\xi b},B] &= \frac{\sinh(\sqrt{2\xi}/2)}{\sqrt{2\xi}\cosh\sqrt{2\xi}},\\
E[e^{\xi H},A] &= \frac{1}{2}e^{\xi/2},\\
E[e^{\xi H},B] &= \int_0^{1/2}e^{\xi x}dx,\\
E[e^{\xi M},A] &= \frac{\sinh(\sqrt{2\xi}/2)}{\sinh\sqrt{2\xi}},
\end{aligned}$$

$$E[e^{\xi M}, B] \;\; = \;\; \frac{\sinh(\sqrt{2\xi}/2)}{(\sinh\sqrt{2\xi})(\cosh\sqrt{2\xi})}.$$

Here $E[\cdot, A]$ denotes the expectation with respect to P on the set A.

References

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