Some path-transformations related to Brownian motion, random matrices and representation theory

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Based on joint work with Ph. Biane and Ph. Bougerol.

UK-Japan Winter School, January 2005

Pitman's Theorem (1975)

Let $(B_t, t \geq 0)$ be a standard real Brownian motion, and define

$$R_t = B_t - 2 \inf_{s \leq t} B_s.$$

Then $(R_t, t \geq 0)$ is a three-dimensional Bessel process: it has the same law as the norm of a three-dimensional Brownian motion.

For continuous $\eta : [0, \infty) \to \mathbb{R}$ with $\eta(0) = 0$, define $\mathcal{P}\eta$ by

$$\mathcal{P}\eta(t) = \eta(t) - 2 \inf_{s \le t} \eta(s).$$

The main (and surprising) content of Pitman's theorem is that the operator $\mathcal{P}$ behaves very nicely with respect to Wiener measure:

It preserves the Markov property.

Connection with Random Matrices

Consider the Hermitian matrix-valued Brownian motion

$$M_t = \begin{pmatrix} X_t & U_t - iV_t \\ U_t + iV_t & -X_t \end{pmatrix},$$

where X, U and V are independent real Brownian motions. The eigenvalues of M_t are $\pm R_t$ where

$$R_t = \sqrt{X_t^2 + U_t^2 + V_t^2}$$

is a three-dimensional Bessel process.

Generalisations

O'C-Yor (ECP 2002) and Bougerol-Jeulin (PTRF 2002):
Give formulae which transform d-dimensional Brownian motion into the eigenvalue process of a Hermitian Brownian motion.

Biane, Bougerol and O'C (DMJ, to appear):
Extend these constructions and show that they are equivalent.

Brownian motion (conditioned to stay) in a cone

Let C be a convex cone in $\mathbb{R}^n$. Let $p_t^0(x, y)$ be the heat kernel on C with Dirichlet boundary conditions, that is, the transition density for Brownian motion killed at the boundary of the cone.

There exists a unique (up to a constant factor) positive p^0-harmonic function h on C. Brownian motion in the cone C is defined to be the corresponding Doob h-transform, with infinitessimal generator

$$\frac{1}{2}\Delta + \nabla(\log h) \cdot \nabla$$

and transition density

$$q_t(x, y) = \frac{h(y)}{h(x)} p_t^0(x, y).$$

Example 1: The three-dimensional Bessel Process

If $n = 1$ and $C = \mathbb{R}_+$ then

$$p^0(x, y) = p_t(x, y) - p_t(x, -y) \qquad h(x) = x.$$

Brownian motion in $\mathbb{R}_+$ is the three-dimensional Bessel Process, with infinitessimal generator

$$\frac{1}{2}\frac{d}{dx^2} + \frac{1}{x}\frac{d}{dx}.$$

Example 2: The general setting

Let W be a finite Coxeter group acting on $\mathbb{R}^n$ with fundamental chamber C (a cone in $\mathbb{R}^n$). Then

$$p_t^0(x,y) = \sum_{w \in W} \varepsilon(w) p_t(x, wy) \qquad h(x) = \prod_{\alpha \in \Phi^+} (\alpha, x).$$

If $W = S_{n+1}$ the Brownian motion in C is distributed as the eigenvalue process of a Brownian motion in $su_{n+1}(\mathbb{C})$.

We will give a formula which transforms Brownian motion in $\mathbb{R}^n$ into Brownian motion in the fundamental chamber C.

Hyperplane reflections and the braid relations

Let V be a finite-dimensional Euclidean space. For $\alpha \in V$ with $(\alpha, \alpha) = 1$, let s_α denote the reflection through $\alpha^\perp$, that is,

$$s_\alpha \lambda = \lambda - 2(\alpha, \lambda)\alpha.$$

Note that $s_\alpha^2 = 1$. For $\beta \in V$ with $(\alpha, \beta) = -\cos(\pi/n)$,

$$(s_\alpha s_\beta)^n = 1.$$

Equivalently,

$$s_\alpha s_\beta s_\alpha \cdots = s_\beta s_\alpha s_\beta \cdots$$

where each side has n terms.

Finite Coxeter groups

$W =$ finite group of isometries in V
$S = \{s_\alpha, \alpha \in \Delta\}$ generating set of 'simple' reflections

If $\pi/n_{\alpha\beta}$ is the angle between hyperplanes $\alpha^\perp$ and $\beta^\perp$ then

$$s_\alpha^2 = 1 \qquad (s_\alpha s_\beta)^{n_{\alpha\beta}} = 1 \qquad \alpha, \beta \in \Delta$$

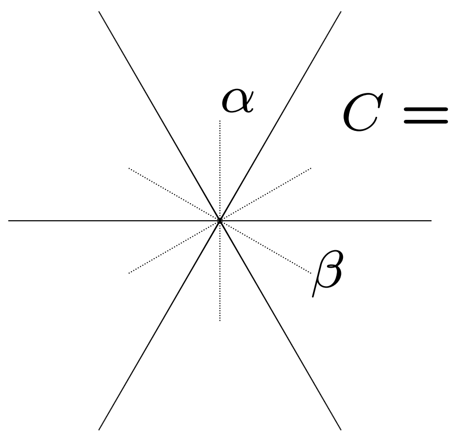
are the defining relations for W. The fundamental chamber

$$C = \{\lambda \in V : \ (\alpha, \lambda) > 0, \ \forall \alpha \in \Delta\}$$

is a fundamental domain for the action of W on V.

Example

$$\alpha = (e_1 - e_2)/\sqrt{2} \qquad W = \langle\, s_\alpha,\, s_\beta \,\rangle \simeq S_3 = \langle\, (12),\, (23) \,\rangle$$
$$\beta = (e_2 - e_3)/\sqrt{2}$$



$$C = \{x_1 > x_2 > x_3\}$$

Generalised Pitman Transformations

Let V be a Euclidean space and $\alpha \in V$ with $(\alpha, \alpha) = 1$. For continuous $\eta : [0, \infty) \to V$ with $\eta(0) = 0$, define

$$\mathcal{P}_\alpha \eta(t) = \eta(t) - 2 \inf_{s \leq t} (\alpha, \eta(s)).$$

Theorem. For $\beta \in V$ with $(\alpha, \beta) = -\cos(\pi/n)$,

$$\mathcal{P}_\alpha \mathcal{P}_\beta \mathcal{P}_\alpha \cdots = \mathcal{P}_\beta \mathcal{P}_\alpha \mathcal{P}_\beta \cdots$$

where each side has n terms.

Corollary. Let W be a finite Coxeter group with generating simple reflections $S = \{s_\alpha, \alpha \in \Delta\}$. For each $w \in W$, we can define

$$\mathcal{P}_w = \mathcal{P}_{\alpha_1} \cdots \mathcal{P}_{\alpha_k}$$

where $w = s_{\alpha_1} \cdots s_{\alpha_k}$ is any reduced (minimal length) decomposition of w.

The longest element

Let W be a finite Coxeter group with generating simple reflections $S = \{s_\alpha, \alpha \in \Delta\}$. The length of an element $w \in W$ is the minimal number of terms required to write w as a product of simple reflections. There is a unique element $w_0 \in W$ of maximal length.

For example, the longest element in S_3 is

$$(13) = (12)(23)(12) = (23)(12)(23).$$

Theorem. Let (W, S) be a finite Coxeter system acting on V with fundamental chamber C. If $X = (X_t, t \geq 0)$ a Brownian motion in V then $\mathcal{P}_{w_0}X$ is a Brownian motion in C.

A remarkable property

The conditional law of X_t given the path $(\mathcal{P}_{w_0}X(s), 0 \leq s \leq t)$ only depends on the endpoint $\mathcal{P}_{w_0}X(t)$.

In the case where W is a Weyl group this is precisely the so-called Duistermaat-Heckman measure associated to $\mathcal{P}_{w_0}X(t)$, and its Fourier transform is given by the Harish-Chandra formula.