Laplacians on uniformly distributed nets and asymptotics of heat semigroups on nilpotent covering manifolds

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1 Introduction

Let M be a compact Reimannian manifold and consider the covering $\pi : \widetilde{M} \to M$ of M. We denote its covering transformation group by Γ. In this talk, we explain long time asymptotics of heat semigroup $e^{-t\Delta_{\widetilde{M}}}$ on $\widetilde{M}$ when Γ is nilpotent.

In 1993, Davies [2] obtained an asymptotic behavior of the heat kernel for a second-order differential operator on $\mathbb{R}$ with periodic coefficients. Then he pointed out that the long time behavior of the heat kernel can be approximated by the heat kernel of a homogenized operator associated with the scaling on $\mathbb{R}$. Then Batty et. al. generalized the result of Davies to the case of a nilpotent Lie group in [1]. After that, Kotani and Sunada proved long time asymptotics of the heat kernel on abelian coveing manifolds in [5].

On the other hand, a long time behavior of the random walks on nilpotent covering graphs are obtained in [4]. Recently, an approximation of the eigenvalues of the Laplacian on a compact Riemannian manifold by the eigenvalues of Laplacians on a sequence of graphs by Otsu [8]. Then we apply these results to obtain our problem.

1.1 Statemants of results

Let $\{X_n \subset M\}_{n\in\mathbb{N}}$ be a sequence of nets, namely a sequence of finite subsets on compact Riemannian manifold M. For each X_n and a positive $r > 0$, we define an oriented graph $X_n(r) = (V_n(r), E_n(r))$ by

$$V_n(r) = X_n, \quad x \sim y \in V_n(r) \iff d_M(x,y) < r$$

(see Otsu [8]). We call it r-net. For a covering $\pi : \widetilde{M} \to M$ of M, we can define the covering graph $\widetilde{X_n}(r)$ in the same way. The discrete Laplacian $\Delta_{\widetilde{X_n}(r)}$ on $X_n(r)$ is defined by

$$\Delta_{\widetilde{X_n}(r)} f(x) = \frac{1}{\deg x} \sum_{x\sim y} \left(f(x) - f(y) \right).$$

From the definiton of $X_n(r)$, $\Delta_{\widetilde{X_n}(r)} f(x)$ is rewritten by

$$\Delta_{\widetilde{X_n}(r)} f(x) = \frac{1}{\#\{y \in B(x,r)\}} \sum_{y \in B(x,r)} (f(x) - f(y)).$$

We consider whether a sequence of discrete Laplacians $\Delta_{\widetilde{X_n}(r)}$ approximates to the Laplacian $\Delta_{\widetilde{M}}$ on $\widetilde{M}$. It is not trivial to find the sequence of nets so that the sequence of Laplacians converges (see Fujiwara [3]). Then, in view of the result of Otsu [8], we choose a sequence of nets $\{X_n\}$ which is *uniformly distributed*, namely the sequence such that for any continuous function $f \in C(M)$,

$$\lim_{n\to\infty} \frac{1}{\#\{X_n\}} \sum_{x \in X_n} f(x) = \frac{1}{\operatorname{vol} M} \int_M f(x) d\operatorname{vol}(x).$$

Since M is compact, there exists such a sequence (see [6]). Then we have the following convergence of the sequence of Laplacian:

Theorem 1 (cf. Otsu[8]) *Let $\{X_n\}$ be an uniformly distributed sequence of nets in m-dimensional closed Riemannian manifold M and $\widetilde{M}$ its covering. Then for any $\epsilon > 0$ and $f \in C_0^\infty(\widetilde{M})$, there exists $r_0 > 0$ such that for $0 < r < r_0$, there exists $n_r \in \mathbb{N}$ such that for $n \geq n_r$*

$$\left\| \frac{2(m+2)}{r^2} \Delta_{\widetilde{X_n}(r)} f - \Delta_{\widetilde{M}} f \right\|_{L^\infty(\widetilde{X_n})} < \epsilon.$$

By the perturbation theory due to Trotter [9], we have

$$\left\| L_{\widetilde{X_n}(r)}{}^{[t \frac{2(m+2)}{r^2}]} f - e^{-t\Delta_{\widetilde{M}}} f \right\|_\infty < \epsilon,$$

where $L_{\widetilde{X_n}(r)}$ is a transition operator for the simple random walk on $\widetilde{X_n}(r)$.

The uniformly distribution is not new notion. Indeed, there are a lot of results for the uniform distribution (see [6]). For example, the following result characterizes the uniform distribution.

Theorem (cf. Kuipers and Niederreiter[6] p. 175) *Let X be a compact metric space with nonnegative regular normed Borel measure μ. Then a sequence of nets X_n in X is uniformly distributed if and only if*

$$\lim_{n\to\infty} \frac{\#\{x \in A \cap X_n\}}{\#\{x \in X_n\}} = \frac{\mu(A)}{\mu(X)}$$

holds for all Borel subset $A \subset X$ with $\mu(\partial A) = 0$.

To prove a central limit theorem on $\widetilde{M}$, we use a central limit theorem on covering graphs. Let X be an oriented finite graph and consider its covering $\pi : \widetilde{X} \to X$. We assume that its covering transformation group Γ is nilpotent.

By a theorem of Mal'cev [7], there exits a connected and simply connected nilpotent Lie group G_Γ such that Γ is identified with a cocompact lattice in G_Γ. Let $\Phi_X : \widetilde{X} \to G_\Gamma$ be a realization, namely a Γ-equivariant map from $\widetilde{X}$ to G_Γ. From the definition of nilpotent Lie group, there exists a decomposition of the Lie algebra $Lie(G_\Gamma) = \mathfrak{g} = \oplus_{1\le k\le r}\mathfrak{g}^k$ such that

$$[\mathfrak{g}^i, \mathfrak{g}^j] \subset \oplus_{k\ge i+j}^r \mathfrak{g}^k,$$

where $[\cdot,\cdot]$ is the Lie bracket of $\mathfrak{g}$. For $\delta > 0$, let $\tau_\delta : G_\Gamma \to G_\Gamma$ be the dilation defined by

$$\tau_\delta x = \exp\left(\sum_{k=1}^r \delta^k \exp^{-1} x\big|_{\mathfrak{g}^k}\right), \quad x \in G_\Gamma.$$

Then we have the following central limit theorem.

Theorem ([4]) *Let $L_{\widetilde{X}}$ be a transition operator for the simple random walk on a nilpotent covering graph $\widetilde{X}$. Then for any $f \in C_\infty(G_\Gamma)$, as $n \uparrow \infty$ and $\delta \downarrow$ with $n\delta^2 \to \mathrm{vol}\,(X)t$ we have*

$$\left\| L_{\widetilde{X}}{}^n \left(\tau_\delta \Phi_X\right)^* f - \left(\tau_\delta \Phi_X\right)^* e^{-t\Omega_X} f \right\|_\infty \to 0,$$

where Ω_X is the sub-Laplacian on G_Γ with respect to the Albanese metric on $\mathfrak{g}^1$.

Next we obtain an approximation of homogenized operator between on graphs and on manifold in order to show the central limit theorem on $\widetilde{M}$. Let Ω_M be a sub-Laplacian on G_Γ w.r.t. $\widetilde{M}$. Then we have the following.

Theorem 2 *Let $\{X_n\}$ be a sequence of uniformly distributed nets in m-dimensional closed Riemannian manifold M and $\widetilde{M}$ its covering. Then for any $\epsilon > 0$ and $f \in C_0^\infty(G_\Gamma)$, there exists $r_0 > 0$ such that for $0 < r < r_0$, there exits $n_r \in \mathbb{N}$ such that for $n \ge n_r$,*

$$\left\| \frac{\mathrm{vol}\,(M)}{\mathrm{vol}\,(X_n)} \frac{2(m+2)}{r^2} \Omega_n f - \Omega_M f \right\|_\infty < \epsilon,$$

where Ω_n is a sub-Laplacian on G_Γ w.r.t $\widetilde{X_n}(r)$. By the perturbation theory, we have

$$\left\| e^{-t \frac{\mathrm{vol}\,(M)}{\mathrm{vol}\,(X_n)} \frac{2(m+2)}{r^2} \Omega_n} f - e^{-t\Omega_M} f \right\|_\infty < \epsilon.$$

By using Theorems 1, 2 and a central limit theorem on nilpotent covering graphs, we prove a central limit theorem on nilpotent covering manifold $\widetilde{M}$. Let $\Phi_M : \widetilde{M} \to G_\Gamma$ be a Γ-equivalent map from $\widetilde{M}$ to a connected and simply connected nilpotent Lie group G_Γ. Then we conclude

Theorem 3 *For any $f \in C_\infty(G_\Gamma)$ and $t > 0$, we have*

$$\left\| e^{-\delta^{-2}\mathrm{vol}\,(M)t\Delta_{\widetilde{M}}} \left(\tau_\delta \Phi_M\right)^* f - \left(\tau_\delta \Phi_M\right)^* e^{-t\Omega_M} f \right\|_i nfty \to 0 \quad (\delta \downarrow 0).$$

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