# On the Poisson manifolds twisted by closed 3-forms

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| infinitesimal objects | | global objects |
|---|---|---|
| Poisson structures | $\leftrightsquigarrow$ | symplectic groupoids |
| twisted Poisson structures | $\leftrightsquigarrow$ | ? |

To what objects does twisted Poisson structures correspond?

$\downarrow$

twisted symplectic groupoids

A **twisted symplectic groupoid**$(G \rightrightarrows M, \omega, \phi)$ is a Lie groupoid which satisfies the following conditions,

1. $m^*\omega = \mathrm{pr_1}^*\omega - \mathrm{pr_2}^*\omega$
2. $d\omega = s^*\phi - t^*\phi$
3. $d\phi = 0$

where, $\omega \in \Omega^2(G)$: non-degenerate, $\phi \in \Omega^3(M)$

$M$: smooth manifold
$\phi$: closed 3-form
A subbundle $L \subset TM \oplus T^*M$ : $\phi$**-twisted Dirac structure**

1. $L$ has rank equal to dim$M$
2. $\langle \cdot, \cdot \rangle_+|_L = 0$, where $\langle \cdot, \cdot \rangle_+$ is non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_+ : \Gamma(L) \times \Gamma(L) \to C^\infty(M)$
$$\langle (X, \xi), (Y, \eta) \rangle_+ := \eta(X) + \xi(Y)$$
3. the space of sections $\Gamma(L)$ is closed under the bracket $[\![\cdot, \cdot]\!]_\phi : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$
$$[\![(X, \xi), (Y, \eta)]\!]_\phi := ([X, Y], \mathfrak{L}_X \eta - i_Y d\xi + \phi(X, Y, \bullet))$$

$\pi \in \Gamma(\wedge^2 TM)$: $\phi$**-twisted poisson structure**
$\overset{\text{def}}{\Longleftrightarrow}$ $L_\pi$ =graph$(\pi^\# : T^*M \to TM) \subset TM \oplus T^*M$ is $\phi$-twisted Dirac structure.

**Def 1** *$(M, \pi, \phi)$ is* $\phi$**-twisted poisson manifold**.

$\pi$ is $\phi$-twisted poisson structure if and only if it satisfies
$$[\pi, \pi] = \wedge^3 \pi^\#(\phi)$$

**Def 2 (Lie algebroid)** • *a vector bundle* $E \to X$

- *a bundle map* $\rho : E \to TX$ *called the anchor map*
- $\Gamma(E) \times \Gamma(E) \to \Gamma(E)$*: Lie bracket*
  1. $\tilde{\rho}([e_1, e_2]) = [\tilde{\rho}(e_1), \tilde{\rho}(e_2)]$ *for any* $e_1, e_2 \in \Gamma(E)$
  2. $[e_1, fe_2] = f[e_1, e_2] + (\tilde{\rho}(e_1)f)e_2$ *for any* $f \in C^\infty(X)$

$L_\pi = \text{graph}(\pi^\# : T^*M \to TM)$ are always associated with Lie algebroids.

$$\begin{array}{ccc} L_\pi \subset TM \oplus T^*M & \xrightarrow{pr_1} & TM \\ \downarrow & & \\ M & & \end{array}$$

For a Lie groupoid $G \rightrightarrows M$ together with

$$\begin{aligned} s &: G \to M, \text{ the source map} \\ t &: G \to M, \text{ the target map} \\ \epsilon &: M \to G, \text{ the identity map} \end{aligned}$$

There exists the associated Lie algebroid $\mathcal{A}(G)$

$$\begin{array}{ccc} \coprod \text{Ker}(ds)_{\epsilon(x)} & \xrightarrow{dt} & TM \\ \downarrow & & \\ M & & \end{array}$$

Lie algebroid $E$: **integrable** $\overset{\text{def}}{\Longleftrightarrow} \exists G$ : Lie groupoid s.t.$E \cong \mathcal{A}(G)$

Given $(M, \pi, \phi)$: $\phi$-twisted poisson manifold, $T^*M$ has a Lie algebroid structure $T^*M_{(\pi,\phi)}$.

- the anchor map: $\pi^{\#} : T^*M \to TM$
- Lie bracket of $\Gamma(\wedge^2 T^*M)$

$$[\omega_1, \omega_2] := \mathcal{L}_{\pi^{\#}(\omega_1)}\omega_2 - \mathcal{L}_{\pi^{\#}(\omega_2)}\omega_1 - d\pi(\omega_1, \omega_2) \\ + \phi(\pi^{\#}(\omega_1), \pi^{\#}(\omega_2), \bullet)$$

**Def 3** *$\phi$-twisted poisson manifold $M$ is* **integrable** *if $T^*M_{(\pi,\phi)}$ is integrable.*

**Theorem 1 (Cattaneo and Xu)** *Integrable twisted Poisson manifolds $\stackrel{1:1}{\longleftrightarrow}$ twisted symplectic groupoids*

- Given a twisted symplectic groupoid,

$$\begin{array}{ll} G & \\ \downarrow & \\ M & \longleftarrow \text{twisted Poisson structure} \end{array}$$

- Given a twisted Poisson manifold,
  There exists a topological groupoid $G(A)$ such that $T^*M_{\pi,\phi} \cong \mathcal{A}(G(A))$
  $G(A) \rightrightarrows M$ is endowed with a twisted symplectic groupoid structure.