A Clark-Ocone type formula under change of measure for canonical Lévy processes

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Introduction

- The Clark-Ocone formula is an explicit stochastic integral representation for random variables in terms of Malliavin derivatives:For $F \in \mathbb{D}^{1,2}(\mathbb{R})$,

$$F = \mathbb{E}[F] + \int_0^T \int_{\mathbb{R}} \mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]Q(dt, dz).$$

- One for Brownian functionals: Clark (1970, 1971, Stochastics **41**, **42**) , Ocone (1984, Stochastics **12**) and Haussmann (1979 ,Stochastics **3**).
- One for pure jump Lévy functionals:
  Løkka (2004, Stochastic Anal. Appl. **22**)
- Clark-Ocone formula under change of measure for Brownian motions:
  Ocone-Karatzas (1991, Stochastics **34**).
- Clark-Ocone formula under change of measure for pure jump Lévy processes: Huehne (2005, Working Paper)
- White noise generalization of the Karatzas-Ocone formula:
  Okur (2010, Stochastic Anal. Appl. **28**)
- Di Nunno et al. (2009, Universitext) and Okur (2012, Stochastics **84**) introduced one for Lévy processes and their results are different from our result(different setting, different representation).

Settings I

Throughout this talk, we consider Malliavin calculus for canonical Lévy processes, based on, Sole et al. (2007, SPA) and Delong-Imkeller (2010, SPA).

- $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$: the usual canonical space for a one-dimensional Brownian motion
- $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$: the canonical space for a pure jump Lévy process
- $(\Omega, \mathcal{F}, \mathbb{P})$: a product of two canonical spaces $(\Omega_W \times \Omega_N, \mathcal{F}_W \times \mathcal{F}_N, \mathbb{P}_W \times \mathbb{P}_N)$
- $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$: the canonical filtration completed for $\mathbb{P}$.
- $X = \{X_t; t \in [0, T]\}$ be a centered square integrable Lévy process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$ with

$$X_t = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz).$$

- $\sigma \geq 0$: constant
- $\{W_t; t \in [0, T]\}$ is a standard Brownian motion on $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$

Settings II

- N is a Poisson random measure on $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ defined by

$$N(A,t) = \sum_{s \le t} 1_A(\Delta X_s), A \in \mathcal{B}(\mathbb{R}_0), \Delta X_s := X_s - X_{s-},$$

  where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$.

- We will denote by $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ the compensated Poisson random measure,
  where $dt\nu(dz) = \lambda(dt)\nu(dz)$ is the compensator of N, $\nu(\cdot)$ the Lévy measure of X and λ the Lebesgue measure on $\mathbb{R}$. Since X is square integrable, the Lévy measure satisfies $\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$.

Settings III

Now we consider the finite measure q defined on $[0,T]\times\mathbb{R}$ by

$$q(E)=\sigma^2\int_{E(0)}dt+\int_{E'}z^2\nu(dz)dt,\quad E\in\mathcal{B}([0,T]\times\mathbb{R}),$$

where $E(0)=\{t\in[0,T];(t,0)\in E\}$ and $E'=E-E(0)$, and the random measure Q on $[0,T]\times\mathbb{R}$ by

$$Q(E)=\sigma\int_{E(0)}dW(t)+\int_{E'}z\tilde{N}(dt,dz),\quad E\in\mathcal{B}([0,T]\times\mathbb{R}).$$

We denote $L^2_{T,q,n}(\mathbb{R}):=\{f:([0,T]\times\mathbb{R})^n\to\mathbb{R}:\|f\|^2_{L^2_{T,q,n}}:=\int_{([0,T]\times\mathbb{R})^n}|f((t_1,z_1),\cdots,(t_n,z_n))|^2q(dt_1,dz_1)\cdots q(t_n,z_n)<\infty\}$. For $n\in\mathbb{N}$ and $f_n\in L^2_{T,q,n}(\mathbb{R})$, a multiple two-parameter integral with respect to the random measure Q can be defined as

$$I_n(f_n):=\int_{([0,T]\times\mathbb{R})^n}f_n((t_1,z_1),\cdots,(t_n,z_n))Q(dt_1,dz_1)\cdots Q(dt_n,dz_n).$$

Wiener-Itô chaos expansion

Next theorem is a key theorem of Malliavin calculus for canonical Lévy processes.

Proposition 1 (Itô, 1956)

Any $\mathcal{F}$-measurable square integrable random variable F on the canonical space has a unique representation

$$F = \sum_{n=0}^{\infty} I_n(f_n), \mathbb{P} - \text{a.s.}$$

with functions $f_n \in L^2_{T,q,n}(\mathbb{R})$ that are symmetric in the n pairs $(t_i, z_i), 1 \leq i \leq n$ and we have the isometry

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! ||f_n||^2_{L^2_{T,q,n}}.$$

Malliavin derivative

Given the chaos expansion we are able to define the Malliavin derivative.

Definition 2

1. Let $\mathbb{D}^{1,2}(\mathbb{R})$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying

$$\sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2_{T,q,n}} < \infty.$$

2. Let $F \in \mathbb{D}^{1,2}(\mathbb{R})$. Then the Malliavin derivative $DF : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ of a random variable $F \in \mathbb{D}^{1,2}(\mathbb{R})$ is a stochastic process defined by

$$D_{t,z}F = \sum_{n=1}^{\infty} n I_{n-1}(f_n((t,z),\cdot)), \quad \text{valid for } q\text{–a.e. } (t,z) \in [0,T] \times \mathbb{R}, \mathbb{P} - a.s.$$

Closability of the Malliavin derivative

Now we establish the following fundamental results: closability of the Malliavin derivative.

Proposition 3

Let $F \in L^2(\mathbb{P})$ *and* $F_k \in \mathbb{D}^{1,2}(\mathbb{R}), k \in \mathbb{N}$ *such that*

1. $\lim_{k\to\infty} F_k = F$ *in* $L^2(\mathbb{P})$,
2. $\{D_{t,z}F_k\}_{k=1}^{\infty}$ *converges in* $L^2(q \times \mathbb{P})$.

Then, $F \in \mathbb{D}^{1,2}(\mathbb{R})$ *and* $\lim_{k\to\infty} D_{t,z}F_k = D_{t,z}F$ *in* $L^2(q \times \mathbb{P})$.

Definition 4

$\mathbb{L}^{1,2}(\mathbb{R})$ denote the space of product measurable and $\mathbb{F}$ -adapted processes $G : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G(s,x)|^2 q(ds,dx)\right] < \infty$, $G(s,x) \in \mathbb{D}^{1,2}(\mathbb{R})$, q–a.e. $(s,x) \in [0,T] \times \mathbb{R}$ and $\mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2} |D_{t,z}G(s,x)|^2 q(ds,dx)q(dt,dz)\right] < \infty$.

Definition 5

1. Let $\mathbb{L}_0^{1,2}(\mathbb{R})$ denote the space of measurable and $\mathbb{F}$ -adapted processes $G : \Omega \times [0,T] \to \mathbb{R}$ satisfying $\mathbb{E}\left[\int_{[0,T]} |G(s)|^2 ds\right] < \infty$, $G(s) \in \mathbb{D}^{1,2}(\mathbb{R})$, $s \in [0,T]$, a.e. and $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \int_{[0,T]} |D_{t,z}G(s)|^2 ds q(dt,dz)\right] < \infty$.
2. Let $\tilde{\mathbb{L}}_1^{1,2}(\mathbb{R})$ denote the space of product measurable and $\mathbb{F}$ -adapted processes $G : \Omega \times [0,T] \times \mathbb{R}_0 \to \mathbb{R}$ satisfying $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_0} |G(s,x)|^2 \nu(dx) ds\right] < \infty$, $[\mathbb{E}\left[\left(\int_{[0,T]\times\mathbb{R}_0} |G(s,x)| \nu(dx) ds\right)^2\right] < \infty$, $G(s,x) \in \mathbb{D}^{1,2}(\mathbb{R}), (s,x) \in [0,T] \times \mathbb{R}_0$, a.e. , $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \left(\int_{[0,T]\times\mathbb{R}_0} |D_{t,z}G(s,x)| \nu(dx) ds\right)^2 q(dt,dz)\right] < \infty$ and $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \int_{[0,T]\times\mathbb{R}_0} |D_{t,z}G(s,x)|^2 \nu(dx) ds q(dt,dz)\right] < \infty$.

Commutation of integration and Malliavin differentiability

Proposition 6 (Imkeller-De Long, 2010)

Let $G : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a predictable process with* $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G(s,x)|^2 q(ds,dx)\right] < \infty$. *Then*

$$G \in \mathbb{L}^{1,2}(\mathbb{R}) \text{ if and only if } \int_{[0,T]\times\mathbb{R}} G(s,x)Q(ds,dx) \in \mathbb{D}^{1,2}(\mathbb{R}).$$

Furthermore, if $\int_{[0,T]\times\mathbb{R}} G(s,x)Q(ds,dx) \in \mathbb{D}^{1,2}(\mathbb{R})$, *then, for* q *-a.e.* $(t,z) \in [0,T] \times \mathbb{R}$, *we have*

$$D_{t,z} \int_{[0,T]\times\mathbb{R}} G(s,x)Q(ds,dx) = G(t,z) + \int_{[0,T]\times\mathbb{R}} D_{t,z}G(s,x)Q(ds,dx), \quad \mathbb{P}\text{-a.s.},$$

and $\int_{[0,T]\times\mathbb{R}} D_{t,z}G(s,x)Q(ds,dx)$ *is a stochastic integral in Itô sense.*

Commutation of integration and Malliavin differentiability

Proposition 7 (Imekeller-De Long (2010), Suzuki (2012))

1. *Let* $G \in \mathbb{L}_0^{1,2}(\mathbb{R})$. *Then,* $\int_{[0,T]} G(s)ds \in \mathbb{D}^{1,2}(\mathbb{R})$ *and*

$$D_{t,z} \int_{[0,T]} G(s)ds = \int_{[0,T]} D_{t,z} G(s)ds$$

holds for q *-a.e.* $(t,z) \in [0,T] \times \mathbb{R}$, $\mathbb{P}$ *-a.s.*

2. *Let* $G \in \tilde{\mathbb{L}}_1^{1,2}(\mathbb{R})$. *Then,* $\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds \in \mathbb{D}^{1,2}(\mathbb{R})$ *and*

$$D_{t,z} \int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds = \int_{[0,T]\times\mathbb{R}_0} D_{t,z} G(s,x)\nu(dx)ds$$

holds for q *-a.e.* $(t,z) \in [0,T] \times \mathbb{R}$, $\mathbb{P}$ *-a.s.*

Chain rules

We introduce chain rules for all processes in $\mathbb{D}^{1,2}(\mathbb{R})$.

Theorem 8 (Chain rule, Viquez (2012))

1. *Let* $F \in \mathbb{D}^{1,2}(\mathbb{R})$, $f \in C_b^1$ *(or a.e. differentiable if* F *has a density). Then* $f(F) \in \mathbb{D}^{1,2}(\mathbb{R})$ *and*
$$D_{t,0} f(F) = f'(F) D_{t,0} F$$
and
$$D_{t,z} f(F) = \frac{f(F + zD_{t,z}F) - f(F)}{z}, z \neq 0$$
2. *Let* $G, H, GH \in \mathbb{D}^{1,2}(\mathbb{R})$. *Then*
$$D_{t,z}(GH) = D_{t,z}G \cdot H + G \cdot D_{t,z}H + z \cdot D_{t,z}H \cdot D_{t,z}G.$$

Assumption 1

Let $\theta(s,x) < 1, s \in [0,T], x \in \mathbb{R}_0$ *and* $u(s), s \in [0,T]$*, be predictable processes with "good" conditions. Moreover we denote*

$$Z(t) := \exp\Big(-\int_0^t u(s)dW(s) - \frac{1}{2}\int_0^t u(s)^2 ds + \int_0^t \int_{\mathbb{R}_0} \log(1-\theta(s,x))\tilde{N}(ds,dx) + \int_0^t \int_{\mathbb{R}_0} (\log(1-\theta(s,x)) + \theta(s,x))\nu(dx)ds\Big), t \in [0,T].$$

Define a measure $\mathbb{Q}$ *on* $\mathcal{F}_T$ *by* $d\mathbb{Q} = Z(T)d\mathbb{P}$*, and we assume that* $Z(T)$ *satisfies the Novikov condition.*

Clark-Ocone type formula under change of measure for Lévy processes (Main Theorem) I: Setting

We denote

$$\tilde{H}(t,z) := \exp\Bigg(- \int_0^T zD_{t,z}u(s)dW_{\mathbb{Q}}(s) - \frac{1}{2}\int_0^T (zD_{t,z}u(s))^2 ds$$
$$+ \int_0^T \int_{\mathbb{R}_0} \left[zD_{t,z}\theta(s,x) + \log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right)(1-\theta(s,x)) \right]$$
$$\times \nu(dx)ds$$
$$+ \int_0^T \int_{\mathbb{R}_0} \log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right)\tilde{N}_{\mathbb{Q}}(ds,dx)\Bigg),$$

and

$$K(t) := \int_0^T D_{t,0}u(s)dW_{\mathbb{Q}}(s) + \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)}\tilde{N}_{\mathbb{Q}}(ds,dx).$$

Clark-Ocone type formula under change of measure for Lévy processes (Main Theorem) I: Setting

Assumption 2

Assume that the following:

1. $F, Z(T), Z(T)F \in \mathbb{D}^{1,2}(\mathbb{R})$,
2. $\log Z(T), \theta(s,x) \in \mathbb{D}^{1,2}(\mathbb{R}), (s,x)$ *-a.e. with densities,*
3. $u, x^{-1}\log(1-\theta(s,x)) \in \mathbb{L}^{1,2}(\mathbb{R})$,
4. $u(s)^2 \in \mathbb{L}_0^{1,2}(\mathbb{R})$ *and* $\theta(s,x), \log(1-\theta(s,x)) \in \tilde{\mathbb{L}}_1^{1,2}(\mathbb{R})$,
5. *and* $F\tilde{H}(t,z), \tilde{H}(t,z)D_{t,z}F \in L^1(\mathbb{Q}), (t,z)$ *-a.e.*

Clark-Ocone type formula under change of measure for canonical Lévy processes (Main Theorem) III

Theorem 9 (Suzuki (2012))

Under Assumption 1 and Assumption 2, we have

$$F = \mathbb{E}_{\mathbb{Q}}[F] + \sigma \int_0^T \mathbb{E}_{\mathbb{Q}}\left[D_{t,0}F - FK(t) \middle| \mathcal{F}_{t-} \right] dW_{\mathbb{Q}}(t) + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z) - 1) + z\tilde{H}(t,z)D_{t,z}F|\mathcal{F}_{t-}]\tilde{N}_{\mathbb{Q}}(dt, dz).$$

Corollary 10

Assume in addition to all assumptions of Theorem 9, $\boldsymbol{u}$ *and* θ *are deterministic functions, then we have*

$$F = \mathbb{E}_{\mathbb{Q}}[F] + \sigma \int_0^T \mathbb{E}_{\mathbb{Q}}[D_{t,0}F|\mathcal{F}_{t-}]dW_{\mathbb{Q}}(t) + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[D_{t,z}F|\mathcal{F}_{t-}]z\tilde{N}_{\mathbb{Q}}(dt, dz).$$

Corollary

Corollary 11

1. *(Pure jump) If* $\sigma = 0$, $u = 0$ *and* $\nu \neq 0$, *then,* $zD_{t,z}F = D_{(t,z)}F$ *and*

$$F = \mathbb{E}_{\mathbb{Q}}[F] + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z) - 1) + \tilde{H}(t,z)D_{(t,z)}F | \mathcal{F}_{t-}]\tilde{N}_{\mathbb{Q}}(dt,dz),$$

where,

$$\tilde{H}(t,z) := \exp\left(\int_0^T\int_{\mathbb{R}_0}\left[D_{(t,z)}\theta(s,x) + \log\left(1 - \frac{D_{(t,z)}\theta(s,x)}{1-\theta(s,x)}\right)(1-\theta(s,x))\right]\nu(dx)ds + \int_0^T\int_{\mathbb{R}_0}\log\left(1 - \frac{D_{(t,z)}\theta(s,x)}{1-\theta(s,x)}\right)\tilde{N}_{\mathbb{Q}}(ds,dx)\right).$$

2. *(Brownian motions) If* $\sigma > 0$, $\theta = 0$, *and* $\nu = 0$, *then,* $D_{t,0}F = \sigma^{-1}D_tF$ *and*

$$F = \mathbb{E}_{\mathbb{Q}}[F] + \int_0^T \mathbb{E}_{\mathbb{Q}}\left[D_tF - F\int_0^T D_tu(s)dW_{\mathbb{Q}}(s)\middle|\mathcal{F}_t\right]dW_{\mathbb{Q}}(t).$$

Applications

Main theorem can be applied to mathematical finance :

- local risk minimization problem,
- mean-variance hedging,
- optimal consumption and trading problem under partial information and shortfall risk, etc ...
- We are preparing a paper concerning the local risk minimization problem as an application of the main theorem of this paper (Arai-Suzuki (2012)).

Thank you for your attention!