Special Legendrian submfds in toric Sasaki-Einstein mfds

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§1. Sasakian geometry

§2. Special Legendrian submfds

§3. Main theorems

§ Sasakian geometry

S : C^∞-mfd

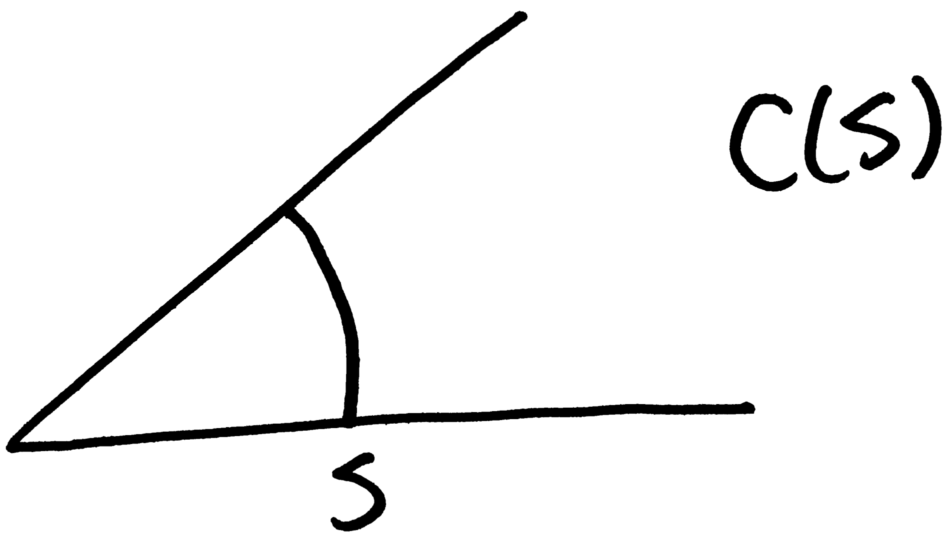
$\dim_{\mathbb{R}} S = 2n+1 \geq 5$

Def (S, g) : Sasaki mfd

$\underset{\text{def}}{\Longleftrightarrow} (C(S), \bar{g}) := (\mathbb{R}_{>0} \times S, dr^2 + r^2 g)$

: Kähler w.r.t. $^\exists J$

$$S \simeq \{1\} \times S \subset C(S)$$



On S

η : contact form
($\Leftrightarrow \eta \wedge (d\eta)^n \neq 0$)

ξ : Reeb vector field
($\Leftrightarrow \eta(\xi) = 1,\ i_\xi d\eta = 0$)

$\Phi \in \Gamma(\mathrm{End}(TS))$ s.t. $\Phi^2 = -\mathrm{id} + \xi \otimes \eta$

(ξ, η, Φ, g) : Sasaki str on S

$$\Updownarrow$$

$(\bar{g}, \omega, J)$: Kähler str on $C(S)$

(S, g) : Sasaki mfd

Def (S, g) : Sasaki-Einstein

$\underset{\text{def}}{\Longleftrightarrow}$ g : Einstein ($\Rightarrow \mathrm{Ric} = 2ng$)

$\Longleftrightarrow$ $(C(S), \bar{g})$: Ricci-flat

($\mathrm{Ric}_{\bar{g}} = 0$)

From now on, we assume that

S : simply connected.

Prop (S,g) : Sasaki-Einstein

$\Leftrightarrow$ ${}^{\exists}(\Omega,\omega)$: weighted Calabi-Yau str on $C(S)$

s.t. $\omega = \frac{1}{2} d(r^2 \eta)$

$\Updownarrow$

Ω : holo. $(n+1)$-form

ω : Kähler form on $C(S)$

s.t. $L_{r\frac{\partial}{\partial r}} \Omega = (n+1)\Omega$

$L_{r\frac{\partial}{\partial r}} \omega = 2\omega$

$\Omega \wedge \overline{\Omega} = c_{n+1} \omega^{n+1}$

§ Special Legendrian

(S, g) : S-E with (Ω, ω)

Def $L \subset S$: special Legendrian

$\underset{\text{def}}{\Longleftrightarrow}$ $C(L) \subset C(S)$: special Lagrangian

Lemma $\tilde{L}$: $(n+1)$-dim submfd in $C(S)$

$\tilde{L} \subset C(S)$: s. Lagrangian cone

$\Longleftrightarrow$ $\Omega^{\mathrm{Im}}|_{\tilde{L}} = 0$, $\eta|_{\tilde{L}} = 0$

Prop $\tau : C(S) \to C(S)$

: anti-holo. involution

$(\tau_* \circ J = -J \circ \tau_*,\ \tau^2 = id)$

If $\tau^* r = r$, $fix(\tau) \neq \phi$

$\Rightarrow fix(\tau) \subset C(S)$: s. Lag. cone

$fix(\tau) \cap S \subset S$: s. Leg.

Example $S^{2n+1} \subset \mathbb{C}^{n+1} \setminus \{0\}$

$r(z) = |z|$

$\tau : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^{n+1} \setminus \{0\}$: cpx conj.

$\Rightarrow \tau^* r = r$, $fix(\tau) = \mathbb{R}^{n+1} \setminus \{0\}$.

$fix(\tau) \cap S^{2n+1} = S^n$

§ Main theorem

Def (S, g) : toric Sasaki
$\Longleftrightarrow (C(S), \bar{g})$: toric Kähler

Real str

$C(S) \curvearrowleft (T^{n+1})_{\mathbb{C}} = (\mathbb{C}^*)^{n+1}$

X_0 : open dense orbit
with coord. $(w^1, \cdots, w^{n+1})$

$\tau : C(S) \to C(S)$: anti-holo. invol.
s.t. $\tau(w) = \overline{w}$ on X_0

$\Rightarrow \tau^* r = r$

Thm (S, g) : cpt. simply conn.
toric S-E

$\Rightarrow \mathrm{fix}(\tau) \cap S \subset S$: s. Leg

Example

${}^{\exists} g_{p,q}$: toric S-E metric on $S^2 \times S^3$

(· Gauntlett-Martelli-Sparks-Waldram '04
· Futaki-Ono-Wang '06)

$S^1 \times S^1 \subset S^2 \times S^3$: s. Leg.

In general, we can prove

Thm (S, g) : cpt toric Sasaki

$\Rightarrow \mathrm{fix}(\tau) \cap S \subset S$: totally geodesic Legendrian

Not toric case

- Link of polynomials
- Homogeneous mfds