Hypersurface geometry and moment map

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I. Motivation and Introduction

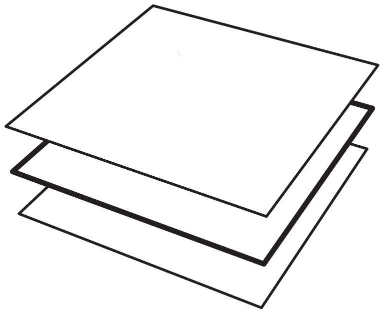
II. Theory of isoparametric hypersurfaces

III. Moment map of the spin action

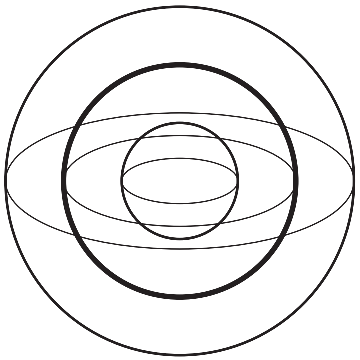
References

1. R. Miyaoka, *Isoparametric hypersurfaces with* $(g, m) = (6, 2)$, Annals of Math. to appear **176**, no.3 (2012) http://annals.math.princeton.edu/toappear
2. R. Miyaoka, *Moment map of the spin action and the Cartan-Münzner polynomial of degree four*, Math. Ann. to appear (2012).

Examples

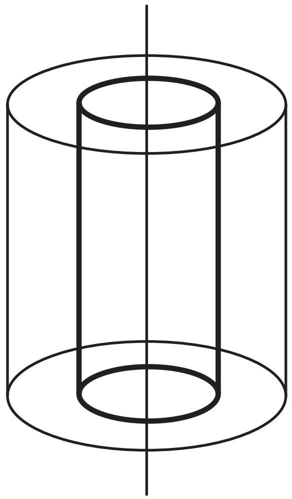


$E^{n-1} \subset E^n$
$H^{n-1} \subset H^n$



$S^{n-1} \subset E^n, H^n$

$S^{k-1} \times E^{n-k}$
$S^{k-1} \times H^{n-k}$



Characterization:

These surfaces have constant principal curvatures:

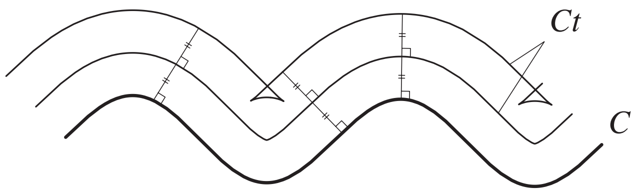
plane: $\lambda_1 = \lambda_2 = 0$
sphere: $\lambda_1 = \lambda_2 = 1/r$
cylinder: $\lambda_1 = 1/r$, $\lambda_2 = 0$.

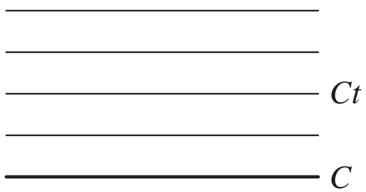
Q. Which surface M has parallel surfaces similar to itself?
(In particular, all regular?)

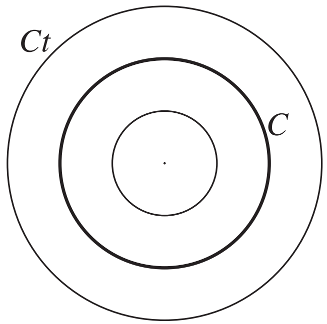
A. In E^3, plane, sphere and cylinder.

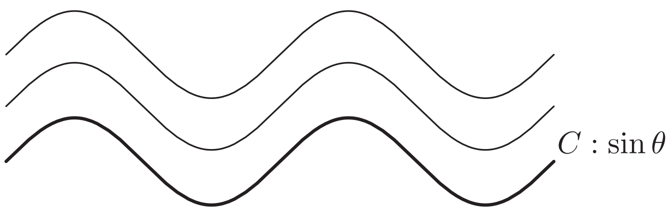
$\mathcal{H} = \{$parallel hypersurfaces$\}$

parallel curves









not parallel

A similar fact holds in H^n and E^n, namely, such M is either totally geodesic, totally umbilic or product of these (cylinder).

Origin: geometric optics, or wave fronts of the evolution of surfaces following Huygens principle.

Q. How do we express M?

Level set expression: $M = f^{-1}(t),\ f : E^3 \to \mathbb{R}$.
(a global expression) is suitable for "surface evolution".
e.g. mean curvature flows

Warning: The function f is not unique.

- $f(x) = |x|$ and $g(x) = \cos|x|$ describe same surfaces.

$\overline{M}$: a complete connected Riemannian manifold

∇ : the Levi-Civita connection, $\triangle$: the Laplacian

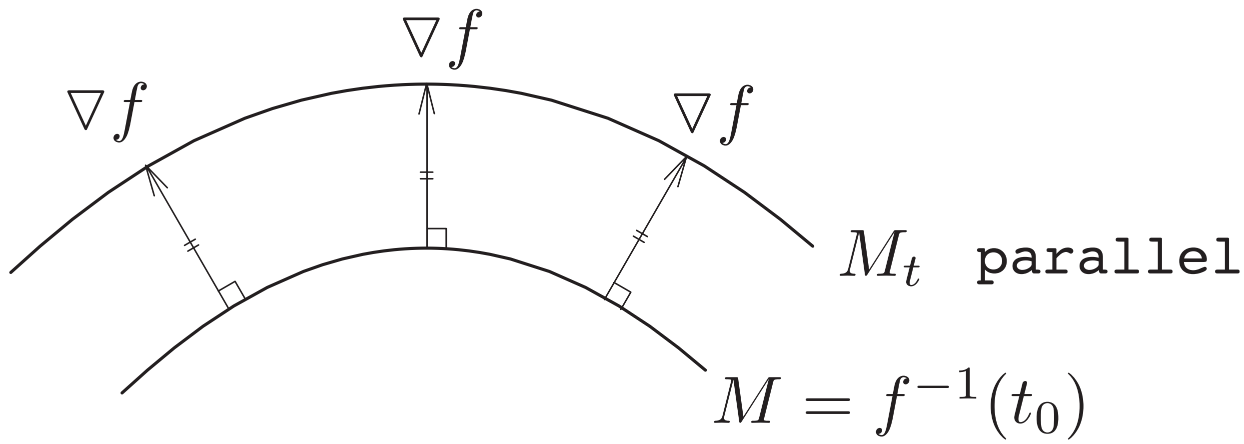
Definition.

(1) A C^2 function $f : \overline{M} \to \mathbb{R}$ satisfying

$$\begin{array}{llll} \text{(I)} & |\nabla f|^2 & = \varphi(f), & \varphi : f(\overline{M}) \to \mathbb{R} : C^2 \\ \text{(II)} & \triangle f & = \psi(f), & \psi : f(\overline{M}) \to \mathbb{R} : C^0 \end{array}$$

is called an **isoparametric function**.

(2) A level set of a regular value of an isoparametric function is called an **isoparametric hypersurface**.



(I) ==> The level sets are mutually parallel.

(II) ==> The level sets have CMC (constant mean curvature)

Fact 1. (É. Cartan)

Let $\overline{M}$ *be a space form* (E^n, S^n or $\mathbb{H}^n$), *and consider a family of parallel hypersurfaces* $\{M_t\}$*. Then the following are equivalent:*

(i) $\{M_t\}$ *is a family of isoparametric hypersurfaces.*

(ii) *All* M_t *have constant mean curvatures.*

(iii) *One of* M_t *has constant principal curvatures.*

Remark. A local notion (iii) implies a global notion (i).

Known examples:

$\overline{M}$	M^{n-1}		
E^n	E^{n-1} or S^{n-1}	$E^k \times S^{n-k-1}$	–
H^n	H_{eq} or S^{n-1}	$H_{eq}^k \times S^{n-k-1}$	–
S^n	S^{n-1}	$S^k \times S^{n-k-1}$	more

H_{eq} : an equidistant h's, including a horosphere.

{homogeneous h'surfaces} $\subset$ {isoparametric h'surfaces}

- The equality holds for E^n and H^n.
- In S^n, $\exists$ **more homogeneous and non-homogeneous examples**. [Ozeki-Takeuchi, Ferus-Karcher-Münzner]

Fact 3. (Münzner, '81) *For isop. h's.* M_t *in* S^n *:*

(a) $g = \sharp\{\textit{distinct principal curvatures}\} \in \{1, 2, 3, 4, 6\}$.

(b) *For the principal curvatures* $\lambda_1 > \lambda_2 > \cdots > \lambda_g$*, the multiplicities* $m_1, m_2, \ldots, m_g$ *satisfy* $m_i = m_{i+2}$*.*

(c) *There exists a* **Cartan-Münzner polynomial** $F : E^{n+1} \to \mathbb{R}$*, homogeneous and of degree* g*, satisfying*

$$\begin{aligned} &(i) \quad \|DF(x)\|^2 = g^2\|x\|^{2g-2} \\ &(ii) \quad \triangle F(x) = \frac{m_2 - m_1}{2} g^2 \|x\|^{g-2}, \end{aligned} \qquad (1)$$

and $M_t = F^{-1}(t) \cap S^n$*,* $-1 < t < 1$*.*

Remark. $M_\pm = f^{-1}(\pm 1)$ are called the focal submanifolds.

Why isoparametric h'surfaces in S^n are interesting?

(partially from the talk in Manchester in Jan. 2010)

• give many explicit examples of special Lagrangian submanifolds in $T\mathbb{R}^{n+1} \cong \mathbb{C}^{n+1}$.

• give many Lagrangian minimal submanifolds in $Q^{n-1}(\mathbb{C})$.

• give many self-similar solutions to the mean curvature flow.

• give a hint to solve Yau's conjecture on the first eigenvalue (recently Tang and Yan solved it for all isoparametric minimal hypersurfaces).

• All the representations of the Clifford algebra are realized geometrically by isoparametric hypersurfaces.

Classification of isoparametric h's. in S^n:

g	1	2	3	4^*	6
M	S^{n-1} hom.	$S^k \times S^{n-k-1}$ hom.	$C_{\mathbb{F}}$ hom.	hom. or OT-FKM	N^6, M^{12} hom.

$g = 3$**:** **Cartan hypersurfaces $C_{\mathbb{F}}^{3d}$**

Theorem. [Cartan '38] *Isoparametric hypersurfaces with $g = 3$ are given by tubes $C_{\mathbb{F}}^{3d}$ over the standard embedding of the projective planes $\mathbb{F}P^2$ in S^4, S^7, S^{13} and S^{25}, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathcal{C}$ (Cayley numbers). ($d = 1, 2, 4, 8$).*

Theorem. [Abresch, ‘83] *When* $g = 6$, $m_i = m \in \{1, 2\}$.

For each case there is a homogeneous example:

$m = 1$: isotropy orbits of $G_2/SO(4)$ in S^7.

$m = 2$: isotropy orbits of $G_2 \times G_2/G_2$ in S^{13}.

Proposition. [M. ‘93] *The homogeneous hypersurface* N^6 *with* $(g, m) = (6, 1)$ *has a fibration* $\pi : N \to S^3$ *with fiber* $C_{\mathbb{R}} = SO(3)/Z_2 \oplus Z_2$.

Proposition. [M. ‘08] *The homogeneous hypersurface* M^{12} *with* $(g, m) = (6, 2)$ *has a Kähler fibration* $\pi : M \to S^6$ *with fiber* $C_{\mathbb{C}} = SU(3)/T^2$.

$m = 1$

$$N^6 \cong SO(4)/Z_2 \oplus Z_2$$

$$\big\downarrow \quad \leftarrow C_{\mathbb{R}} \cong SO(3)/Z_2 \oplus Z_2$$

$$S^3 \cong SO(4)/SO(3)$$

$m = 2$

$$M^{12} \cong G_2/T^2$$

$$\big\downarrow \quad \leftarrow C_{\mathbb{C}} \cong SU(3)/T^2$$

$$S^6 \cong G_2/SU(3)$$

Remark. The focal submanifolds $M_\pm$ of $(g, m) = (6, 2)$ are related to Bryant's twistor fibrations:

(ii) $M_+ \cong \mathbb{Q}^5 \to S^6 = G_2/SU(3)$ with fiber $\mathbb{C}P^2$. This is diffeomorphic to the twistor fibration over S^6.

(iii) $M_- \cong \mathbb{Q}^5 \to G_2/SO(4)$ with fiber $\mathbb{C}P^1$. This is diffeomorphic to the twistor fibration over the quaternionic Kähler manifold $G_2/SO(4)$.

Theorem. [Dorfmeister-Neher, '85, M. '09] *Isoparametric hypersurfaces with* $(g, m) = (6, 1)$ *are homogeneous, i.e., isotropy orbits of* $G_2/SO(4)$.

Theorem 1. (M. to appear in Ann. Math.) The isoparametric hypersurfaces with $(g, m) = (6, 2)$ are homogeneous, i.e., isotropic orbits of $G_2 \times G_2/G_2$.

Key Proposition. (M. '93, '98) *Isoparametric hypersurfaces with* $g = 6$ *are homogeneous* $\Leftrightarrow$ **Condition A** *is satisfied, namely, the shape operators of a focal submanifold have the kernel independent of the normal directions.*

(To prove Condition A is extremely difficult.)

Non-homogeneous case occurs only when $g = 4$.

Known isoparametric hypersurfaces in S^n with $g = 4$:

<table>
<tr><td rowspan="3">OT-FKM type</td><td colspan="2">non-homogeneous $(m_1, m_2) = (3, 4k), (7, 8k), \ldots$</td></tr>
<tr><td rowspan="4">homogeneous:
isotropy orbits
of G/K</td><td>G/K : non-Hermitian
$(4, 4k-1)$</td></tr>
<tr><td>*Hermitian
$(1, k), (2, 2k-1), (9, 6)$</td></tr>
<tr><td rowspan="2">non OT-FKM</td><td>*Hermitian $(4, 5)$</td></tr>
<tr><td>non-Hermitian $(2, 2)$</td></tr>
</table>

They are all classfied except for $(m_1, m_2) = (7, 8)$ (Cecil-Chi-Jensen, Immervoll, and Chi, 2007∼2012).

Clifford systems and h's of OT-FKM type

$O(n)$: the orthogonal group, $\mathfrak{o}(n)$: its Lie algebra.

Definition. $P_0, \ldots P_m \in O(2l)$ is called a Clifford system

$$\Leftrightarrow P_i P_j + P_j P_i = 2\delta_{ij}\mathrm{id}, \quad 0 \leq i, j \leq m.$$

• Clifford system corresponds to a representation of Clifford algebra in a one-to-one way.

Remark. (1) The possible pairs (m, l) :

m	1	2	3	4	5	6	7	8	$\cdots$	$m+8$	$\cdots$
$l = \delta(m)$	1	2	4	4	8	8	8	8	$\cdots$	$16\delta(m)$	$\cdots$

(2) W.r.t. the inner product $\langle P, Q\rangle = \dfrac{1}{2l}\mathrm{Tr}(P^t Q)$, $P_0, \ldots, P_m$ give an orthonormal basis of the linear space V of symmetric orthogonal operators, which they span.

Fact 4. (Ferus-Karcher-Münzner '81)

When a Clifford system $P_0, \ldots, P_m$ *is given,*

$$F(x) = \langle x, x \rangle^2 - 2 \sum_{i=0}^{m} \langle P_i x, x \rangle^2 \tag{2}$$

is a Cartan-Münzner polynomial of degree 4. If $l - m - 1 > 0$, $F|_{S^{2l-1}}$ *defines isoparametric hypersurfaces in* S^{2l-1} *with* $g = 4$ *and* $m_1 = m$, $m_2 = l - m - 1$.

Goal: We express $F(x)$ via the moment map of a spin action.

$P_0, \ldots, P_m$: Clifford system $\Rightarrow P_iP_j$, $0 \le i < j \le m$, are skew, and generate a Lie subalgebra of $\mathfrak{o}(2l)$ isomorphic to $\mathfrak{o}(m+1)$.

Fact 5. **[FKM, '81]** *Spin$(m+1)$ acts on $\mathbb{R}^{2l}$, and preserves $F(x)$, namely, $F(x)$ is constant on each Spin$(m+1)$ orbit.*

Remark. *Spin*$(m+1)$ action is small, and in general, never transitive on the hypersurface.

Review of symplectic geometry

Definition.

(1) (P^{2n}, ω) is **a symplectic manifold**

$\Leftrightarrow$ ω is a non-degenerate closed 2-form on P.

(2) **The Hamiltonian vector field** H_f **of** $f \in \mathbb{C}^\infty(P)$

$\Leftrightarrow df = \omega(H_f, \ )$.

Put $\mathrm{Ham}(P) = \{H_f \mid f \in \mathbb{C}^\infty(P)\}$.

K : a compact Lie group acting on P.

Definition.

(1) **a fundamental vector field** on P

$$\Leftrightarrow X_\zeta = \frac{d}{dt}\Big|_{t=0} (\exp t\zeta)x, \quad \zeta \in \mathfrak{k}$$

(2) $K \curvearrowright P$ is **a symplectic action**

$$\Leftrightarrow k^*\omega = \omega, \forall k \in K.$$

(3) $K \curvearrowright P$ is **a Hamiltonian action**

$$\Leftrightarrow X_\zeta \in \mathrm{Ham}(P), \forall \zeta \in \mathfrak{k}.$$

i.e., $\exists \mu_\zeta \in \mathbb{C}^\infty(P)$ s.t. $d\mu_\zeta = \omega(X_\zeta, \ )$.

(4) With respect to the coadjoint action of K on $\mathfrak{k}^*$,

$\mu : P \to \mathfrak{k}^*$ is **a moment map**

$$\Leftrightarrow \begin{array}{l} \text{(i) } \mu \text{ is } K \text{ equivariant} \\ \text{(ii) } d\mu(\zeta) = \omega(X_\zeta, \ ) \end{array}$$

- $K \curvearrowright P$ is Hamiltonian

$$\Leftrightarrow \exists \mu : P \to \mathfrak{k}^*, \text{ the moment map}$$

$\Rightarrow$ for $\zeta \in \mathfrak{k}$, $\mu_\zeta(p) = \mu(p)(\zeta) \in C^\infty(P)$

and $H_{\mu_\zeta} = X_\zeta$.

Example. (1) $(\mathbb{C}^n, J, \omega)$ with $\omega(X, \ ) = -\langle JX, \ \rangle$

$K \curvearrowright \mathbb{C}^n$: Hamiltonian $\Rightarrow d\mu_\zeta(Y) = \omega(X_\zeta, Y) = -\langle JX_\zeta, Y\rangle$

$\Rightarrow X_\zeta = J\nabla\mu_\zeta$.

(2) G/K : a Hermitian symmetric space,

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: the Cartan decomposition,

$\exists$ a center $\mathfrak{c}$ of $\mathfrak{k} \Rightarrow \exists$ a Kähler structure J on $\mathfrak{p}$ given by

$$Jx = \mathrm{ad}\mathfrak{z}(x) = -\mathrm{ad}x(\mathfrak{z}), \quad \mathfrak{z} \in \mathfrak{c},\ x \in \mathfrak{p}.$$

$\Rightarrow$ the isotropy action $K \curvearrowright \mathfrak{p}$ is a Hamiltonian action with **the moment map:** $\mu^H(x) = \frac{1}{2}(\mathrm{ad}x)^2\mathfrak{z}$ **(Ohnita)**.

Remark. In general, there does not exist symplectic (nor Kähler) structure on $\mathfrak{p}$ of symmetric spaces.

Symplectic structure on $T\mathbb{R}^n$

A complex structure $\tilde{J}$ on $T\mathbb{R}^n$ is given by

$$\tilde{J}(U,V) = (-V,U), \quad (U,V) \in T_{(x,Y)}(T\mathbb{R}^n) \cong \mathbb{R}^n \oplus \mathbb{R}^n.$$

$\Rightarrow T\mathbb{R}^n$: a symplectic manifold with a symplectic form

$$\omega(\tilde{Z},\tilde{W}) = -\langle \tilde{J}\tilde{Z},\tilde{W}\rangle, \quad \tilde{Z},\tilde{W} \in T_{(x,Y)}(T\mathbb{R}^n),$$

$\tilde{J}$: parallel $\Rightarrow \omega$ is a non-degenerate closed 2-form.

$T\mathbb{R}^n$ has a standard symplectic structure.

Hamiltonian action on $T\mathbb{R}^n$

$K \subset O(n)$: acting on $\mathbb{R}^n$

Extend $K \curvearrowright \mathbb{R}^n$ naturally to $T\mathbb{R}^n$, then for $\zeta \in \mathfrak{o}(n)$,

$$X_\zeta = \zeta x.$$

Proposition. $K \curvearrowright T\mathbb{R}^n$ is a Hamiltonian action with the moment map $\mu : T\mathbb{R}^n \to \mathfrak{k}^*$ given by

$$\mu(x, Y)(\zeta) = -\langle \zeta x, Y \rangle.$$

e.g. $n = 3$, $\zeta_1, \zeta_2, \zeta_3 \in \mathfrak{o}(3)$ is an o.n.basis, then for $(x, Y) \in T\mathbb{R}^3$,

$$\mu(x, Y)(\zeta_i) = -\langle \zeta_i x, Y \rangle$$

is the angular momentum.

In particular, we have

$$\mu(x, Y) = -\sum_{i=1}^{3} \langle \zeta_i x, Y \rangle \zeta_i.$$

$Spin(m+1)$ **action on** $T\mathbb{R}^{2l}$

Let $P_0, \ldots,, P_m$ be a Clifford system on $\mathbb{R}^{2l}$:

$\Rightarrow \zeta_{ij} = P_iP_j \in \mathfrak{o}(2l)$, $0 \leq i < j \leq m$, generate $\mathfrak{o}(m+1)$, acting on $\mathbb{R}^{2l}$.

Apply the previous argument to the $Spin(m+1)$ action on $\mathbb{R}^{2l}$ given by $(\exp tP_iP_j)x$ for $x \in \mathbb{R}^{2l}$.

We may regard $\zeta_{ij} = P_iP_j$ as an orthonormal frame of $\mathfrak{o}(m+1)$, and hence obtain:

Proposition 1. The moment map of the $Spin(m+1)$ action on $T\mathbb{R}^{2l}$ is given by

$$\mu(x,Y) = -\sum_{0\leq i<j\leq m} \langle \zeta_{ij}x, Y\rangle \zeta_{ij} \in \mathfrak{o}(m+1) \cong \mathfrak{o}^*(m+1).$$

And thus it follows $\|\mu(x,Y)\|^2 = \sum_{0\leq i<j\leq m} \langle P_i P_j x, Y\rangle^2$.

Since the $U(1) \curvearrowright T\mathbb{R}^{2l}$ associated with J is commutes with ω, this action is synplectic, and moreover, Hamiltonian.

Theorem 2. (M, to appear in Math. Ann.)

$P_0, \dots, P_m$ on $\mathbb{R}^{2l}$: a Clifford system,

$Y : \mathbb{R}^{2l} \to T\mathbb{R}^{2l}$: (not necessarily continuous) vector field;

$$Y_x = \begin{cases} P_0 x, & \text{if } \langle P_0 x, x\rangle = 0 \\ \dfrac{\langle P_1 x, x\rangle P_0 x - \langle P_0 x, x\rangle P_1 x}{\sqrt{\langle P_1 x, x\rangle^2 + \langle P_0 x, x\rangle^2}}, & \text{if } \langle P_0 x, x\rangle \neq 0. \end{cases}$$

Then the Cartan Münzner polynomial is given by

$$F(x) = \|\mu_0(x, Y_x)\|^2 - 2\|\mu(x, Y_x)\|^2$$

where $\mu_0 + \mu$ is the moment map of $U(1) \times Spin(m+1) \curvearrowright T\mathbb{R}^{2l}$.

Remark. (1) The RHS is determined by $x \in \mathbb{R}^{2l}$.

(2) P_0, P_1 can be replaced by any two orthogonal unit elements of V. This corresponds to that there is no standard choice of a principal vector for λ_1 if $m_1 > 1$.

(3) $C = \{(x, Y_x) \in T\mathbb{R}^{2l}\}$ is a $2l$ dimensional cone (outside x such that $\langle P_0 x, x\rangle = 0$). However, C is not a Lagrangian cone of $T\mathbb{R}^{2l}$.

(4) For isotropy orbits of the Hermitian symmetric spaces, an expression of $F(x)$ via moment map was first given by S. Fujii (2011), and Fujii and H. Tamaru.

Remaining case

$F(x)$ for the OT-FKM type has been expressed by μ:

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</table>

The last two homogeneous cases are not of OT-FKM type.

Review of homogeneous case

Fact. (Hsiang-Lawson,'69) Every homogeneous hypersurface in S^n is given by an isotropy orbit of a rank two symmetric space.

G/K : a rank two symmetric space

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$: the Cartan decomposition

Extend the isotropy action $K \curvearrowright \mathfrak{p}$ to $T\mathfrak{p}$ in a natural way: $k \cdot (x, Y) = (\mathrm{Ad}k(x), \mathrm{Ad}k(Y)), \quad (x, Y) \in T\mathfrak{p},\ k \in K.$

- Since $\mathfrak{p} \cong \mathbb{R}^n$, we can apply the previous argument to this case.

Proposition 2. *G/K : a rank two symmetric space,*
$\Rightarrow U(1) \times K \curvearrowright T\mathfrak{p}$ is a Hamiltonian action with the moment map $\mu_0 + \mu : T\mathfrak{p} \to \mathfrak{u}(1)^* \oplus \mathfrak{k}^*$;

$$\begin{aligned} \mu_0(x, Y) &= \frac{1}{2}(\|x\|^2 + \|Y\|^2)\eta, \\ \mu(x, Y) &= -\mathrm{ad}x(Y), \quad (x, Y) \in T\mathfrak{p}. \end{aligned}$$

Corollary. *If G/K is a Hermitian symmetric space, for $\mathfrak{z} \in \mathfrak{c} \subset \mathfrak{k}$ s.t. $J = \mathrm{ad}\mathfrak{z}$*

$$\Rightarrow \mu(x, \tfrac{1}{2}Jx) = \mu^H(x) = \tfrac{1}{2}(\mathrm{ad}x)^2\mathfrak{z}$$

Remark. The proposition holds not only for $g = 4$, but also for all the homogeneous hypersurfaces.

In our case, $G/K = SO(5) \times SO(5)/SO(5)$ $((m_1, m_2) = (2, 2))$, or $SO(10)/U(5)$ $((m_1, m_2) = (4, 5))$.

Put $G_{ij} = E_{ij} - E_{ji} \in \mathfrak{o}(5) \subset \mathfrak{u}(5)$, $1 \leq i < j \leq 5$, where, E_{ij} is the matrix with (i, j) component equal to one and all other components equal to 0.

Theorem 3. (M.)
When $(m_1, m_2) = (2,2)$, $(4,5)$ which are not of OT-FKM, using $\tau = G_{25} + G_{45} \in \mathfrak{k}$, put $Y_H = [H, \tau] \in \mathfrak{p}$ for $H \in \mathfrak{a}$, and extend it to a vector field Y_x on $\mathfrak{p}$ by the action of K. Restricting the moment map $\mu_0 + \mu$ of the action of $U(1) \times K$ to the cone $C = \{(x, Y_x) = \mathrm{Ad}k(H, Y_H)\} \subset T\mathfrak{p}$, we can express

$$F(x) = p\|\mu_0(x, Y_x)\|^2 - q\|\mu(x, Y_x)\|^2,$$

where $(p, q) = (3, 4)$ for $(m_1, m_2) = (2, 2)$, and $(p, q) = (\frac{3}{4}, 1)$ for $(m_1, m_2) = (4, 5)$.

Summary

Finally, the Cartan-Münzner polynomials with $g = 4$ *are expressed by the square norm of the moment map on* $T\mathbb{R}^{2l}$ *of a certain group action restricted to the* $2l$ *dimensional cone, in both homogeneous and non-homogeneous cases.*

Problem. The extension to $T\mathbb{R}^n \cong \mathbb{C}^n$ is in other words the complexification of the object.

If we "complexify" the Cartan-Münzner polynomial suitably to a homogeneous polynomial on $T\mathbb{R}^n \cong \mathbb{C}^n$, then what follows about hypersurfaces in $\mathbb{C}^n$ or in $\mathbb{C}P^{n-1}$ given by the level set of F?

THANK YOU FOR YOUR ATTENTION!