Convergence of Stochastic Processes and Collapsing of Manifolds

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Overview

This is a report on work in progress. We try to understand the following problem. Consider a differentiable manifolds equipped with a sequence of Riemannian metrics, that converges to a manifold of lower dimension in the Gromov-Hausdorff distance. We discuss the dynamic picture of this collapsing. In this talk we investigate in detail the example of the Hopf vibration and Berger's spheres, where the scaling of the round metric on S^3 induces a parametrized family of differential operators. We discuss such operators and also construct singular perturbations to geodesic flows.

The Hopf Fibration

Let $SU[2] = \left\{ \left(\begin{array}{cc} z & -\bar{w} \\ w & \bar{z} \end{array} \right) \middle| z, w \in \mathcal{C}, |z|^2 + |w|^2 = 1 \right\}$.

• There is a right action by $S^1 \sim U[1]$ on $SU[2]$:

$$\left(\begin{array}{c} z \\ w \end{array} \right) \mapsto \left(\begin{array}{c} e^{i\theta} z \\ e^{i\theta} w \end{array} \right).$$

The action is smooth, proper, effective. The orbits are the circles.

• There is a unique manifold structure on left cosets $SU[2]/U[1]$ s.t. the projection $\pi : SU[2] \to SU[2]/U[1]$ is smooth, surjective, a submersion (Tp is surjective), and a fibration with fibre S^1.

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "\text{diffeo}"] \arrow[d, "\pi"'] & U_i \times S^1 \arrow[dl, "\text{Proj}"] \\
U_i \subset SU[2]/U[1] \sim S^2 &
\end{tikzcd}$$

Hopf Fibration : $\begin{array}{c} S^3 \\ \pi \big\downarrow \\ S^2 \end{array}$

Identify $U(1)$ with S^1. Identify $SU(2)$ with S^3, the latter with quaternion multiplication in not a commutative group and identify.

- Let S^3 be given the standard Riemannian structure, that of sub-manifold of $\mathbf{R}^4$.
  There is a unique Riemmanian structure on S^2 such that π is a Riemannian submersion. Let $T_uS^3 = VT_u\pi \oplus HT_uS^3$, the orthogonal decomposition, $T_u\pi : HT_u\pi \to T_{\pi(u)}S^2$ is an isometry. With this metric, S^2 has constant sectional curvature $\frac{1}{4}$.

Hopf constructed this example to compute the homotopy groups of S^2, $H_3(S^2) = Z$.

Hopf Fibration

S^3 is a simply connected Lie group with structural constants $\{-2,-2,-2\}$. Let X_1, X_2, X_3 be an o.n.b. of its Lie algebra and by the same left the left invariant vector fields.

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ia & \beta \\ -\bar{\beta} & -ia \end{pmatrix}, \qquad a \in \mathbf{R}, \beta \in \mathcal{C} \right\}, \ \langle A, B \rangle = \frac{1}{2} trace(AB^*)$$

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

$X_1 \sim \frac{d}{d\theta}$ is the action field of S^1. Left translations: $X_i^*(u) = uX_i$.

- $$[X_2, X_3] = -2X_1, [X_3, X_1] = -2X_2, [X_1, X_2] = -2X_3.$$

  The horizontal distributions are not integrable.
- $\pi : S^3 \to S^2$ is a harmonic map. Totally geodesic fibres.
- The holonomy of fibers is S^1.

Brownian Motions and SDEs

A Brownian motion on a Riemannian manifold is a sample continuous strong Markov process with generator $\frac{1}{2}\Delta$. They can be constructed as solutions of SDE's, $dx_t = \sum_{i=1}^m \sigma_i(x_t) \circ db_t^i + \sigma_0(x_t)dt$, $m \geq d$. It takes $m \geq d$ vector fields to construct a Brownian motion. On S^3, a Lie group, can take $m = 3$.

On S^2, we need three vector fields and hence a 3-dimensional Brownian motion. [We'll see later, we can construct a Brownian motion with 2 vector fields.]

Some interesting SDE's:

- Solution to the SDE below is Brownian motion on S^3, projection of solution is a Brownian motion on S^2.

$$dx_t = X_1(x_t) \circ db_t^2 + X_2(x_t) \circ db_t^2 + X_3(x_t) \circ db_t^3.$$

- SDE's satisfying Hörmander conditions. The first two equations has an hypoelliptic Laplacian as a Markovian generator.
  $dx_t = X_2(x_t) \circ db_t^2 + X_3(x_t) \circ db_t^3$,

  $dx_t = X_1(x_t) \circ db_t^1 + X_2(x_t) \circ db_t^2$. $dx_t = X_1(x_t) \circ db_t^1 + X_2(x_t)dt$.

- J. Milnor [Mil76] for a discussion on classifications of three dimensional Lie groups.
- See Urbantke [Urb03] for Hopf fibration in physics.
- See J.-M. Bismut [Bis11, Bis08] on hypoelliptic Laplacian and Bott-Chern cohomologies, see [ABGR09] for hypoelliptic Laplacians on unimodular Lie groups, [BB09] for discussion on hypoelliptic operators on $SU(2)$.

Berger's Spheres

- Define left invariant Riemannian metric m_ϵ on S^3 by keeping X_1, X_2, X_3 orthogonal, but scaling the circle direction by ϵ: $|X_1|_{m_\epsilon} = \epsilon$. The spaces (S^3, m_ϵ) are Berger's spheres.
- Brownian motion on (S^3, m_ϵ) can be constructed as solution to:

$$dx_t^\epsilon = \frac{1}{\epsilon} X_1(x_t^\epsilon) \circ db_t^1 + X_2(x_t^\epsilon) \circ db_t^2 + X_3(x_t^\epsilon) \circ db_t^3.$$

- Hypoelliptic SDEs: $dx_t = X_2(x_t) \circ db_t^2 + X_3(x_t) \circ db_t^3$, $dx_t = \frac{1}{\sqrt{\epsilon}} X_1(x_t^\epsilon) \circ db_t^1 + X_2(x_t^\epsilon) \circ db_t^2$,

$$dx_t = \frac{1}{\sqrt{\epsilon}} X_1(x_t^\epsilon) \circ db_t + X_2(x_t^\epsilon) dt.$$

Collapsing of (S^3, m_ϵ)

- The diameter of the orbits of Berger's spheres is ϵ, which shrinks to zero. The injectivity radius of $(S^3, m_\epsilon) \to 0$ as $\epsilon \to 0$. The volume of S^3 shrinks to zero.
- Berger: (S^3, m_ϵ) converges to S^2 with constant sectional curvature $\frac{1}{4}$. The limit space is a lower dimensional manifold.
- Gromov-Cheeger, [CG86], would like to see collapsings of manifold sequences while keeping sectional curvatures uniformly bounded. For Berger's spheres,

$$K^\epsilon(X_1, X_2) = \epsilon^2, K^\epsilon(X_1, X_3) = \epsilon^2, K^\epsilon(X_2, X_3) = 4 - 3\epsilon.$$

Notion of convergence of manifolds

Strong Convergence of Riemannian manifolds $(M_n, g_n) \to (M, g)$: there are diffeomorphisms $\phi_n : M_n \to M$ such that $(\phi_n)^* g_n \to g$.

Let us look at an example for some intuition on the requirement 'bounded sectional curvature': Consider Riemannian manifold (M, g_t), where $g_t \in (\wedge^2 T_M)^*$ satisfies:

$$\dot{g}_t = -2Ric_{g_t}, \qquad g_0 \text{ smooth.}$$

R. S. Hamilton 82 proved short time existence and uniqueness. Let $g_t, t \in (0, T)$ be a maximal flow.

- For $t < T$, the metrics are equivalent:

  $$e^{-2Ct} g(0) \leq g(t) \leq e^{2Ct} g(0).$$

  The metrics do not 'collapse' as in Berger's spheres.
- The norm of the Riemannian curvature blows up as $t \uparrow T$ unless $T = \infty$ (Hamilton).

Gromov-Hausdorff Convergence

- Gromov Distance: Let A, B be sets in a metric space (X, d), denote by d_H the Hausdorff distance:

$$d_H(A, B) = \inf\{\epsilon > 0 : B \subset A_\epsilon, A \subset B_\epsilon\}.$$

  For any point x in A there is a point y in B such that $d(x, y) \leq \epsilon$.

- Gromov-Hausdorff distance between metric spaces: $(X_1, d_1), (X_2, d_2)$:

$$d_{GH}((X_1, d_1), (X_2, d_2)) = \inf_{(\phi_i:(X_i,d_i)\to(X,d))} \{d_H(\phi_1(X_1), \phi_2(X_2))\}.$$

  Here ϕ are isometric embeddings.

- Gromov-Hausdorff distance equals zero implies that the two spaces are isometric.
- The set of equivalent classes of compact metric spaces with diameter bounded above is compact.

Measured G-H convergence

If $(M_n, g_n) \to (M, g)$ how about the spectral of the Laplacian?

- K. Fukaya introduced Measured Gromov-Hausdorff convergence: consider the metric spaces (M_n, g_n, μ_n) where μ_n is a probability measure. $\lim_{n\to\infty}(M_n, g_n) = (M, g)$ if there is a family of measurable maps: $M_n \to M$ and positive numbers $\epsilon_n \to 0$ such that $|d(\psi_n(p), \psi_n(q)) - d(p, q)| < \epsilon_n$, $(\psi_n(M_n))_{\epsilon_n} = M$ and $(\psi_n)_*(\mu_n) \to \mu$ weakly.
- One a Riemannian manifold of finite volume, we take the measure to be the volumee measure normalised to 1. Kukaya[Fuk87]: Let $\mathcal{D}M(n, D)$ be the closure of the class of Riemannian manifolds whose sectional curvature K is bounded between -1 and 1 in the measured Gromov-Hausdorff distance. Let $\lambda_k(M)$ be the *kth*-eigenvalue of a manifold $M \in \mathcal{D}M(n, D)$. Then λ_k can be extended to a continuous function on $\mathcal{D}M(n, D) - \{(point, 1)\}$. For each element (M, μ), λ_k is the *kth* eigenvalue of an unbounded selfadjoint operator on $L^2(X, \mu)$.

G-H-Wasserstein convergence and the spectral-distance Convergence of Kasue-Kumura

- Villani's formulation for Measured Gromov-Hausdorff convergence: the Gromov-Hausdorff-Wasserstein distance

$$d_{GHW}(X_1, X_2) = \inf_{\phi_i X_i \to X \text{ isometric}} \{d_H(\phi_1(X_1, \phi_2(X_2)) + d_{W_p}((\phi_1)_*\mu_1, \phi_2)_*\mu_2)\}.$$

- Spectral-distance (Kasue-Kumura): compare heat kernels at time t (weighted by $e^{-(t+1/t)}$).Examples: Warped product spaces.
- Y. Ogura [Ogu01], Y. Ogura-S. Taniguchi [OT96]: Assume convergence of (M_n, g_n) in spectral distance, Let $X^n(t)$ be the Brownian motions and $\Phi_n : M_n \to M$ are suitable isometric maps. Let $\{t^n\} \subset [0, 1]$, with $\Delta t_n \to 0$ Let $\tilde{X}^n$ the piecewise interpolations. Then $\Phi_n(\tilde{X}^n(t^n))$ is tight.

The Spectrum on (S^3, m_ϵ)

$$
\begin{aligned}
S^3 &: \lambda_k = k(k+2) = 0, 3, 8, \dots \\
S^2 &: \lambda_k = k(k+1) = 0, 2, 6, 12, \dots \\
S^1 &: \lambda_k = k^2 = 0, 1^2, 4, 9, \dots .
\end{aligned}
$$

$\Delta^\epsilon = \frac{1}{\epsilon}\mathcal{L}_{X_1^*}\mathcal{L}_{X_1^*} + \mathcal{L}_{X_2^*}\mathcal{L}_{X_2^*} + \mathcal{L}_{X_3^*}\mathcal{L}_{X_3^*} = \Delta^\epsilon_{S^1} + \Delta_h$. Facts:

- Δ_{S^3}, Δ_h, $\Delta^\epsilon_{S^1}$ commute.
  The fibre is totally geodesic implies that Δ_{S^1} commutes with any basic vector fields (horizontal lifts of vector fields below) if and only if the fibre is totally geodesic. (L. Bérard-Bergery and J.-P. Bourguignon[BBB82], O'Neill [O'N67].
- $\Delta(f \circ \pi) = \Delta_{S^2} f \circ \pi$. c.f. [ELJL99] for intertwined Laplacians.
- $\lambda_1(\Delta^\epsilon) \to \lambda_1(S^2(\frac{1}{2}))$,

$$
\lambda_1(\Delta^\epsilon) = \min\{8 + 0, 2 + \frac{1}{\epsilon}1^2\} = 8, \text{ when } \epsilon^2 < \frac{1}{6}.
$$

S. Tanno [Tan80][BBB82].

Convergence related to the Brownian Motions

What can we say about the Brownian motion u_t^ϵ on (S^3, m_ϵ), from u_0, as $\epsilon \to 0$? They solve the SDE

$$du_t^\epsilon = \frac{1}{\sqrt{\epsilon}} X_1^*(u_t^\epsilon) \circ db_t^1 + X_2^*(u_t^\epsilon) \circ db_t^2 + X_3^*(u_t^\epsilon) \circ db_t^3.$$

This provides a naturally occurring singularly perturbed SDE. We first analyse the convergence a the level of the processes.

Introduce Tools and Notation

The orthogonal splitting of the tangent space $T_u S^3 = H_u TS^3 \oplus \ker(T_u\pi)$ induces a S^1-invariant connection on S^3. If σ is a semi-martingale on S^3 (in general not C^1 in t), denote by $\tilde\sigma$ its horizontal lift, c.f. [ELJL10].

This is well known for the horizontal lifts to the orthonormal frames of semi-martingales: related to the stochastic parallel transport (K. Itô) and to the stochastic development map (J. Eells-D. Elworthy)).

Let $x_t = \pi(u_t^\epsilon)$ and $\tilde x_t^\epsilon$ its horizontal lift.

Proposition ([Li12a])

1) x_t^ϵ is BM on S^2, 2) $\tilde x_t^\epsilon$ is a diffusion with generator the hypoelliptic Laplacian $\frac{1}{2}\Delta_H$:

$$d\tilde x_t^\epsilon = \tilde x_t^\epsilon \mathrm{Ad}(a_t^\epsilon) X_2 \circ db_t^2 + \tilde x_t^\epsilon \mathrm{Ad}(a_t^\epsilon) X_3 \circ db_t^3$$

$a_t^\epsilon \in S^1$ satisfies $da_t^\epsilon = \frac{1}{\sqrt{\epsilon}} a_t^\epsilon X_1 \circ db_t^1$.

Note that *span*$\{X_2, X_3\}$ is $\mathrm{Ad}(S^1)$ invariant and the metric is Ad-invariant.

Generating hypoelliptic diffusion with one vector field

In the proposition,

$$d\tilde{x}_t^\epsilon = \tilde{x}_t^\epsilon \mathrm{Ad}(a_t^\epsilon)X_2 \circ db_t^2 + \tilde{x}_t^\epsilon \mathrm{Ad}(a_t^\epsilon)X_3 \circ db_t^3, \; da_t^\epsilon = \frac{1}{\sqrt{\epsilon}} a_t^\epsilon X_1 \circ db_t^1$$

the Ad-invariant property of the horizontal distribution together with an Ad-invariant metric meant that for all ϵ, x_t^ϵ has the same distribution.

An interesting singularly perturbed SDE:

$$du_t^\epsilon = u_t^\epsilon Y_0 g_t^\epsilon dt + \frac{1}{\sqrt{\epsilon}} u_t^\epsilon X_1 \circ db_t, \qquad dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} g_t^\epsilon X_1 \circ db_t.$$

If $g \in S^1$, $Y_0 \in span\{X_2, X_3\}$. Then $Y_0 g \in span\{X_2, X_3\}$.

Formulation of the Theorem

Recall $Y_0 \in span\{X_2, X_3\}$.

Theorem ([Li12b])

Take $u_0 \in SU(2)$. Let $(u_t^\epsilon, g_t^\epsilon)$ be the solution to the following SDE on $SU(2) \times U(1)$, with $u_0^\epsilon = u_0$ and $g_0^\epsilon = 1$,

$$du_t^\epsilon = (Y_0 g_t^\epsilon)^*(u_t^\epsilon)dt + \frac{1}{\sqrt{\epsilon}} X_1^*(u_t^\epsilon) \circ db_t, \qquad dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} X_1^L(g_t^\epsilon) \circ db_t.$$

Let $x_t^\epsilon = \pi(u_t^\epsilon)$ and $\tilde{x}_t^\epsilon$ its horizontal lift. Then $\tilde{x}_t^\epsilon$ converges in probability to the hypoelliptic diffusion with generator $\bar{\mathcal{L}} F = \frac{1}{2}|Y_0|^2 \Delta_H$. If Y_0 is a unit vector, x_t^ϵ converges in law to the Brownian motion on S^2.

Multi-scale analysis

We compute the generator $\mathcal{L}^\epsilon$ of $(u_t^\epsilon, g_t^\epsilon)$. Let $F : SU(2) \times U(1) \to \mathbf{R}$ be C^∞, constant in the second variable. Let $Z = (Y_0 g)^*$, $Z_1^g = \frac{1}{2}(Y_0 g X_1)^*$, $\mathcal{L}_0 = \frac{1}{2}\mathcal{L}_{X_1^*}\mathcal{L}_{X_1^*}$.

$$\mathcal{L}^\epsilon(g)F(u) = \frac{1}{\epsilon}\mathcal{L}_0 F(u) + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_{Z_1^g} F(u) + \mathcal{L}_Z F.$$

The middle term comes from interaction between u and g. Let F be solution to $\frac{\partial F}{\partial t} = \mathcal{L}^\epsilon(g)F$. Expand F in ϵ,

$$F = F_0 + \sqrt{\epsilon}F_1 + \epsilon F_2 + o(\epsilon).$$

Multi scale analysis

$$\frac{\partial F}{\partial t} = \mathcal{L}^{\epsilon}(g)F, \quad F = F_0 + \sqrt{\epsilon}F_1 + \epsilon F_2 + o(\epsilon)$$

Expand the equation in $\sqrt{\epsilon}$,
$\dot{F}_0 + \sqrt{\epsilon}\dot{F}_1 + \epsilon \dot{F}_2 + o(\epsilon) = (\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_{Z_1^g} + \mathcal{L}_Z)(F_0 + \sqrt{\epsilon}F_1 + \epsilon F_2 + o(\epsilon))$.

$$\mathcal{L}_0 F_0 = 0 \qquad \Longrightarrow F_0 \text{ does not depend on the } \theta\text{-variable}$$

$$\mathcal{L}_{Z_1} F_0 = -\mathcal{L}_0 F_1 \qquad \Longrightarrow F_1 = \mathcal{L}_0^{-1}(\mathcal{L}_{Z_1} F_0).$$

$$\dot{F}_0 = \mathcal{L}_Z F_0 + L_{Z_1} F_1 + \mathcal{L}_0 F_2. \qquad \int \mathcal{L}_0 F_2 d\theta = 0$$

$\mathcal{L}_0 = \frac{1}{2}\mathcal{L}_{X_1^*}\mathcal{L}_{X_1^*}$, Integrate lat equation with respect to $d\theta$,
$\int \dot{F}_0 = \int \mathcal{L}_Z F_0 + \int L_{Z_1} F_1 + \int \mathcal{L}_0 F_2$. Define $\bar{F}_0 = \int F_0$.

$$\frac{d}{dt}\bar{F}_0 = \mathcal{L}_Z \bar{F}_0 + L_{Z_1}\bar{F}_1 = \mathcal{L}_Z \bar{F}_0 + L_{Z_1}\mathcal{L}_0^{-1}(L_{Z_1}\bar{F}_0),$$

We have a second order differential operator.

Idea of Proof

The multi scale analysis given earlier confirms the scaling is correct. However it does not seem to help us to prove the theorem.

- Since $\tilde{x}_t^\epsilon$ and u_t^ϵ live in the same fibre, there is an element $a_t^\epsilon \in S^1$ such that $u_t^\epsilon = \tilde{x}_t^\epsilon a_t^\epsilon$. We compute a_t^ϵ and prove that
$$d\tilde{x}_t^\epsilon = (g_t^\epsilon Y_0)^*(\tilde{x}_t^\epsilon)dt.$$
- We prove the tightness of relevant measures for the weak convergence.
- Let $F : S^3 \to \mathbf{R}$ be any smooth function. Since $Y_0 \in span\{X_2, X_3\}$,
$$F(\tilde{x}_t^\epsilon) = F(u_0) + \sum_{j=2}^{3} \int_0^t dF(\tilde{x}_s^\epsilon X_j)\langle X_j, g_s^\epsilon Y_0\rangle ds.$$
- Note the right hand side is bounded variation term. However we seek an approximate 'semi-martingale' decomposition, of the form, 'martingale +drift+ $o(\epsilon)$-terms. To the drift term we may apply the ergodic theorem.
- Use Stroock-Varadhan's martingale method to identify the limits.

Bensoussan-Lions-Papanicolaou [BLP76]

Collapsing with bounded Ricci curvature? Convergence with bounded derivative flows?

Question: How do we deal with the concept of sectional curvature not blowing up?
Let $P_t^\epsilon f(u_0) = \mathbf{E} f(u_t^\epsilon)$. Then $d(P_t^\epsilon) f(v) = \mathbf{E} df(v_t^\epsilon)$ where v_t^ϵ is the derivative flow with initial value v_0.

Observation: $|\nabla P_t f|^2 \le e^{2Kt} |\nabla f|^2$ if and only if $Ric \ge K$.
The if part is simple: $d(P_t f)(v) = \mathbf{E} df(W_t)$ where W_t solves $\frac{DW_t}{dt} = -\frac{1}{2} Ric^{\#}(W_t)$ and $W_0 = v$.
We propose to replace the' boundedness in sectional curvature' by (1) $|\nabla P_t^\epsilon f|_\epsilon^2 \le e^{2Kt} |\nabla f|_\epsilon^2$ or (2) the derivative flows $|\mathbf{E}\{v_t^\epsilon | \mathcal{F}_t^\epsilon\}|_\epsilon \le C |v_0|_\epsilon$.

Some Estimates

Using the Levi-Civita connection we may compute explicitly: $\nabla_{X_i} X_j$, e.g. $\nabla_{X_2} X_1 = \epsilon X_3$, $\nabla_{X_3} X_1 = -\epsilon X_2$,... Hence

$$DV_t = \frac{1}{\sqrt{\epsilon}} \nabla_{v_t} X_1 \circ db_t^1 + \sum_{i=2}^{3} \nabla_{v_t} X_i \circ db_t^i.$$

Since $/\!\!/_t$ is an isometry, we see that $|V_t^\epsilon|$ is nicely bounded. This can also follow from the following computation:

$$Ric^\epsilon(X_2) = 4 - 2\epsilon, Ric^\epsilon(X_1, X_1) = 2\epsilon^2.$$

Proposition

With respect to the round metric, $|v_t^\epsilon|_1$ is a constant in t. The eigenvalues of $(u_t^\epsilon)^{-1} v_t^\epsilon$ is constant in t.

Remark: Suppose that $v_0 \neq 0$. The derivative flow of the equation for $\tilde{x}_t^\epsilon$ satisfies that $(\tilde{x}_t^\epsilon)^{-1} v_t^\epsilon$ is hypoelliptic on the unit sphere $\{h \in \mathfrak{su}(2) : |h| = |v_0|\}$.

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