The expected number of fixed points of stochastic flows and Kusuoka's McKean-Singer formula.

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Based on: K.D.E & AbdulRahman Al-Hussein " Infinite Dimensional Degree Theory & Stochastic Analysis" JFPTA (2010)

Following: S.Kusuoka "Degree Theory in certain Wiener Riemannian manifolds" LNM 1322 (1988)

Set up

M a compact connected smooth manifold, dimension n.

$X^1, ..., X^m, A$ smooth vector fields on M.

$\mathcal{L}_A$ Lie differentiation in the direction A.

Lie differentiation on differential forms

For ϕ a smooth q-form, $\phi \in \Gamma \wedge^q T^*M$,

$\mathcal{L}_A(\phi)$ is the q-form given by:

$$\mathcal{L}_A(\phi) = \frac{d}{dt}(\eta_t^A)^*(\phi)|_{t=0},$$

where $\{\eta_t^A\}_t$ is the flow of A.

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$\mathcal{A} = \frac{1}{2}\sum_{j=1}^{m} \mathcal{L}_{X^j}\mathcal{L}_{X^j} + \mathcal{L}_A$ on functions.

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$\mathcal{A}^q = \frac{1}{2}\sum_{j=1}^{m} \mathcal{L}_{X^j}\mathcal{L}_{X^j} + \mathcal{L}_A$ on q-forms.

Use $\mathcal{A}^*$ for the operator on the complex of forms.

Euler characteristic, Lefschetz number

χ_M is:

- alternating sum of Betti numbers of M, which equals:
- algebraic number of fixed points of a smooth map $\eta : M \to M$, homotopic to the identity, when fixed points are non-degenerate:

$$\chi_M = \sum_{\{x:\eta(x)=x\}} sgn\{\det(I - T_x\eta)\}.$$

To discuss: Generalised McKean-Singer formula

Let $\{P_t^*\}_{t\geq 0}$ be the semi-group on forms generated by $\mathcal{A}^*$.

$$\chi_M = STr(P_t^*) \quad any \;\; t > 0.$$

Supertraces

$$
\begin{aligned}
STr(P_t^*) : &= \sum_0^n (-1)^q Tr\ P_t^q \\
&= \sum_0^n (-1)^q \int_M trace\ k_t^q(x,x)dx \\
&= \int_M Str\ k_t^*(x,x)dx
\end{aligned}
$$

for fundamental solution $k_t^q(x,y) : \wedge^q T_x^* M \longrightarrow \wedge^q T_y^* M$.

To discuss: Generalised McKean-Singer formula

Let $\{P_t^*\}_{t\geq 0}$ be the semi-group on forms generated by $\mathcal{A}^*$.

$$\chi_M = STr(P_t^*) \quad any \;\; t > 0.$$

$\mathcal{A}$ elliptic.

"History"

McKean -Singer 1967 for $\mathcal{A}^* = -\frac{1}{2}\Delta = -\frac{1}{2}(d+d^*)$, usual Hodge-Kodaira Laplacians. "Supersymmetric" proofs by Patodi, Getzler,....

Method uses χ_M as the index of the elliptic operator $(d + d*)$ from odd forms to even forms, with eigenfunction counting.

Method does not work for general elliptic $\mathcal{A}^*$???

(Though $\mathcal{A}^* = d\hat{\delta} + \hat{\delta} d$, El-LeJan-Li LNM 1720.)

To discuss: Rice type formula

Let $\xi_t : \Omega \times M \to M$ be solution flow of SDE on M:

$$dx_t = \sum_j X^j(x_t) \circ dB_t^j + A(x_t)dt$$

and $T\xi_t : TM \to TM$ the derivative flow.

$$\mathbf{E}\sharp\{x : \xi_t(x) = x\} = \int_M \mathbf{E}\Big\{|det\,(I - T_x\xi_t)|\,\Big|\xi_t(x) = x\Big\}k_t^0(x,x)dx$$

Decomposed formulation

$$\begin{aligned}\mathbf{E}\sharp\{x : \xi_t(x) &= x\} = \int_M \mathbf{E}\Big\{(|det\ (I - T_x\xi_t)|)\ \Big|\xi_t(x) = x\Big\} k_t^0(x,x)dx \\ &= \int_M \int_{Diff_x M} |det\ (I - T_x\xi)|\ k_t^0(x,x) d\nu_t^x(\xi) dx\end{aligned}$$

where ν_t^x is the conditional law of ξ_t on the space of diffeomorphisms $Diff_x$ of M which fix x.

"History"

Rice Formulae go back to Rice, 1944/45: for Gaussian random $\Theta : \mathbf{R} \to \mathbf{R}$, stationary, variance 1, $\mathbf{E}\sharp\{x \in I : \Theta(x) = u\} = const.\, e^{-u^2/2}|I|$

More generally, (Azaïs & Wschebor) : if $x \to \Theta(x)$ is C^1 , non-degenerate, Gaussian or...,

$$\mathbf{E}\sharp\{x \in I : \Theta(x) = u\} = \int_I \mathbf{E}\big\{|\Theta'(x)|\big|\Theta(x) = u\big\}p(x,u)dx$$

$p(x,u)du$ law of $\Theta(x)$, assumed cts.

reference

See "Level Sets and Extrema of Random Processes and Fields" Azaïs& Wschebor, Wiley 2009.

Link: Path integral version

Path integral for P_t^*

$$P_t^q \phi = \mathbf{E}(\xi_t)^*(\phi) = \mathbf{E}\phi \circ \wedge^q (T\xi_t).$$

Kusuoka's path integral formulation

Generalised M-S is:

$$\chi_M = \int_M \int_{\{\xi_t(x)=x\}} \det(I - T_x\xi_t)\, d\nu_x(\xi_.)\, k_t^0(x,x)\, dx$$

$$= \int_M \int_{\{\xi_t(x)=x\}} \sum_{q=1}^{n} (-1)^q \mathrm{tr}(\wedge^q(T_x\xi_t))\ d\nu_x(\xi_.) k_t^0(x,x)\, dx$$

Rice & McKean Singer

$$\mathbf{E}\sharp\{x : \xi_t(x) = x\} = \int_M \int_{Diff_x M} |det\,(I - T_x\xi)|\, d\nu_x(\xi)\, k_t^0(x,x)\, dx$$

$$\chi_M = \int_M \int_{Diff_x M} \det(I - T_x \xi_t)\; d\nu_x(\xi_.)\; k_t^0(x,x)\; dx$$

Comments and Questions on McKean-Singer

It was the main tool for heat equation proofs of the Gauss-Bonnet-Chern theorem /Atiyah -Singer Index Theorems. Take the limit of the supertrace as $t \to 0$. Using Weitzenbock formula $\mathcal{A}^q = trace\, \nabla.\nabla(\phi) + \mathcal{L}_A - \mathcal{R}^q$ this limit gives a form in terms of the curvature which is the Euler form. Can be done using probabilistic techniques eg as Ikeda & Watanabe. This is when $\mathcal{A}^*$ is the Hodge-Kodaira operator.

For general elliptic $\mathcal{A}$ the same should work. For hypoelliptic $\mathcal{A}$?

Example: Gradient flow on S^n. Picture.

<u>Fig. 2:</u> A sample flow, shown with 10 distinct initial points, of the equation $d\theta_t = (\cos \theta_t)dB_t^1 + (\sin \theta_t)dB_t^2$. The solution from each point is a Brownian motion on S^1, (with angle as coordinate, time drawn radially outwards).
{*Computer simulation by P. Townsend and D. Williams*}.

Fig. 2: A sample flow, shown with 10 distinct initial points, of the equation $d\theta_t = (\cos \theta_t) dB_t^1 + (\sin \theta_t) dB_t^2$. The solution from each point is a Brownian motion on S^1, (with angle as coordinate, time drawn radially outwards).

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(See cover of Rogers, L. C. G.; Williams, David *Diffusions, Markov processes, and martingales. Vol. 2. Itô calculus*.)

Gradient flow on S^n

$$T_{x_0}\xi_t = e^{\beta_t - \frac{1}{2}nt} /\!/_t$$

for $\beta_\cdot$ a BM($\mathbf{R}$) independent of $\xi_\cdot(x_0)$.

Case n=1

$$T_{x_0}\xi_t = e^{\beta_t - \frac{1}{2}t} /\!\!/_t$$

$$\begin{aligned} \mathbf{E}\sharp\{x : \xi_t(x) = x\} &= \mathbf{E}\{|1 - e^{\beta_t - \frac{1}{2}t}|\} \\ &\to 2 \quad \text{as } t \to 0 \end{aligned}$$

Case n>1

$$T_{x_0}\xi_t = e^{\beta_t - \frac{1}{2}nt} /\!/_t$$

Need to estimate holonomy $\mathbf{E}\{\wedge^q /\!/_t\}$ over a Brownian bridge. By symmetry it has the form $\mathbf{E}\{/\!/_t\} = c_t^q \wedge^q (Id)$ where $c_t^q \in \mathbf{R}$. From eigenvalue formulae $c_t^q \to 0$ exponentially , $q \neq 0, 1$. Again:

$$\mathbf{E}\sharp\{x : \xi_t(x) = x\} \to 2 \quad \text{as } t \to 0$$

Degree for maps of manifolds

$$F : P \to Q$$

continuous, proper, P,Q both n-dimensional smooth, connected, oriented.

Degree: Deg(F) $\in \mathbb{Z}$

$= \sum_{\{x:F(x)=z)\}} sgn(\det DF(x))$ for F a C^1 map and z a *regular value* of F.

Sard's Theorem

For F as above the set $\mathcal{R}eg(F)$ of regular values is dense, and its complement $\mathcal{C}rit(F)$ has measure zero.

Note: properness not needed, proper implies $\mathcal{R}eg(F)$ open.

Leray-Schauder degree: E a Banach space

$$F : \overline{U} \longrightarrow E$$

$$F(x) = x + u(x)$$

u continuous, compact, $U \subset E$, open , $z \in E - F[\partial U]$.

Degree Deg$(F, U, z) =$"algebraic number of points in $F^{-1}(z)$".

Fredholm operators of index zero

$A \in \mathbf{L}(G; E)$ is a Φ_0-operator if $A = S + \alpha$ where $S \in \mathbf{L}(G; E)$ is a linear isomorphism and α is compact.

If $H_1 \subset H_2 \subset ... \subset E$ has each H_n finite dimensional and $\bigcup_n H_n$ is dense in E we can choose α with range in H_n some n.

Fredholm maps $F : P \to Q$; P,Q Banach manifolds

A C^1 map $F : P \to Q$ is Φ_0 if each derivative

$$T_x F : T_x P \to T_{F(x)} Q$$

is Φ_0.

Smale-Sard: The regular values of a Φ_0-map are dense

For proper Φ_0-maps Smale defined a mod 2 degree.

NB: All manifolds asssumed separable metrisable, usually connected.

Fredholm & Layer structures

Given a Φ_0 map $F : P \to E$ there exists an atlas $\{(U_j, \phi_j)\}_{j=1}^{\infty}$ modelled on E such that locally

$$F \circ \phi_j^{-1}(x) = x + \alpha_j(x)$$

for $\alpha(x) \in H_n$ some fixed $n = n_j$, i.e. a layer map.

Consequently each change of co-ordinates $\phi_i \circ \phi_j^{-1}$ is a layer map. A layer atlas. May be orientable. Elworthy-Tromba

Oriented degree

For a proper Φ_0 map $F : P \to E$ can define $Deg(F) \in \mathbb{Z}$ given an orientation.

$$Deg(F) := \sum_{\{x : F(x) = z\}} sgn(\det T_x F)$$

for z a *regular value* of F. Also for suitable $F : P \to Q$.

Elworthy, Tromba, Eells, Mukherjea, Borisovich, Ratiner, Zvyagin, Fitzpatrick, Pejsachowicz, Benevieri, Furi, S.Wang ,....

References

See:

- Kokarev, Gerasim; Kuksin, Sergei: Quasi-linear elliptic differential equations for mappings of manifolds. II. Ann. Global Anal. Geom. 31 (2007), no. 1, 59–113.
- Weitsman, Jonathan: Measures on Banach manifolds and supersymmetric quantum field theory. Comm. Math. Phys. 277 (2008), no. 1, 101–125.

Example from Kokarev-Kuksin

M and N finite dimensional, Riemannian,
$\mathcal{F} = \mathcal{F}(M, N)$ a space of maps from M to N;
E a suitable Banach space of "non-autonomous" vector fields on N.

$$P := \{(f, v) \in \mathcal{F} \times E : \triangle(f) = v(x, f(x))\}$$

Take the projection $F : P \to E$. In certain cases it is a proper Φ_0-map, giving an orientable structure.

Diffeomorphism group example

M compact; $Diff^{(0)}(M)$ the identity component of its diffeomorphism group.

$$P := \{(x,\theta) \in M \times Diff^{(0)}(M) : \theta(x) = x\}.$$

Take the projection $F : P \to Diff^{(0)}(M)$. It is proper Φ_0 and Deg(F) = $\sum_{\{x:\theta(x)=x\}} sgn \det(I - T_x\theta)$. = fixed point index of θ = Euler characteristic $\chi(M)$.

USING STOCHASTIC ANALYSIS WE CAN FIND INTEGRAL FORMULAE:

General pull back measures

For a measure μ on Q and $F : P \to Q$ a Φ_0-map with $\mu\left(\mathit{Crit}F\right) = 0$:

define $F^*(\mu)$ on P by

1. $F^*(\mu)$(critical points of F) $= 0$

2. If $U \subset P$ is open and F maps U diffeomorphically onto an open V in Q, then F is measure preserving from U to V.

General degree formula

For a measure μ on Q and $F : P \to Q$ a proper Φ_0-map with $\mu\,(\mathit{Crit}F) = 0$ and an orientation:

$$\int_P \lambda(F(x))\mathrm{sgn}(T_x F)\; d(F^*(\mu))(x) = \mathrm{Deg}(F) \int_Q \lambda\; d\mu$$

provided $\lambda \circ F : P \to \mathbf{R}$ is integrable

Area Formula, Jacobi's formula, Banach's formula

Let $F : P \to Q$ be a Φ_0-map and μ a locally finite Borel measure on P for which the critical values of F have measure zero. Suppose $f : P \to \mathbf{R}$ is measurable. Then

$$\int_P f(x)\, dF^*(\mu)(x) = \int_Q \sum_{\{x:F(x)=y\}} f(x)\, d\mu(y).$$

Both formulae

$$\int_P \lambda(F(x))\mathrm{sgn}(T_xF)\ d(F^*(\mu))(x) = \mathrm{Deg}(F)\int_Q \lambda\ d\mu$$

$$\int_P f(x)\ dF^*(\mu)(x) = \int_Q \sum_{\{x:F(x)=y\}} f(x)\ d\mu(y).$$

Wiener manifolds

Pull backs of non-degenerate Gaussian measures are locally absolutely continuous with respect to Gaussian measures when represented in the special layer atlas induced by the Φ_0-map.

This uses the Gross-Kuo-(Ramer-Kusuoka) Theorem.

Measure Theoretic Sard's Theorem for Φ_0-maps

For $F : P \to E$ a Φ_0-map the critical values of F have (non-degenerate) Gaussian measure 0.

This uses the Gross-Kuo-(Ramer-Kusuoka) Theorem. Noted by Eells-Elworthy (1971).

Paths on Diff (M), (after Kusuoka)

M compact, connected.
As before F is the projection:

$$P := \{(x, \xi) \in M \times C_{id}Diff(M) : \xi_T(x) = x\} \to C_{id}Diff(M).$$

It is proper Φ_0, and Deg$(F) = \chi(M)$
Kusouka obtained an integral formula, in a similar situation, related to the McKean-Singer formula.
PROBLEM: what can we get from above?

First define μ on $C_{id}Diff(M)$:

Paths on $Diff(M)$

Take SDE $dx_t = X(x_t) \circ dB_t + A(x_t)dt$ on M for $B_.$ canonical BM on $\mathbf{R}^m$.

Flow SDE on $Diff(M)$:

$$d\xi_t = X(\xi_t(-)) \circ dB_t + A(\xi_t(-))dt$$

giving Itô map $\mathcal{I} : C_0(\mathbf{R}^m) \to C_{id}Diff(M)$.

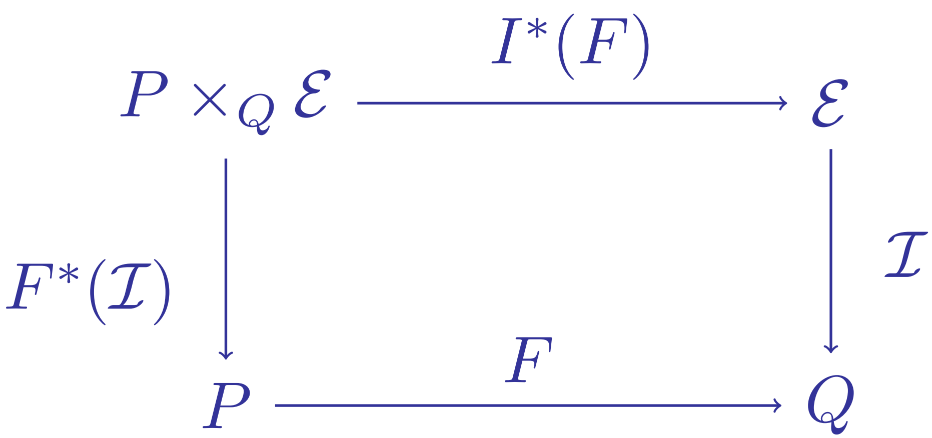
Set $\mu = \mathcal{I}_*(\mathbf{P})$.

A problem

For $F : P \to C_{id}Diff(M)$ is $\mu(Crit(F)) = 0$?

μ is degenerate in general

Sard's Theorem holds for $\mathcal{I}$ transversal to F



- The inverse image under $\mathcal{I}$ of the critical values of F is the set of critical values of $\mathcal{I}^*(F)$.

- If F is Φ_0 then so is $\mathcal{I}^*(F)$.

Approximate

On $\mathbf{R}^m$ take OU position process $\{b_t^\beta : 0 \le t \le T\}$ with $\dot{b}_t^\beta = v_t^\beta$ and

$$dv_t^\beta = -\beta v_t^\beta + 2\beta dB_t$$

with $b_0^\beta = v_0^\beta = 0$ and $\beta > 0$. It has C^1 paths and for it $\mathcal{I}$ is well defined and smooth. As $\beta \to \infty$ so $\mathcal{I}(b^\beta)$ tends to $\xi_.$ on $Diff(M)$, i.e. to our stochastic flow on M. {R.Dowell,1980, see also Bismut & Lebeau 2008

The degree of F is defined independently of any measure.

$$Deg\ F = \int_{(C^1_{id}Diff(M)\times M)\cap P} \mathrm{sgn}\,\det(TF)\ d(F^*(\mu^\beta))$$

Decomposition lemma

For $i : H \to E$ an AWS measure γ, M an n-dimensional Riemannian manifold. If $\phi : U \to M$ is a C^1 submersion from an open U of E, then $\phi_*(\gamma)$ has a continuous density with respect to the volume measure λ^M of M. The fibre measures γ_x^ϕ are given by continuous Wiener densities for the strong layer structures given on the submanifolds $U_x^\phi := \phi^{-1}(x)$ of E.

Let $\psi : E \to M$ be another C^1 submersion with a normalised decomposition, fibre measures γ_x^ψ, base measure given by $\rho^\psi(x)d\lambda^M(x)$. Choose the decomposition of γ with respect to ϕ to have base measure $\rho^\psi\lambda^M$ and let $\gamma_x^{\phi,\rho}$ denote the corresponding fibre measures. Suppose $x_0 \in M$ has $\psi^{-1}(x_0) \cap U = \phi^{-1}(x_0)$. Then on $U^\phi_{x_0}$ we have

$$\mathbf{P}1^{\phi,\rho}_{x_0} = (\det \mathcal{M}^\phi(w_0))^{-\frac{1}{2}}(\det \mathcal{M}^\psi(w_0))^{\frac{1}{2}}\,\mathbf{P}1^{\psi,\rho}_{x_0},$$

where $\mathcal{M}^\phi : U \to \mathbf{R}$ and $\mathcal{M}^\psi : E \to \mathbf{R}$ are the Malliavin covariance matrices of ϕ and ψ defined by

$$\mathcal{M}^\phi(w) = (T_w^H\phi)(T_w^H\phi)^*,$$

where $T_w^H\phi : H \to T_{x_0}M$ is the restriction of the derivative of ϕ at w to H.

Gaussian result

Let $p_t^\beta(x,y)dy$ be law of $\xi_t(x)$ on M under μ^β. Then, using Berezin's formula

$$
\begin{aligned}
Deg\,F &= \int_M \int_{\{\xi_t^\beta(x)=x\}} \det(I - T_x\xi_t^\beta)\; d\nu_t^x(\xi_.)\; p_t^\beta(x,x)\,dx \\
&= \int_M \int_{\{\xi_t^\beta(x)=x\}} \sum_{q=1}^{n} (-1)^q \mathrm{tr}(\wedge^q(T_x\xi_t^\beta))\;\; d\nu_t^x(\xi_.) p_t^\beta(x,x)\,dx \\
&= \;\; StrP_t^{\beta,*} \qquad\qquad \text{for all } t>0 \text{ and } \beta>0,
\end{aligned}
$$

agreeing with McKean & Singer in the limit as $\beta \to \infty$.

Conclusions, Questions

- There are interesting classes of examples of proper Fredholm maps; how about K&K's examples?

- Gaussian integration theory may be applied, to give integral formulae; Sometimes it really does not matter what Gaussian measures you use

- Why analytically does the generalised Singer - McKean formula hold?

- The geometric analysis of measures such as μ needs further development.
- How about the hypoelliptic case?
- Do such Rice formulae give interesting information about the long time behaviour of the flow?
- Nielsen numbers?