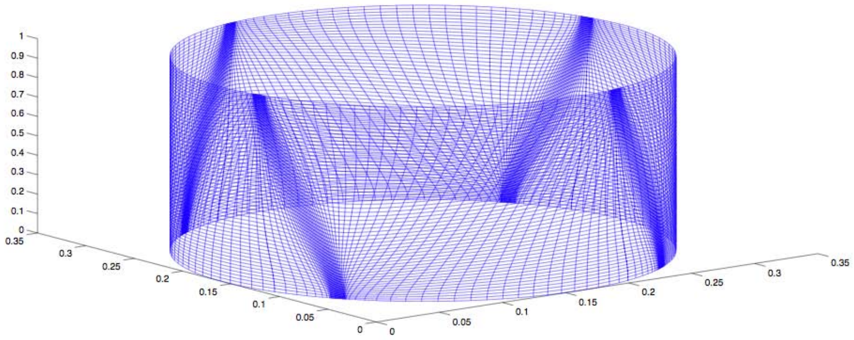
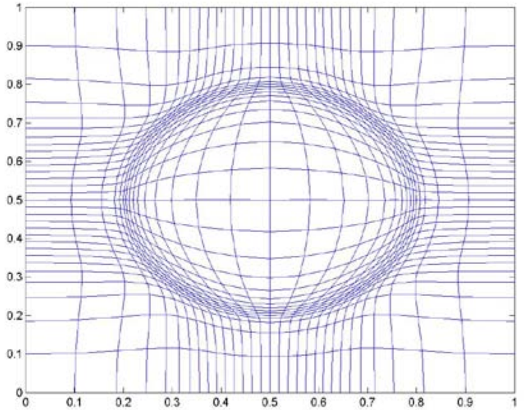
Geometry Based Numerical Methods for Weather Forecasting

Chris Budd (Bath),
Emily Walsh (Bath,SFU), Phil Browne (Bath)





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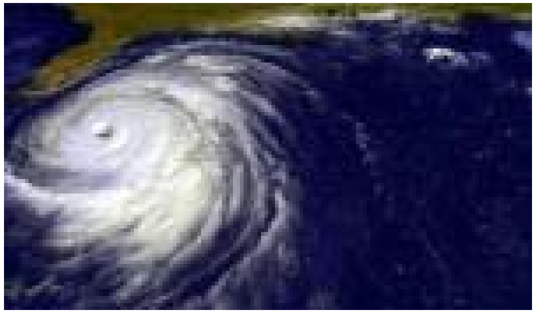
Weather forecasting is an important application of mathematics



Forecasts are typically made solve this using a numerical method based on a computational mesh

Often need to locally refine a mesh to capture small scale features

(i) For accurate numerical computation eg. Storms

(ii) For accurate assimilation of observed data

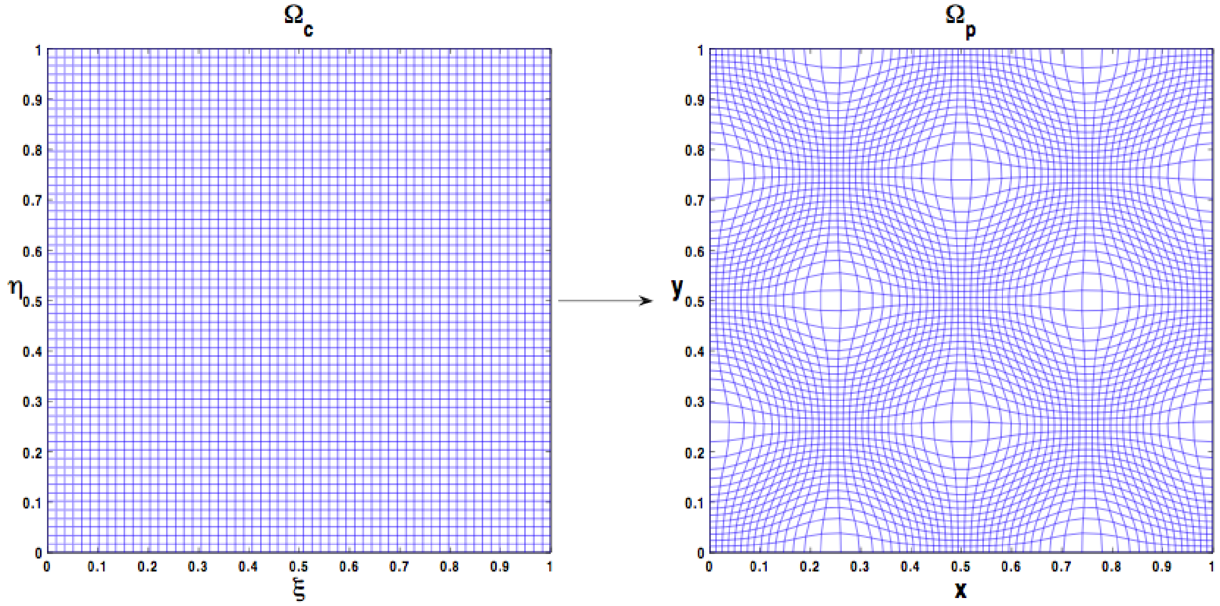
Talk will describe a method for doing this based upon geometrical ideas: optimal transport

Geometrical strategy

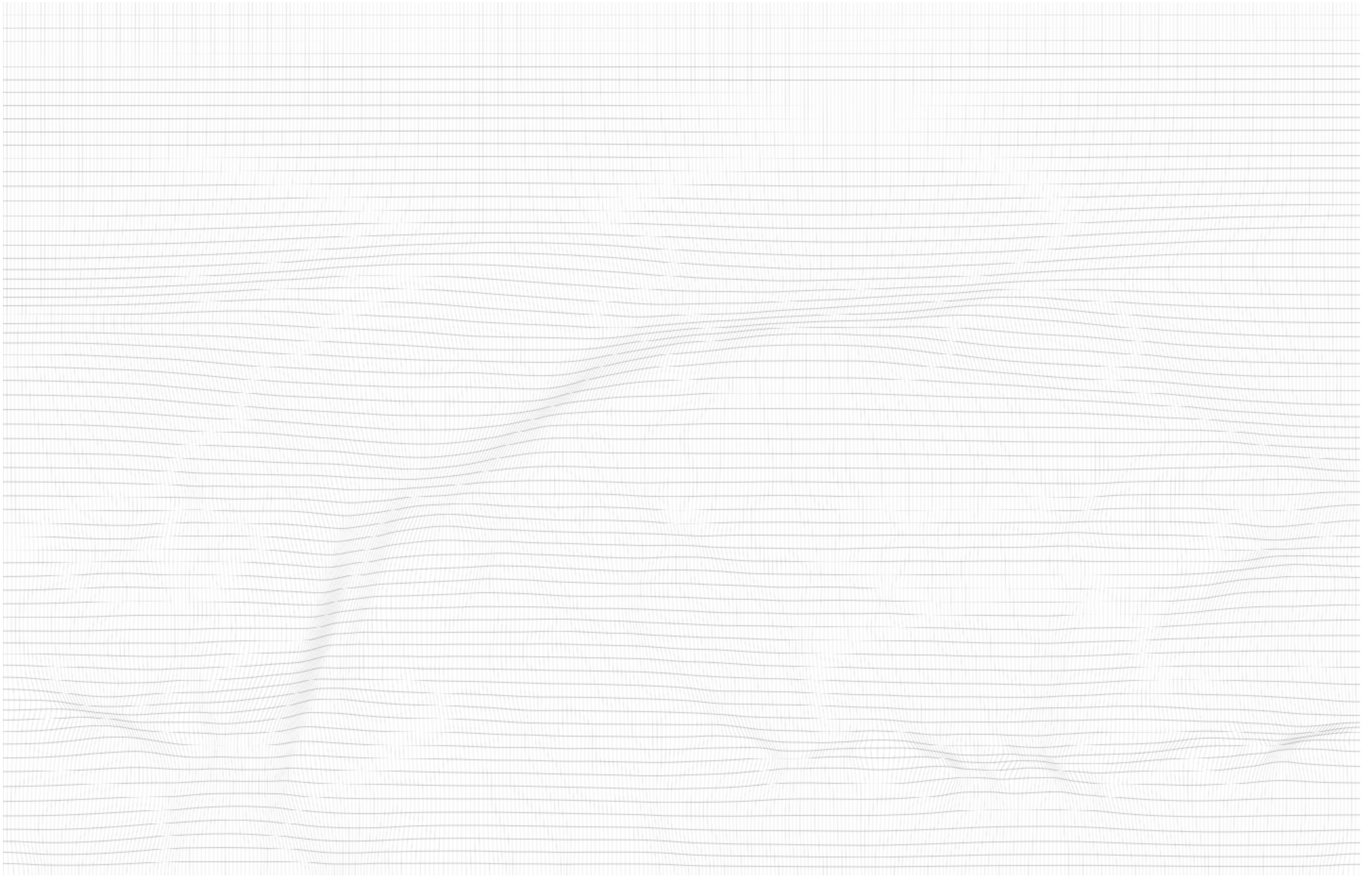
Have a computational domain $\Omega_C(\xi,\eta,\varsigma)$

Physical domain $\Omega_P(x,y,z)$

Identify a map $F:\Omega_C(\xi,\eta,\varsigma)\rightarrow\Omega_P(x,y,z)$



Map a regular mesh, to a mesh used for computation

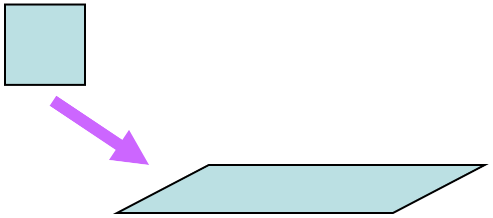


Mesh used to compute a 3D weather front

Determine F by Equidistribution

Introduce a positive unit measure $M(x,y,z,t)$ in the physical domain which controls the mesh density

A : set in computational domain



F(A) : image set

Equidistribute image with respect to the measure

$$\int_A d\xi\, d\eta\, d\varsigma = \int_{F(A)} M(x,y,z,t)\, dx\, dy\, dz$$

Differentiate to give:

$$M(x,y,z,t)\frac{\partial(x,y,z)}{\partial(\xi,\eta,\varsigma)}=1$$

Basic, nonlinear, equidistribution mesh equation

Choose M large to concentrate points where needed without depleting points elsewhere

Note: All meshes equidistribute some function M

[Radon-Nikodym]

Choice of the monitor function M(X)

- Physical reasoning

eg. Vorticity, arc-length, curvature,

- A-priori mathematical arguments

eg. Scaling, symmetry, simple error estimates

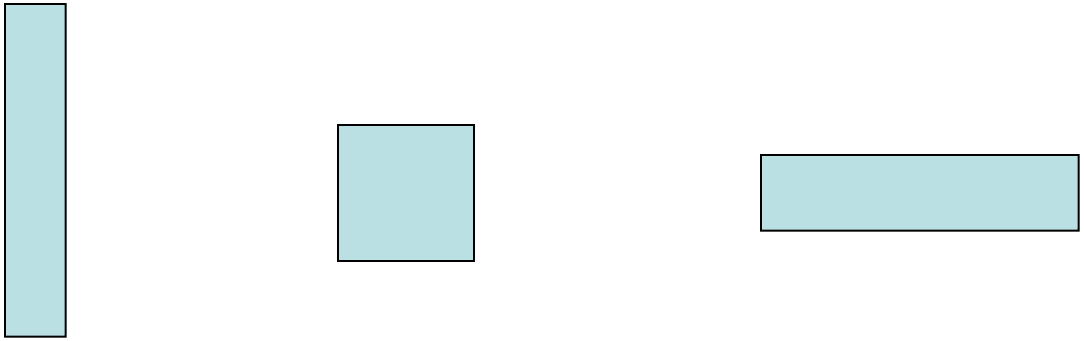
- A-posteriori error estimates

eg. Residuals, super-convergence

- Data correlation estimates

Mesh construction

Problem: in two/three -dimensions equidistribution does NOT uniquely define a mesh!

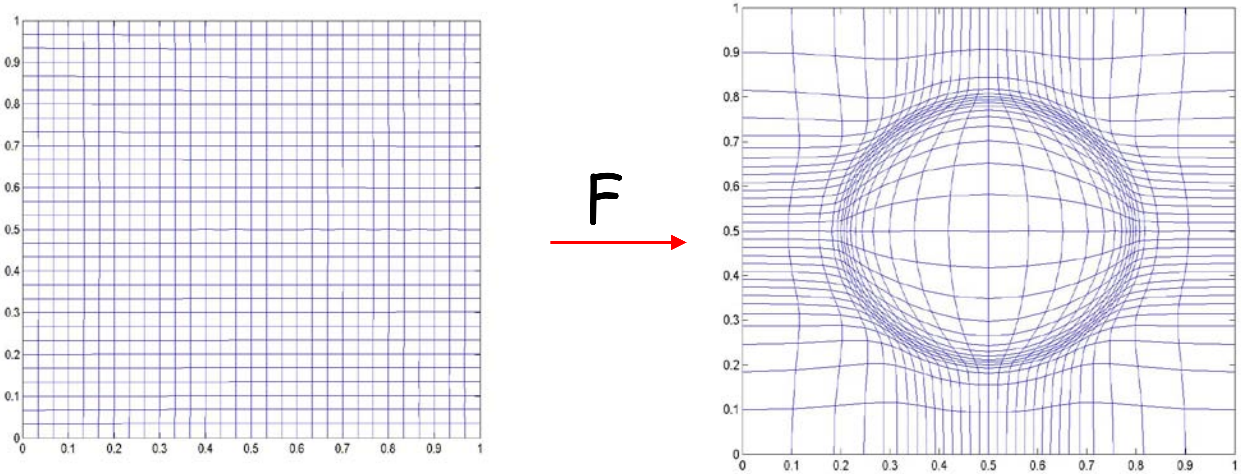


All have the same area

Need additional conditions to define the mesh uniquely

Also want to avoid mesh tangling and long thin regions

Optimally transported meshes



$\Omega_C(\xi,\eta)$ $\qquad$ $\Omega_P(x,y)$

Argue: A good mesh for solving a pde is often one which is as close as possible to a uniform mesh

Monge-Kantorovich optimal transport problem

Minimise

$$I(x,y,z) = \int_{\Omega_C} \left|(x,y,z)-(\xi,\eta,\varsigma)\right|^2 d\xi\, d\eta\, d\varsigma$$

Subject to

$$M(x,y,z,t)\frac{\partial(x,y,z)}{\partial(\xi,\eta,\varsigma)} = 1$$

Also used in image registration, meteorology .....

Optimal transport helps to prevent small angles, reduce mesh skewness and prevent mesh tangling.

Key result which makes everything work!!!!!

Theorem: [Brenier]

(a)There exists a unique optimally transported mesh

(b) For such a mesh the map F is the gradient of a convex function $P(\xi,\eta,\varsigma)$

P : Scalar mesh potential

Map F is a Legendre Transformation

Some 2D corollaries of the Polar Factorisation Theorem

$$(x,y) = \nabla_{\xi} P = (P_{\xi}, P_{\eta})$$

Gradient map

$$x_{\eta} = y_{\xi}$$

Irrotational mesh

Same construction works in all dimensions

It follows immediately in 2D that

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = H(P) = \det\begin{pmatrix} P_{\xi\xi} & P_{\xi\eta} \\ P_{\xi\eta} & P_{\eta\eta} \end{pmatrix} = P_{\xi\xi}P_{\eta\eta} - P^2{}_{\xi\eta}$$

Hence the mesh equidistribution equation becomes

$$M(\nabla P, t)H(P) = 1 \qquad \text{(MA)}$$

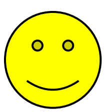
Monge-Ampere equation: fully nonlinear elliptic PDE

Properties of the mesh can be deduced from the regularity of the solution of the MA equation

Basic idea: Solve (MA) for P with appropriate (Neumann or Periodic) boundary conditions

Good news: Equation has a unique solution

Bad news: Equation is very hard to solve



Good news: We don’t need to solve it exactly, and can instead use parabolic relaxation

$Q \longrightarrow P$

Alternatively: Use Newton [Chacon et. al.]

Use a variational approach [van Lent]

Relaxation in n Dimensions

$$\varepsilon\left(I-\alpha\Delta_{\xi}\right)Q_t = \left(\overline{M}(\nabla Q)H(Q)\right)^{1/n} \quad \text{(PMA)}$$

Spatial smoothing [Hou]

(Invert operator using a spectral method)

Averaged monitor

Ensures right-hand-side scales like Q in nD to give global existence

Parabolic Monge-Ampere equation

Solution Procedure

If M is prescribed then the PMA equation can be discretised in the computational domain and solved using a forward Euler method (this is a fast procedure)

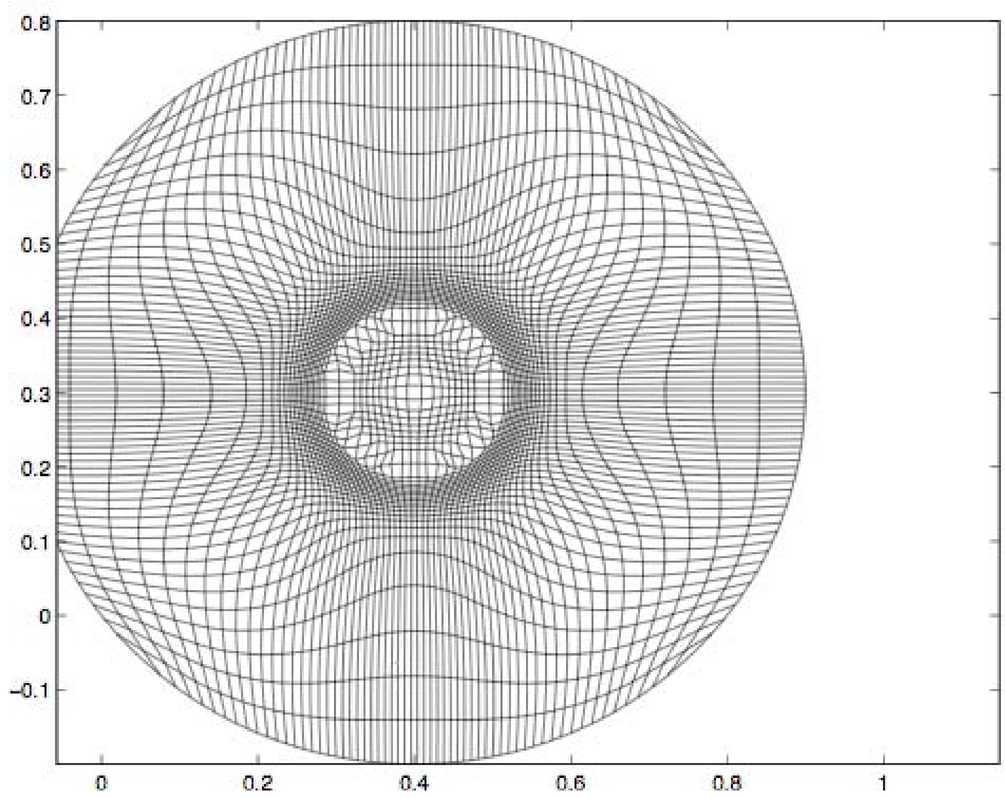
Applications

- Image processing
- Meteorological Data assimilation:

Take M to be the Potential Vorticity of the 3D flow

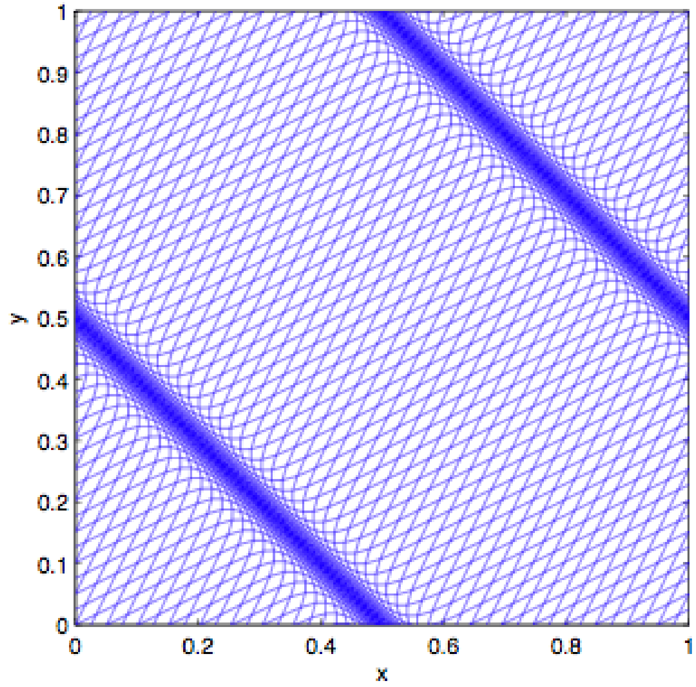
Because PMA is based on a geometric approach, it has a set of useful regularity properties

1. System invariant under translations, rotations, periodicity



Lemma 1: CJB, EJW [2012]

The solutions of the MA equation exactly align with global linear features



Alignment follows from a close coupling between the local structure of the solution and the global structure. This is NOT a property of other mesh generation methods

2. Convergence properties of PMA

Lemma 2: [Budd,Williams 2006]

(a) If $M(x,t) = M(x)$ then PMA admits the solution

$$Q(\xi,t) = P(\xi) + \Lambda t \qquad x(\xi) = \nabla_\xi Q = \nabla_\xi P$$

(b) This solution is locally stable/convergent and the mesh evolves to an equidistributed state

Proof: Follows from the convexity of P which ensures that PMA behaves locally like the heat equation

This result is important when initializing a mesh to the initial data for a PDE

Lemma 3: [B,W 2006]

If M(x,t) is slowly varying then the grid given by PMA is epsilon close to that given by solving the Monge Ampere equation.

Lemma 4: [B,W 2006]

The mapping is 1-1 and convex for all times:

No mesh tangling or points crossing the boundary

4. For appropriate choices of M the coupled system is scale-invariant

Lemma 5: [B,W 2005] Multi-scale property

If the PDE has certain continuous group invariants then meshes can be constructed with the same invariance.

This leads to discrete Noether type theorems

Lemma 6: [B,W 2009] Self-similarity

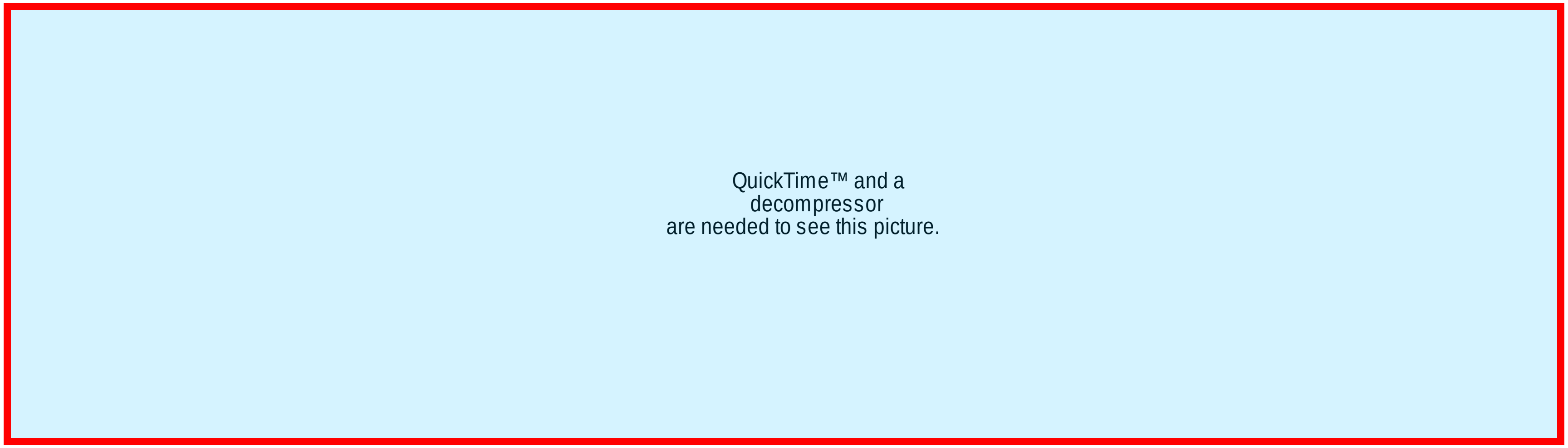
Such constructions can admit discrete self-similar solutions

Extremely useful properties when working with PDEs which have natural scaling laws

Coupling to a PDE

More usually M is a function of the solution of a PDE

- Carefully discretise PDE & PMA in the computational domain



Solve the coupled mesh and PDE system either

Method One

As one large system (stiff!)

Velocity based Lagrangian approach. Works well for parabolic blow-up type problems (JFW)

Advantages:

No need for interpolation

Mesh and solution become one large dynamical system and can be studied as such

Disadvantage: Equations are very hard to solve especially when the PDE is strongly advective

Method 2

By alternating between PDE and mesh

1. Time march the PDE
2. Construct a new mesh
3. Interpolate solution onto the new mesh
4. Repeat from 1.

Advantages:

Very flexible, can build in conservation laws

Disadvantage: Interpolation is difficult and expensive

Example 1: Parabolic blow-up

$$u_t = u_{xx} + u_{yy} + u^3, \quad u \to \infty \quad t \to T$$

Length scale:

$$L(t) = (T-t)^{1/2} \left|\log(T-t)\right|^{1/2}$$

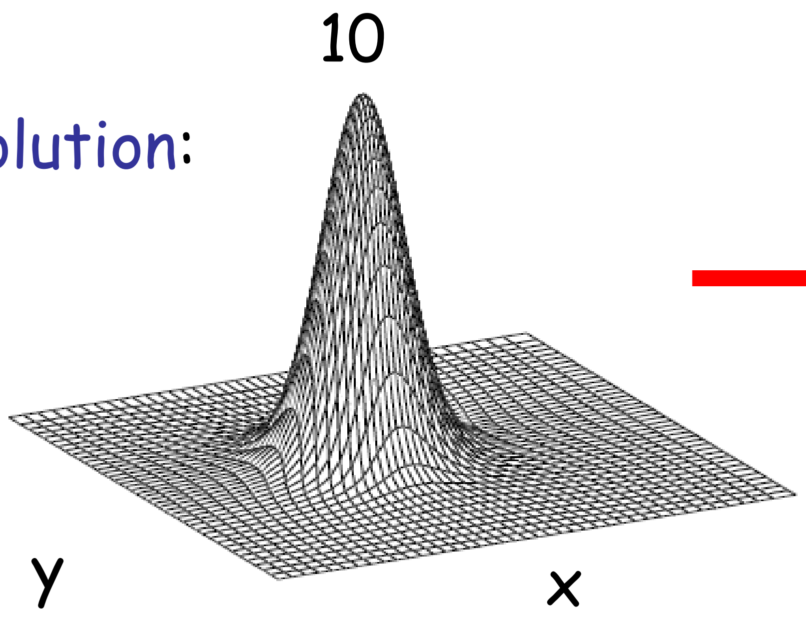
$$M(x,y,t) = \frac{1}{2}\frac{u(x,y)^4}{\int u^4 \, dx\, dy} + \frac{1}{2}$$

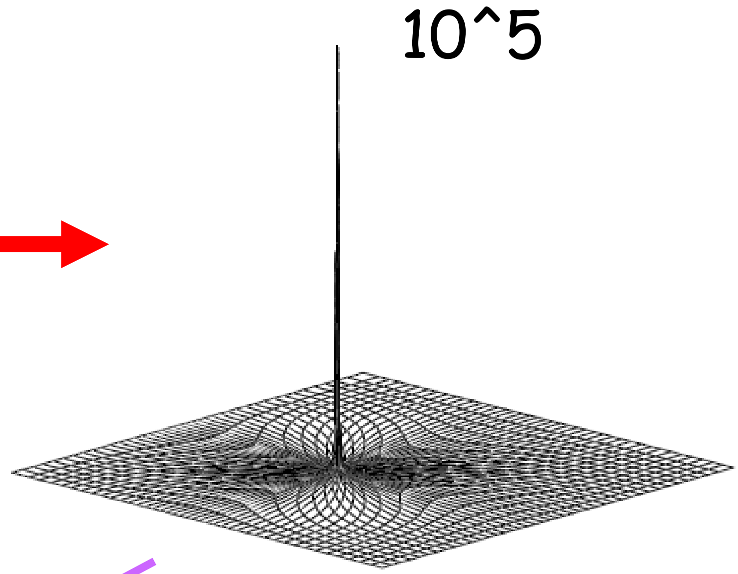
M is locally scale-invariant, concentrates points in the peak and keeps 50% of the points away from the peak

Solve using Method 1

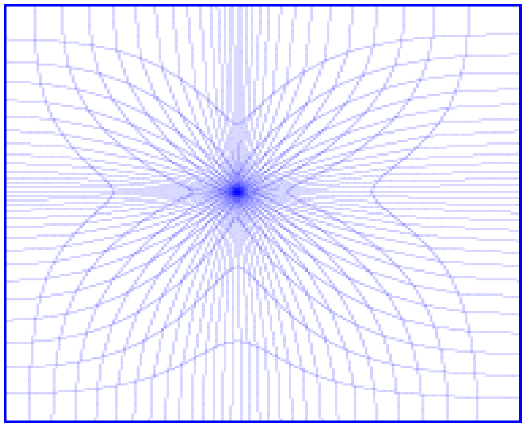
$$u_t = u_{xx} + u_{yy} + u^3$$

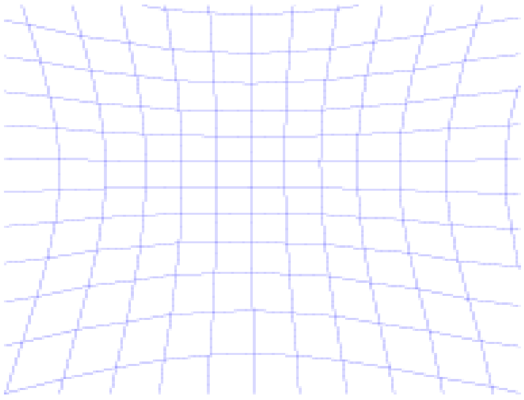
Solution:



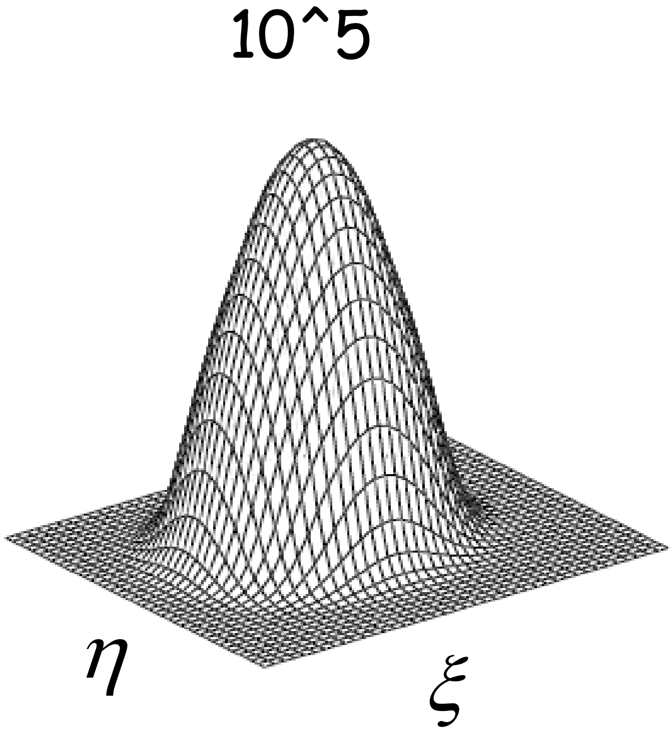


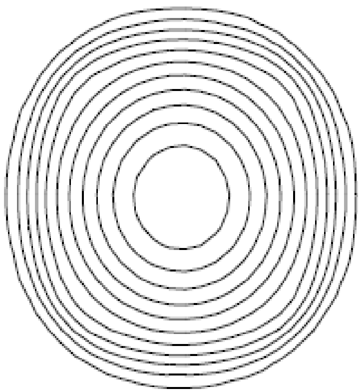
Mesh:



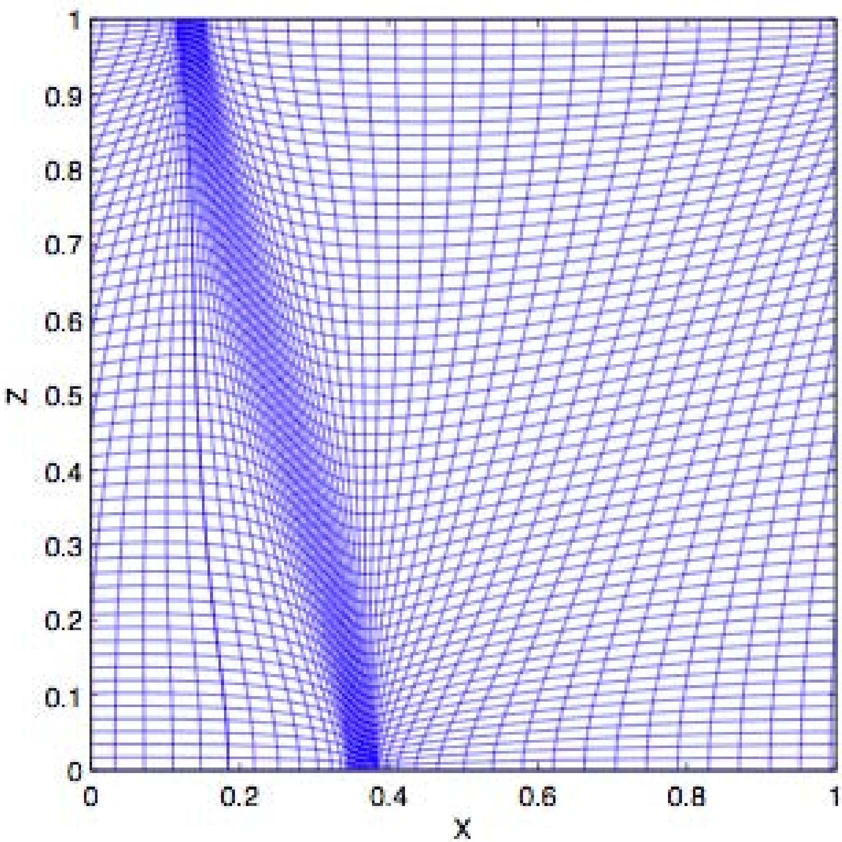


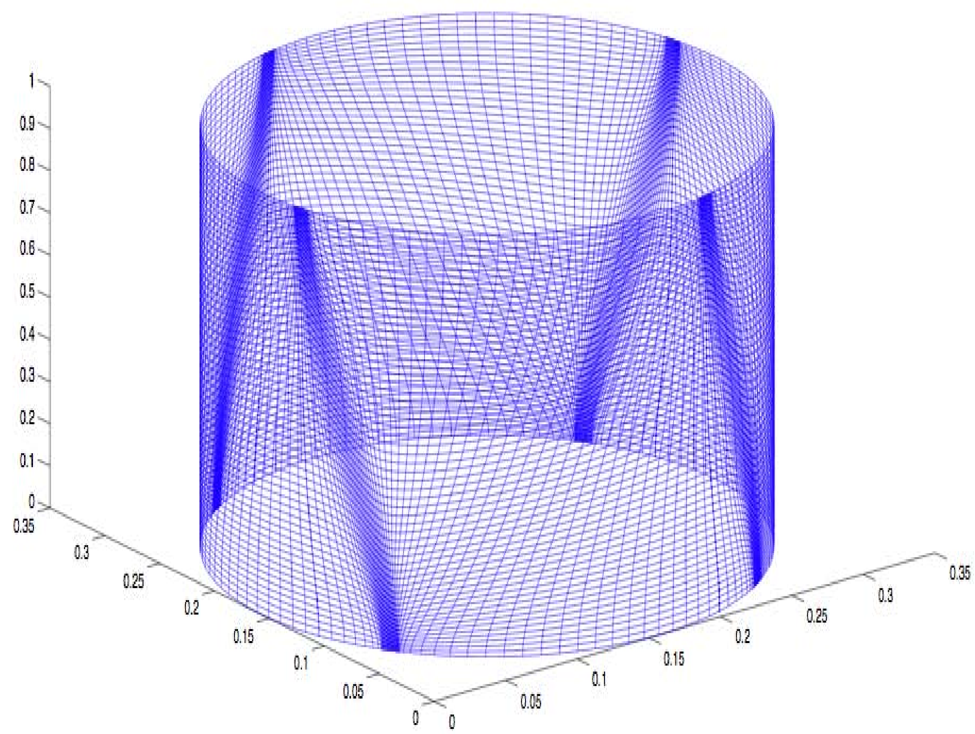
Solution in the computational domain





Example 2: Tropical storm formation (Eady problem)





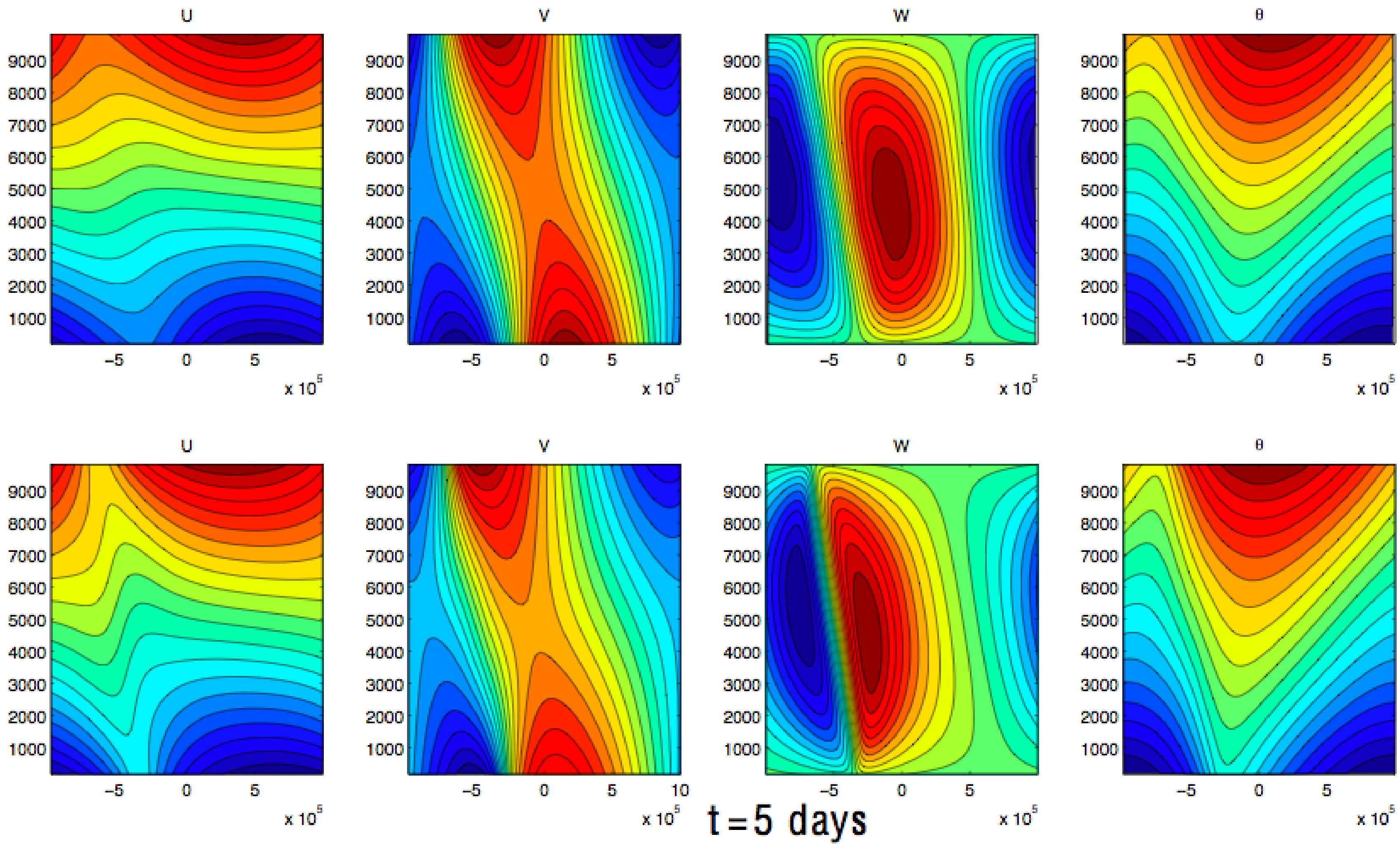
QuickTime™ and a
decompressor
are needed to see this picture.

M : Maximum eigenvalue of Potential Vorticity R

$$R = \begin{pmatrix} f^2 + f v_x & f v_z \\ g\theta_0^{-1}\theta_x & g\theta_0\theta_z \end{pmatrix}$$

Solve using Method 2, with pressure correction

- Update solution every 10 mins
- Update mesh every hour
- Advection and pressure correction on adaptive mesh
- Discontinuity singularity after 6.3 days



U
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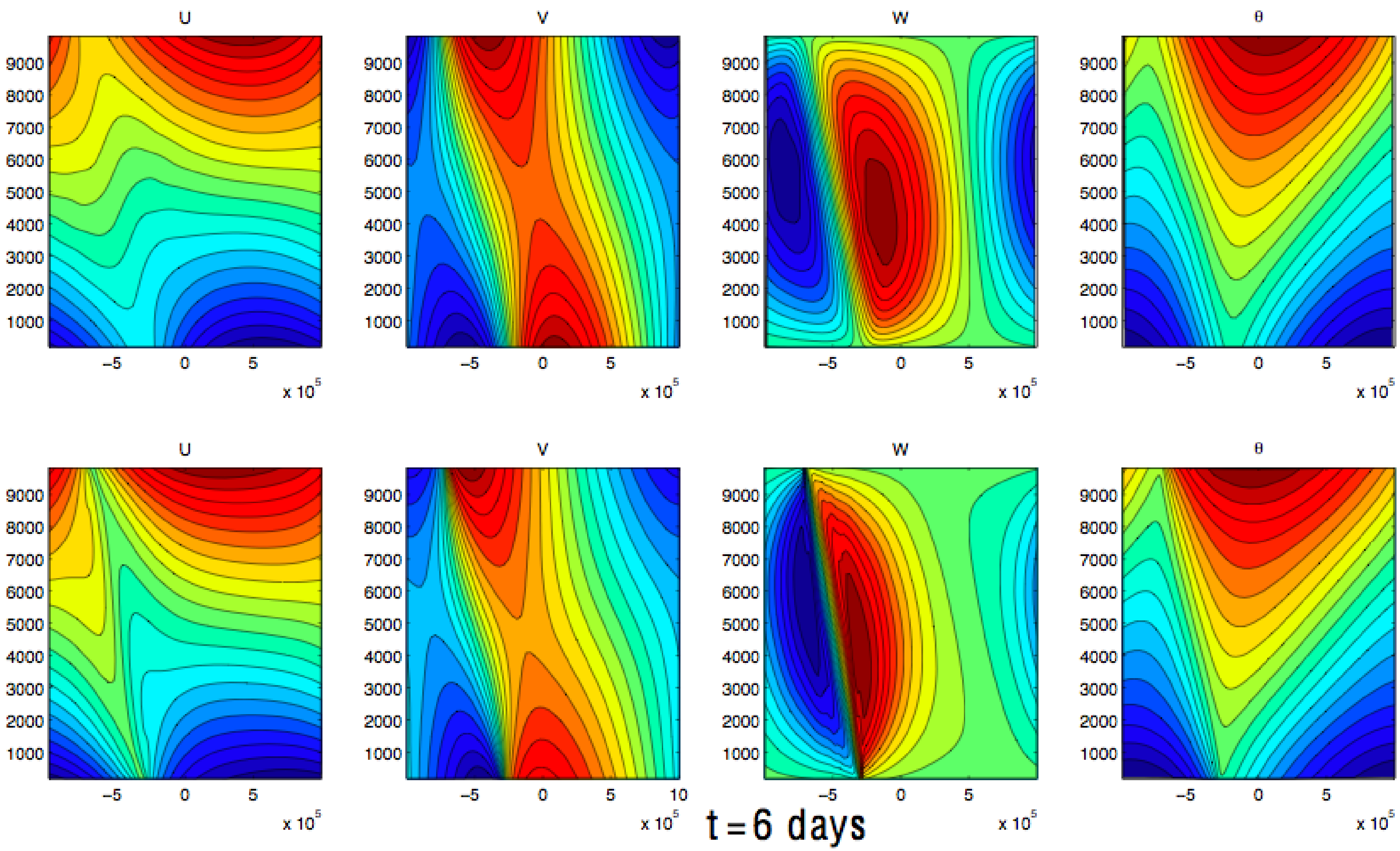
10
x 10^5
t = 5 days


U
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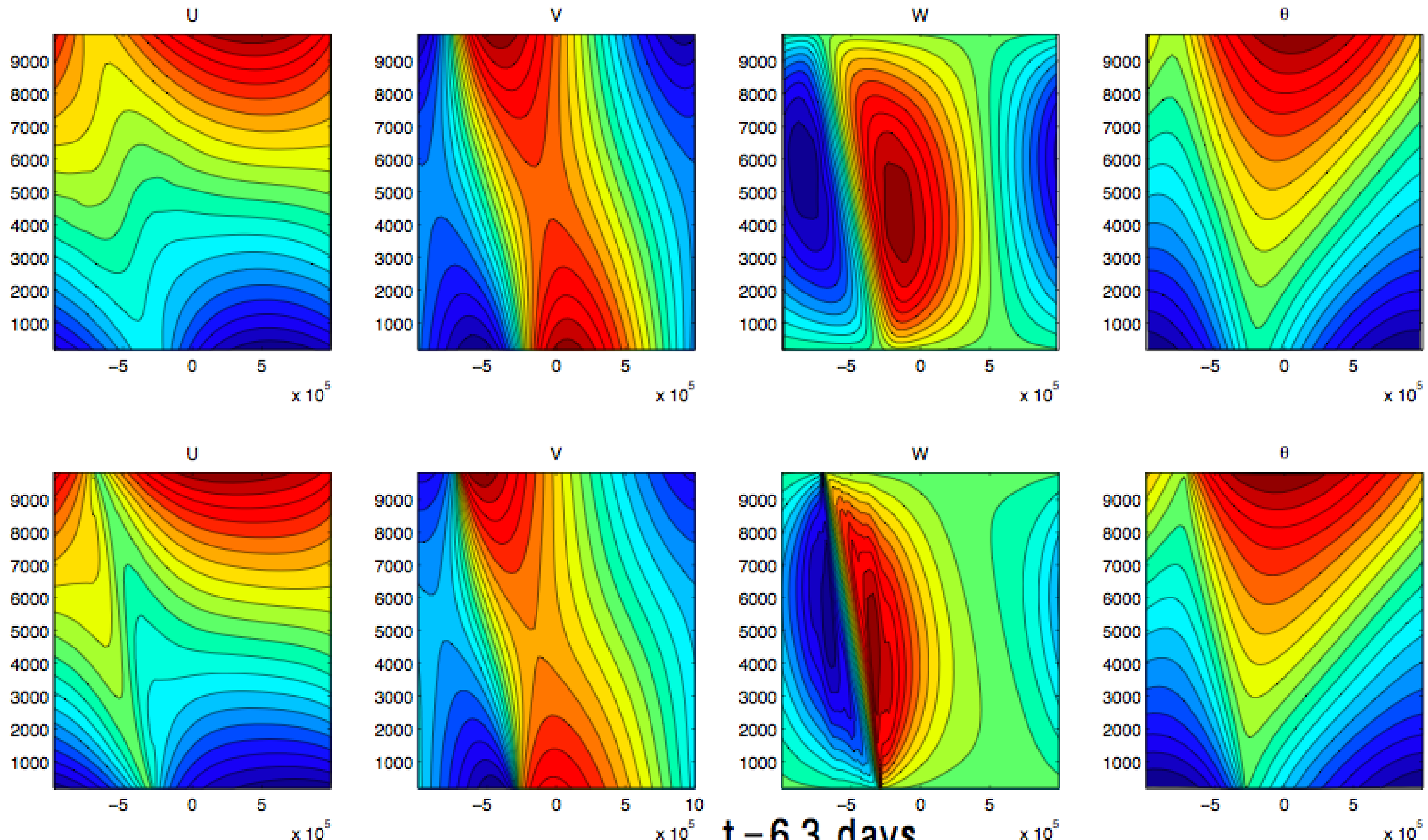
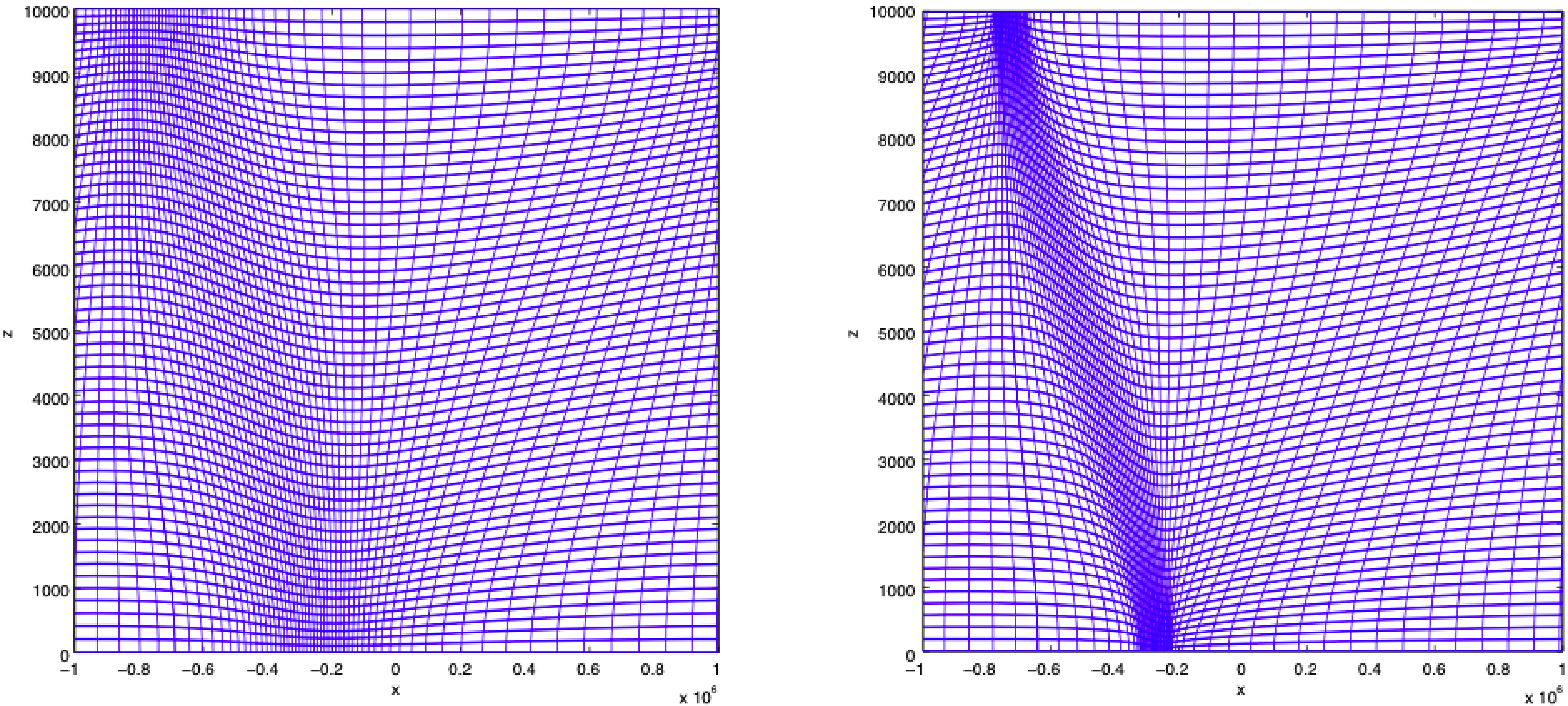
4000
3000
2000
1000
−5
0
5
10
x 10^5
t = 6 days



t = 6.3 days



10000
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4000
3000
2000
1000
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z
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
x
x 10^6
10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
z
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
x
x 10^6

Conclusions

- Optimal transport is a natural way to determine meshes in dimensions greater than one
- It can be implemented using a relaxation process by using the PMA algorithm
- Method works well for a variety of problems, and there are rigorous estimates about its behaviour
- Looking good on meteorological problems
- Still lots of work to be done eg. Finding efficient ways to couple PMA to the underlying PDE