Almost complex surfaces in the product of two 3-spheres

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The receiving space

- Let S^3 be the unit 3-sphere regarded as the group of unit quaternions, so

$$S^3 = \{x + yi + zj + wk \ : \ x^2 + y^2 + z^2 + w^2 = 1\}.$$

- We will look at *almost complex surfaces* (ie 2 real dimensions) in the product $S^3 \times S^3$, when this latter is equipped with
  (a) a special almost complex structure J,
  (b) a special Riemannian metric g (not the product metric).

The almost complex structure

- To define J (and several other structures too), we first define it at the identity $(1,1)$, and then use the group structure on $S^3 \times S^3$ to move it round the whole space.
- We first define J on tangent vectors $(\alpha, \beta) \in T_{(1,1)}(S^3 \times S^3)$ by

$$J(\alpha, \beta) = \frac{1}{\sqrt{3}}(2\beta - \alpha, -2\alpha + \beta).$$

- Then, if $(X, Y) \in T_{(p,q)}(S^3 \times S^3)$, we translate back to $(1,1)$, ie

$$(X, Y) \mapsto (p^{-1}X, q^{-1}Y),$$

- then apply J as defined above to give

$$(X, Y) \mapsto \frac{1}{\sqrt{3}}(2q^{-1}Y - p^{-1}X, -2p^{-1}X + q^{-1}Y)$$

- and then translate back to (p, q) to give (Butruille),

$$J(X, Y) = \frac{1}{\sqrt{3}}(2pq^{-1}Y - X, -2qp^{-1}X + Y).$$

- An easy check shows that

$$J^2 = -Id.$$

.

- The standard product metric $<,>$ on $S^3 \times S^3$ is not J-invariant.
- Define a J-invariant metric g on $S^3 \times S^3$ in a natural way by taking

$$g(U,V) = \frac{1}{2}(< U, V > + < JU, JV >), \quad U, V \in T_{(p,q)}(S^3 \times S^3).$$

- One can work out the Riemannian connection $\tilde{\nabla}$ for g, and it turns out that $(S^3 \times S^3, g, J)$ is a *nearly Kahler* manifold in that

$$J^2 = -Id, \quad g(JU, JV) = g(U,V), \quad U, V \in T_{(p,q)}(S^3 \times S^3),$$

and

$$(\tilde{\nabla}_U J)\, U = 0.$$

- In fact, $(S^3 \times S^3, g, J)$ is a homogeneous nearly-Kaehler manifold, with nearly-Kahler isometries given by

$$(p, q) \mapsto (apc^{-1}, bqc^{-1}), \quad a, b, c \text{ being unit quaternions.}$$

- Recently, Butruille has shown that the only homogeneous 6-dimensional, non-Kaehler nearly-Kaehler manifolds are S^6, $\mathbb{C}P^3$, $S^3 \times S^3$ and $SU(3)/T^2$ (the manifold of flags in $\mathbb{C}^3$).
- Luc, Franki and I have looked at almost complex surfaces (2 real dimensions) in the nearly-Kaehler S^6, so it was not surprising that we'd have a go at looking at almost complex surfaces in one or more of the above. It looked to us that $S^3 \times S^3$ was the simplest to deal with (after S^6), so that's the one we went for!

The almost product structure

- One of the important things you want to know is the curvature tensor $\tilde{R}$ of $(S^3 \times S^3, g)$, and to do this (and for other things too) it is convenient to define a new tensor P. Proceeding as we did with J, we first define P at $(1,1)$ and then use the group structure to move it round the whole space.
- So, take

$$P(\alpha, \beta) = (\beta, \alpha), \quad (\alpha, \beta) \in T_{(1,1)}(S^3 \times S^3),$$

and then, if $(X, Y) \in T_{(p,q)} S^3 \times S^3$, we define

$$P(X, Y) = (pq^{-1} Y, qp^{-1} X).$$

- We call P an *almost product* structure (because it reflects the product structure but it's not parallel). Easy checks show that

$$P^2 = \mathrm{Id}, \quad PJ = -JP,$$

$$g(PU, PV) = g(U, V) \quad \text{for } U, V \text{ tangential to } S^3 \times S^3.$$

The curvature tensor

- It then turns out that

$$\tilde{R}(U,V)W = \frac{5}{12}\big(g(V,W)U - g(U,W)V\big) + \frac{1}{12}\big(g(JV,W)JU - g(JU,W)JV + 2g(U,JV)JW\big) + \frac{1}{3}\big(g(PV,W)PU - g(PU,W)PV + g(JPV,W)JPU - g(JPU,W)JPV\big).$$

- A calculation using the expression for $\tilde{R}$ above shows that

Lemma

Let U, V span a J-invariant 2-plane $\mathcal{W}$. If $P(\mathcal{W})$ is perpendicular to $\mathcal{W}$, then $\mathcal{W}$ has sectional curvature $\tilde{K}$ equal to $2/3$. On the other hand, if $P(\mathcal{W}) = \mathcal{W}$, then $\mathcal{W}$ has $\tilde{K} = 0$.

Almost complex surfaces

- **Definition** A smooth immersion $\phi : M \to S^3 \times S^3$ of a surface M is said to be an *almost complex surface* if the image of the derivative $d\phi$ is J-invariant at each point.
- There are no 4-dimensional almost complex submanifolds in a compact non-Kaehler, nearly-Kaehler 6-manifold (Podesta and Spiro (2010)).
- Standard arguments show:

Lemma

An almost complex surface in a nearly-Kaehler manifold is minimal, and is totally geodesic if and only if $\tilde{K} = K$ (where $\tilde{K}$ is the sectional curvature of the tangent plane as a plane in $S^3 \times S^3$, and K is the sectional curvature of the induced metric).

- This is really useful because it doesn't seem easy to compute the second fundamental form directly.

Totally geodesic ac surfaces

- Let's look for almost complex surfaces which are also *totally geodesic*. The almost product structure P plays a large role here.

Theorem

If an almost complex surface is totally geodesic, then either
(i) $P(\text{tgt space}) \perp \text{tgt space}$*, in which case* $K = \tilde{K} = 2/3$*,*
or
(ii) $P(\text{tgt space}) = \text{tgt space}$*, in which case* $K = \tilde{K} = 0$*.*

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Proof.

- The Codazzi equation shows that if U is a unit tangent vector to the almost complex surface, then $\tilde{R}(U, JU)U$ is also tangential to the almost complex surface, and so is a scalar multiple of JU.
- The form of $\tilde{R}$, and a careful choice of U, shows either $P(\text{tgt space}) \perp \text{tgt space}$ or $P(\text{tgt space}) = \text{tgt space}$.
- Lemmas 1 and 2 show $\tilde{K} = K = \frac{2}{3}$ in former case and $\tilde{K} = K = 0$ in latter case.

Two examples

We now give two examples, one to illustrate each of the above possibilities.

Example 1 Let $\phi : \mathbb{R}^2 \to S^3 \times S^3$ be given by

$$\phi(s,t) = (\cos s + i \sin s, \cos t + i \sin t).$$

- A short calculation shows that this immersion is almost complex and $P(\text{tgt space}) = \text{tgt space}$.
- It is also quick to check that $g(\phi_s, \phi_s) = g(\phi_t, \phi_t) = 4/3$, and $g(\phi_s, \phi_t) = -2/3$. In particular, all are constant so that the induced metric is flat (ie has sectional curvature $K = 0$).
- That ϕ is totally geodesic now follows from Lemma 1 and Lemma 2.
- This gives a flat, almost complex, totally geodesic torus in $S^3 \times S^3$.

- **Example 2** Let S^2 be the 2-sphere of unit imaginary quaternions, and let $\phi : S^2 \to S^3 \times S^3$ be given by

$$\phi(x) = \frac{1}{2}(1 - \sqrt{3}x, 1 + \sqrt{3}x).$$

- Calculations similar to those in the previous example show this this is an almost complex surface with $P(\text{tgt space}) \perp \text{tgt space}$. The induced metric is $3/2$ times the standard metric on S^2, so that the induced sectional curvature K is $2/3$. It now follows from Lemma 1 and Lemma 2 that this almost complex 2-sphere is totally geodesic.
- This gives an almost complex, totally geodesic, constant curvature 2-sphere in $S^3 \times S^3$.

What do I want to achieve?

- Classify (ie, find) all almost complex 2-spheres in $S^3 \times S^3$.
- Find all totally geodesic almost complex surfaces in $S^3 \times S^3$.

What tools can I hope to use?

Complex variable techniques have been very useful in the study of minimal and CMC surfaces (and ac surfaces). For example:

- The Weierstrass representation of minimal surfaces in $\mathbb{R}^3$.
- Hopf's theorem: An immersed CMC 2-sphere in $\mathbb{R}^3$ must be embedded as a round sphere. Proved by:
  - using the second fundamental form to construct a holomorphic differential on any CMC surface in $\mathbb{R}^3$.
  - using the Hopf Index Theorem to deduce that all such differentials vanish on a 2-sphere.
  - This then implies that every point is an umbilic point, which in turn implies that we have the round 2-sphere.

So, we are looking for holomorphic data.

Isothermal Coordinates

- Let's now explore the maths of an almost complex surface ϕ using isothermal coordinates (u, v).
- So, write $\phi(u, v) = (p(u, v), q(u, v)) \in S^3 \times S^3$ with $J(p_u, q_u) = (p_v, q_v)$. We may then write

$$p^{-1}p_u = \alpha, \quad p^{-1}p_v = \beta, \quad q^{-1}q_u = \gamma, \quad q^{-1}q_v = \delta,$$

  where α, β, γ, δ are tangent vectors to the set S^3 of unit quaternions at 1. That is to say, α, β, γ, δ take values in the imaginary quaternions.
- Then $(p_u, q_u) = (p\alpha, q\gamma)$ and $(p_v, q_v) = (p\beta, q\delta)$, so the almost complex condition $J(p_u, q_u) = (p_v, q_v)$ enables us to find γ, δ in terms of α, β. This then enables us to show that the metric induced on the almost complex surface is

$$(\alpha.\alpha + \beta.\beta)(du^2 + dv^2),$$

  where "." denotes the standard inner product in $\mathbb{R}^3$.

Integrability conditions

- We now consider the integrability condition $p_{uv} = p_{vu}$.
- This gives that

$$\alpha_v - \beta_u = \alpha\beta - \beta\alpha = 2\alpha \times \beta \quad \text{(vector cross product)}.$$

- The similar condition for q, after substituting for γ, δ in terms of α, β gives

$$\alpha_u + \beta_v = \frac{2}{\sqrt{3}}\alpha \times \beta.$$

- A short calculation now gives

$$(\alpha.\beta)_u = \frac{1}{2}(\alpha.\alpha - \beta.\beta)_v \text{ and } (\alpha.\beta)_v = -\frac{1}{2}(\alpha.\alpha - \beta.\beta)_u$$

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- Ooh - the Cauchy-Riemann equations!! What is the geometry of the this holomorphic data?

Theorem

Let $\phi : M \to S^3 \times S^3$ be an almost complex surface and let $\Lambda = g(P\phi_z, \phi_z)$. Then Λdz^2 is a holomorphic differential, and the following three conditions are equivalent.
(i) $\Lambda dz^2 = 0$
(ii) $\alpha.\alpha - \beta.\beta = 0$, *and* $\alpha.\beta = 0$. *(cf previous CR equations)*
(iii) P maps the tangent spaces of ϕ to normal spaces.

The previous CR equations were:

$$(\alpha.\beta)_u = \frac{1}{2}(\alpha.\alpha - \beta.\beta)_v \text{ and } (\alpha.\beta)_v = -\frac{1}{2}(\alpha.\alpha - \beta.\beta)_u \ ,$$

so (ii) above is just the condition that the holomorphic function with real part $\alpha.\beta$ and imaginary part $\frac{1}{2}(\alpha.\alpha - \beta.\beta)$ is zero.

- Let's now change direction slightly and look again at the equations

$$\alpha_v - \beta_u = 2\alpha \times \beta, \quad \alpha_u + \beta_v = \frac{2}{\sqrt{3}}\alpha \times \beta.$$

- If we precede α, β by rotation in the tangent spaces of M through angle $2\pi/3$ to give $\tilde{\alpha} = -\frac{1}{2}\alpha + \frac{\sqrt{3}}{2}\beta$ and $\tilde{\beta} = -\frac{\sqrt{3}}{2}\alpha - \frac{1}{2}\beta$, then the above equations become

$$\tilde{\alpha}_v = \tilde{\beta}_u, \quad \tilde{\alpha}_u + \tilde{\beta}_v = -\frac{4}{\sqrt{3}}\tilde{\alpha} \times \tilde{\beta}.$$

- The first equation is an integrability condition. It shows that the form $\tilde{\alpha} du + \tilde{\beta} dv$ is closed.
- Hence, if the surface M is simply-connected, there exists an immersion $\varepsilon : S \to \mathbb{R}^3$ with $\varepsilon_u = \tilde{\alpha}$ and $\varepsilon_v = \tilde{\beta}$.
- Moreover, the other equation gives

$$\varepsilon_{uu} + \varepsilon_{vv} = -\frac{4}{\sqrt{3}}\varepsilon_u \times \varepsilon_v.$$

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$$\varepsilon_{uu} + \varepsilon_{vv} = -\frac{4}{\sqrt{3}}\varepsilon_u \times \varepsilon_v.$$

- Ooh - the equation for a surface in $\mathbb{R}^3$ with constant mean curvature when the coordinates are isothermal for ε; the surface having been scaled to have CMC $= -2/\sqrt{3}$.

Theorem

Let M be a simply connected surface, and let $\phi : M \to S^3 \times S^3$ be an almost complex surface with $\Lambda dz^2 = 0$. Then there exists a corresponding immersion $\varepsilon : M \to \mathbb{R}^3$ with $\varepsilon_u = \tilde{\alpha}$ and $\varepsilon_v = \tilde{\beta}$, and this immersion has constant mean curvature equal to $-2/\sqrt{3}$.

Proof.

- We need to prove that (u, v) are isothermal coordinates for ε (ie., $\tilde{\alpha}.\tilde{\alpha} - \tilde{\beta}.\tilde{\beta} = 0$ and $\tilde{\alpha}.\tilde{\beta} = 0$).
- A quick calculation shows that this holds if and only if $\alpha.\alpha - \beta.\beta = 0$ and $\alpha.\beta = 0$.
- Theorem 2 now shows that if $\Lambda dz^2 = 0$ then (u, v) are isothermal coordinates for ε, as required.

□

- Note that, in this case, the metric induced by ε is equal to $\tilde{\alpha}.\tilde{\alpha}(du^2 + dv^2) = \alpha.\alpha(du^2 + dv^2)$, ie, half that induced on the ac surface (which was $(\alpha.\alpha + \beta.\beta)(du^2 + dv^2)$.)

Theorem

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- Also, P maps the tangent spaces of ϕ to normal spaces, so, by Lemma 1, the sectional curvature $\tilde{K}$ is also equal to $2/3$.
- It now follows from Lemma 2 that the almost complex surface is totally geodesic.
- The uniqueness part essentially follows from the uniqueness of CMC 2-spheres in $\mathbb{R}^3$ (and the double cover $S^3 = SU(2)$ over $SO(3)$).

- We can use similar techniques to classify other types of almost complex surface in $S^3 \times S^3$.

Theorem

Let M be an almost complex surface in $S^3 \times S^3$ such that $P(tgt\ space) = tgt\ space$. Then M may be obtained by applying a nearly-Kaehler isometry to the flat totally-geodesic torus example we gave earlier.

Theorem

If M is an almost complex surface in $S^3 \times S^3$ with parallel second fundamental form, then M is totally geodesic. In fact, M may be obtained by applying a nearly-Kaehler isometry to one of the two totally-geodesic examples (torus or 2-sphere) we gave earlier.