Geometry Of Weakly Symmetric Spaces

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1. Introduction

Weakly symmetric spaces are particular Riemannian homogeneous spaces which have been introduced by Selberg [21] in 1956 in the framework of his trace formula. They attracted only little interest until the author and Vanhecke [7] found a simple geometric characterization of weakly symmetric spaces which lead to a large number of new examples. The purpose of this note is to present a survey about this topic.

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2. Definition

Let M be a connected smooth Riemannian manifold and $I(M)$ its isometry group. According to the definition of Selberg given in [21] M is a *weakly symmetric space* if there exists some closed subgroup G of $I(M)$ and some isometry $\mu \in I(M)$ such that $\mu^2 \in G$ and for all $p, q \in M$ there exists some $f \in G$ such that $f(p) = \mu(q)$ and $f(q) = \mu(p)$. To be more precise we will say in this situation that M is weakly symmetric with respect to (G, μ). It is easy to see that if M is weakly symmetric with respect to some pair (G, μ) then M is also weakly symmetric with respect to $(I(M), \mathrm{id}_M)$. This leads to the geometric characterization that M is weakly symmetric if and only if any two points in it can be interchanged by some isometry [7].

The reason that Selberg involved the pair (G, μ) in the definition, rather than just $(I(M), \mathrm{id}_M)$, is that the algebra of all G-invariant differential operators on M is commutative for a weakly symmetric space with respect to (G, μ). It was exactly this property that Selberg was interested in. In the following, when we say just weakly symmetric we mean weakly symmetric with respect to $(I(M), \mathrm{id}_M)$.

3. Characterizations and geometric properties

Let M be a weakly symmetric space, γ a geodesic in M, and $m \in \gamma$. Choose points $p, q \in \gamma$ such that m is the midpoint on γ between p and q. There exists some isometry $f \in I(M)$ with $f(p) = q$ and $f(q) = p$. If p and q are sufficiently close than f is an involution on γ which fixes m. Conversely, suppose that for each geodesic γ in M and each

$m \in \gamma$ there exists an isometry $f \in I(M)$ with $f(m) = m$ and which is an involution on γ. Such Riemannian manifolds have been introduced by Szabó [22] as *ray symmetric spaces*. Let $p, q \in M$ and γ some geodesic in M connecting p and q. Such a geodesic exists since ray symmetric spaces can be shown to be complete. Let m be the midpoint on γ between p and q. There exists some isometry $f \in I(M)$ which fixes m and is an involution on γ. Then clearly f interchanges p and q. Thus ray symmetry is equivalent to weak symmetry [3]. The motivation for Szabó to introduce ray symmetric spaces lies in spectral theory of operator families. He realized that Gelfands proof for the commutativity of the algebra is isometry-invariant differential operators on symmetric spaces works also under the weaker hypothesis of ray symmetry.

Let M be a weakly symmetric space and K the isotropy subgroup of $I(M)$ at some point $o \in M$. Now consider the isotropy representation $\chi : K \to GL(T_oM)$. Using the characterization of ray symmetry one sees that for each $X \in T_oM$ there exists some $k \in K$ such that $\chi(k)X = -X$. In fact, this property of the isotropy representation characterizes weakly symmetric spaces among the homogeneous spaces [25]. This characterization is quite useful for proving that certain homogeneous spaces are weakly symmetric.

Weakly symmetric spaces have a couple of nice features:

(1) Each local geodesic symmetry is volume-preserving up to sign (D'Atri spaces) [3];
(2) Every complete geodesic is the orbit of a one-parameter group of isometries (g.o. spaces) [2];
(3) The principal curvatures, counted with multiplicities, of any distance sphere are the same at antipodal points ($\mathfrak{SC}$-spaces) [3];
(4) The Jacobi operator has constant eigenvalues along geodesics ($\mathfrak{C}$-spaces) [3];
(5) The algebra of all isometry-invariant differential operators on the manifold is commutative (commutative spaces) [21].

A natural question is whether any of these features characterizes weakly symmetric spaces. The answer is no in all five cases. For (1)-(4) one may find appropriate examples among generalized Heisenberg groups [6]. It was proved by the author, Ricci and Vanhecke [4] that property (5) characterizes the weakly symmetric spaces among the generalized Heisenberg groups. The first example of a homogeneous space which is weakly symmetric but not commutative was found by Lauret [12]. This will be discussed in a broader context in Section 6.

4. Examples - via fibre bundles

It is a simple exercise, using again some midpoint argument, to show that each symmetric space is weakly symmetric. Selberg [21] provided only one example of a weakly symmetric space which is not symmetric, namely the three-dimensional special linear group $SL(2, \mathbb{R})$ equipped with some particular left-invariant Riemannian metric. One might view this space as a circle bundle over the upper-half plane. Later this example was generalized to certain circle bundles over the Siegel half-space [16] and over Hermitian symmetric spaces of non-compact type [18]. Explictly, this circle bundle construction is as follows. Let G be some connected, simply connected, real semisimple Lie group and K a maximal compact subgroup of G such that G/K is a Hermitian symmetric space. Write $K = K_s \cdot Z_K^o$, where K_s is the semisimple part of K and Z_K^o is the identity component of the center of K. Then G/K_s is a weakly symmetric space [18]. The manifolds G/K_s arising in this way are certain φ-symmetric spaces, the analoga of symmetric spaces in contact geometry. Using geometric arguments it was shown by the author and Vanhecke [8] that simply connected φ-symmetric spaces are weakly symmetric. This includes also compact examples of weakly symmetric spaces G/K, where the metric on G/K can be any G-invariant one, for instance $SU(p+q)/(SU(p)\times SU(q))$, $SO(2n)/SU(n)$, $SO(p+2)/SO(p)$, $Sp(n)/SU(n)$, $E_6/SO(10)$, E_7/E_6. The special case of Sasakian space forms has been treated in [5]. This approach has been generalized further by González-Dávila and Vanhecke [9,10] to Killing-transversally symmetric spaces, which are Riemannian manifolds which admit a complete unit Killing vector field X and for which the reflections in the flow lines of X are isometries.

Another interesting circle bundle was discussed by Ziller in [25]. Consider the Riemannian product

$$\mathbb{C}P^2 \times \mathbb{C}P^1 = \frac{SU(3)\times SU(2)}{U(2)\times U(1)} .$$

Consider $U(2)$ as $SU(2)U(1)$ (as manifolds), and embed a circle T into $U(1)\times U(1) \subset SU(2)U(1)\times U(1)$ in a suitable way "with slope p/q". Then

$$M_{pq} = \frac{SU(3)\times SU(2)}{SU(2)T}$$

is a circle bundle over $\mathbb{C}P^2 \times \mathbb{C}P^1$. Any such M_{pq} is weakly symmetric with respect to any $SU(3)\times SU(2)$-invariant Riemannian metric. It was shown by Kreck and Stolz that among these circle bundles there are some which are homeomorphic but not diffeomorphic. Consequently, there exist weakly symmetric spaces which are homeomorphic but not diffeomorphic. This cannot happen for symmetric spaces, since two symmetric spaces which are homeomorphic are also diffeomorphic to each other. In fact, this example belongs to a broader class of weakly symmetric spaces found by Ziller, namely certain principal torus bundles over products of complex projective spaces.

Another interesting example is the following. Fix a maximal torus T^2 in $SU(2)\times SU(2) = S^3\times S^3$ and embed a circle T^1 into T^2 with slope p/q. Then $N_{pq} = (SU(2)\times SU(2))/T^1$ is diffeomorphic to the product $S^3\times S^2$. It was shown by Kowalski and Marinosci [11] as well as by Ziller [25] that N_{pq} is weakly symmetric. Since the isotropy representation of $(SU(2)\times SU(2))/T^1$ has three irreducible components there exists a three-dimensional family of non-isometric weakly symmetric $SU(2)\times SU(2)$-invariant Riemannian metrics on N_{pq}. Different slopes provide non-isometric metrics and therefore $S^3\times S^2$ carries infinitely many three-dimensional families of non-isometric weakly symmetric $SU(2)\times SU(2)$-invariant Riemannian metrics. Such a feature is impossible for symmetric metrics.

All the above examples fit into a general scheme. Let G be some connected semisimple Lie group, K some closed subgroup of G, and H some closed subgroup of K. This gives us a fibre bundle $G/H \to G/K$ with fibre K/H. In the above examples G/K and K/H are symmetric spaces. This suggests the general question: Given such a fibre bundle such that G/K and K/H are (weakly) symmetric spaces, when is G/H weakly symmetric? For further examples of weakly symmetric spaces which can be obtained via this fibre bundle approach we refer to [17,23,24,25].

5. Examples - via nilpotent Lie groups; and commutativity of differential operators

It was shown by Selberg [21] that on any weakly symmetric space the algebra of all isometry-invariant differential operators is commutative. And Selberg posed the question whether the converse is true. Let us first assume that G is a connected compact simple Lie group and K is some closed subgroup of G such that $M = G/K$ is weakly symmetric with respect to (G, μ) for some $\mu \in I(M)$. Here we consider M equipped with any G-invariant Riemannian metric. Then the algebra of all G-invariant differential operators on M is commutative. Since G is connected this just means that (G, K) is a spherical pair, that is, every unitary representation of G contains at most one H-fixed vector. The spherical pairs of connected compact simple Lie groups are all known, and Nguyen [20] studied the isotropy representations of all the homogeneous spaces arising in the list. He showed that all these spaces are weakly symmetric. It follows that in the class of homogeneous spaces of connected compact simple Lie groups the weakly symmetric spaces are precisely the commutative spaces. This result has also been obtained by Akhiezer and Vinberg [1] as a corollary from a more general result about spherical algebraic varieties.

A classical result by E. Cartan says that if (G, K) is a Riemannian symmetric pair then the algebra of integrable functions on G which are biinvariant under K is commutative, that is, (G, K) is a Gelfand pair. This was extended by Nguyen [19] to weakly symmetric spaces. If M is weakly symmetric with respect to (G, μ) and K is the isotropy subgroup of G at some point, then (G, K) is a Gelfand pair. It is worthwhile to mention that one does not have to assume here that G is connected.

We now come to weakly symmetric spaces in the framework of 2-step nilpotent Lie groups. Our motivation for this is that the first answer to the question of Selberg whether any weakly symmetric space is commutative was provided in this framework. We start with a brief introduction to generalized Heisenberg groups. Let $m \in \mathbb{N}$, q the standard negative definite quadratic form on $\mathfrak{z} := \mathbb{R}^m$, and $J : Cl(\mathfrak{z}, q) \to \mathrm{End}(\mathfrak{v})$, $Z \mapsto J_Z$ a real representation of the Clifford algebra $Cl(\mathfrak{z}, q)$ on some finite-dimensional real vector space $\mathfrak{v}$. If $m \not\equiv 3 (\mathrm{mod}\ 4)$, then there exists a unique irreducible Clifford module $\mathfrak{d}$ over $Cl(\mathfrak{z}, q)$ and $\mathfrak{v}$ is the k-fold direct sum of $\mathfrak{d}$ for some $k \in \mathbb{N}$. If $m \equiv 3 (\mathrm{mod}\ 4)$, then there are two inequivalent irreducible Clifford modules $\mathfrak{d}_1$ and $\mathfrak{d}_2$ over $Cl(\mathfrak{z}, q)$ and $\mathfrak{v} = (\oplus^{k_1}\mathfrak{d}_1) \oplus (\oplus^{k_2}\mathfrak{d}_2)$ for some $k_1, k_2 \in \mathbb{N}$. On the direct sum $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ of the vector spaces $\mathfrak{v}$ and $\mathfrak{z}$ we define an inner product $\langle \cdot, \cdot \rangle$ as follows. On $\mathfrak{z}$ the inner product is just minus the polarization of the quadratic form q. The vector spaces $\mathfrak{v}$ and $\mathfrak{z}$ are supposed to be perpendicular with respect to the inner product. Finally we require that for any unit vector $Z \in \mathfrak{z}$ the map J_Z is an orthogonal map on $\mathfrak{v}$ with respect to the induced inner product on $\mathfrak{v}$. It can be shown that such an inner product exists and is unique. We then define a skew-symmetric bilinear map $[.,.] : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$ by $<[U + X, V + Y], W + Z> = <J_Z U, V>$ for all $U, V, W \in \mathfrak{v}$ and $X, Y, Z \in \mathfrak{z}$. This turns $\mathfrak{n}$ into a two-step nilpotent Lie algebra with m-dimensional center $\mathfrak{z}$, a so-called *generalized Heisenberg algebra*. Choosing $m = 1$ yields the classical Heisenberg algebras. The connected and simply connected Lie group N with Lie algebra $\mathfrak{n}$ and equipped with the left-invariant Riemannian metric induced from the inner product on $\mathfrak{n}$ is called a *generalized Heisenberg group*. We refer to [6] for an introduction to generalized Heisenberg groups.

Recall that the author, Ricci and Vanhecke [4] proved that among generalized Heisenberg groups the weakly symmetric spaces are precisely the commutative spaces. The explicit classification of the weakly symmetric generalized Heisenberg groups is as follows: $\dim \mathfrak{z} \in \{1,2,3\}$, or $\dim \mathfrak{z} \in \{5,6,7\}$ and $\mathfrak{v}$ irreducible, or $\dim \mathfrak{z} = 7$ and $\mathfrak{v} = \mathfrak{v}_o \oplus \mathfrak{v}_o$ with $\mathfrak{v}_o$ irreducible. The clever idea of Lauret [12] was to modify the metrics on generalized Heisenberg groups in the following manner. Let S be a positive definite symmetric transformation on $\mathfrak{n}$ which is the identity on $\mathfrak{v}$. Then define a new positive definite inner product on $\mathfrak{n}$ by $\langle X, Y\rangle_S = \langle SX, Y\rangle$ for all $X, Y \in \mathfrak{n}$. The new inner product leads to some 2-step nilpotent Lie group, a so-called modified generalized Heisenberg group. Among these nilpotent Lie groups there are commutative spaces which are not weakly symmetric. For instance, the choice $\dim \mathfrak{z} = 3$, $\mathfrak{v}$ isotypic, and S with three distinct positive eigenvalues gives such a space. Lauret classified in [13] all weakly symmetric spaces among these modified generalized Heisenberg groups.

Lauret considered in [14,15] another interesting class of 2-step nilpotent Lie groups. Let (π, V) be a real representation of some compact Lie algebra $\mathfrak{g}$ and put $\mathfrak{n} = \mathfrak{g} \oplus V$. Let $\langle \cdot, \cdot \rangle$ be some $\mathfrak{g}$-invariant positive definite inner product on $\mathfrak{n}$. By this we mean a positive definite inner product on $\mathfrak{n}$ whose restriction to $\mathfrak{g}$ is invariant under the adjoint representation of $\mathfrak{g}$ and whose restriction to V is invariant under the representation π, and such that $\mathfrak{g}$ and V are orthogonal. By defining $\langle [v, w], x\rangle = \langle \pi(x)v, w\rangle$ for all $v, w \in V$ and $x \in \mathfrak{g}$ we turn $\mathfrak{n}$ into a 2-step nilpotent Lie algebra. This Lie algebra, together with the inner product, determines a connected, simply connected 2-step nilpotent Lie group $N(\mathfrak{g}, V)$. One of the results of Lauret says that any $N(\mathfrak{g}, V)$ with $\mathfrak{g}$ semisimple, V irreducible of real type and $\dim V > 3\mathrm{rk}(\mathfrak{g})$ is a naturally reductive Riemannian homogeneous space which is not weakly symmetric. This result leads to the first examples of naturally reductive spaces which are not weakly symmetric.

A relation between the fibre bundle approach and the approach via nilpotent Lie groups was discovered by Tamaru [24]. Consider a fibre bundle $G/H \to G/K$ with fibre K/H, as decribed above. Suppose that (G, K) is an almost effective Riemannian symmetric pair such that G/K is a compact irreducible Riemannian symmetric space, and assume that $\dim H < \dim K$. Let $\mathfrak{m}_B$ (resp. $\mathfrak{m}_F$) be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ (resp. of $\mathfrak{h}$ in $\mathfrak{k}$) with respect to the Killing form of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_B = \mathfrak{h} \oplus \mathfrak{m}_F \oplus \mathfrak{m}_B$, and $\mathfrak{m}_F$, $\mathfrak{m}_B$ and $\mathfrak{m}_F \oplus \mathfrak{m}_B$ is canonically isomorphic to the corresponding tangent space of K/H, G/K and G/H, respectively. Now put $\mathfrak{n} = \mathfrak{m}_F \oplus \mathfrak{m}_B$ and define a Lie bracket on $\mathfrak{n}$ by $[Z + X, Z' + X'] = [X, X']_{\mathfrak{m}_F}$ for all $Z, Z' \in \mathfrak{m}_F$ and $X, X' \in \mathfrak{m}_B$, where the subscript denotes the $\mathfrak{m}_F$-component. This turns $\mathfrak{n}$ into a 2-step nilpotent Lie algebra with center $\mathfrak{m}_F$. Denote by N the connected, simply connected, Lie group with Lie algebra $\mathfrak{n}$ equipped with the left-invariant Riemannian metric which is induced by the negative of the Killing form of $\mathfrak{g}$. If, for instance, (G, K) is Hermitian symmetric and H is such that $K = U(1)H$

then N is a generalized Heisenberg algebra and the fibre bundle is the one of a φ-symmetric space over an Hermitian symmetric space. It was shown by Tamaru that if G/H is weakly symmetric then the corresponding 2-step nilpotent Lie group N is weakly symmetric. Moreover, H is always a subgroup of the automorphism group $A(N)$ of N, and if H is equal to $A(N)$, then the converse is true. This results leads to further examples of weakly symmetric spaces. For instance, the triple $(Spin(8), Spin(7), G_2)$ leads to a singular 2-step nilpotent Lie group which is weakly symmetric. This is the first example of such a kind.

6. Classifications

We consider now the classification problem of weakly symmetric spaces. First of all it is clear that the Riemannian product $M_1 \times M_2$ of two Riemannian manifolds M_1 and M_2 is weakly symmetric if and only if both M_1 and M_2 are weakly symmetric. We therefore have to consider only irreducible manifolds.

We next discuss the classification of connected, simply connected, irreducible, weakly symmetric spaces in low dimensions. For this denote by $S^n(c)$ and $\mathbb{R}H^n(-c)$ the n-dimensional sphere and real hyperbolic space with constant curvature c and $-c$, respectively, and by $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(-c)$ the n-dimensional complex projective and hyperbolic space with constant holomorphic sectional curvature c and $-c$, respectively, where $c > 0$. Any homogeneous two-dimensional Riemannian manifold has constant curvature, which shows that we get only $S^2(c)$ and $\mathbb{R}H^2(-c)$. In dimensions three, four and five one may use the fact that every weakly symmetric space is a g.o. space and that the classification of connected, simply connected, g.o. spaces is known. In dimension three this leads to the following classification of connected, simply connected, irreducible weakly symmetric spaces [7]:

(a) The symmetric spaces $S^3(c)$ and $\mathbb{R}H^3(-c)$;
(b) The horospheres in $\mathbb{C}H^2(c)$, or equivalently, the 3-dimensional Heisenberg group equipped with any left-invariant Riemannian metric;
(c) The distance spheres in $\mathbb{C}P^2(c)$ and $\mathbb{C}H^2(-c)$, or equivalently, the special unitary group $SU(2)$ equipped with suitable left-invariant Riemannian metrics;
(d) The universal covering of tubes around totally geodesic $\mathbb{C}H^1(-c)$ in $\mathbb{C}H^2(-c)$, or equivalently, the universal covering of the special linear group $SL(2,\mathbb{R})$ equipped with suitable left-invariant Riemannian metrics.

Surprisingly, in dimension four the connected, simply connected, irreducible weakly symmetric spaces are the symmetric spaces $S^4(c)$, $\mathbb{R}H^4(-c)$, $\mathbb{C}P^2(c)$ and $\mathbb{C}H^2(-c)$ [7]. The classification in dimension five is more complicated, as the example of $S^3 \times S^2$ which we already discussed above illustrates. We therefore refer to the paper by Kowalski and Marinosci [11]. The classification of connected, simply connected, weakly symmetric spaces in any dimension greater than five is still an open problem.

The universal covering space of a weakly symmetric space is also weakly symmetric. To prove this one uses the fact that on any connected, simply connected, complete, real analytic Riemannian manifold each local isometry can be extended to some global isometry. Conversely, given a connected, simply connected, weakly symmetric space, one may wonder which subcoverings of it are also weakly symmetric. If Γ denotes the group of deck transformations of such a covering $M \to M/\Gamma$, one obviously needs that the points in M can be interchanged by isometries which lie in the normalizer of Γ in $I(M)$. In theory this condition yields all weakly symmetric subcoverings, but in practice this is a difficult problem.

7. References

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