Real hypersurfaces with constant principal curvatures in complex space forms

By Jürgen Berndt

The purpose of this note is twofold. At first I want to give a survey on results concerning the classification problem of real hypersurfaces with constant principal curvatures in complex projective space $\mathbb{C}P^m$ and complex hyperbolic space $\mathbb{C}H^m$. Recently Kimura [Ki] obtained the classification of Hopf hypersurfaces (a class of real hypersurfaces which is introduced in section 2) with constant principal curvatures in $\mathbb{C}P^m$. Here, I want to present the analogous classification in $\mathbb{C}H^m$. The proof of this result will be omitted here (see [Be1] for a proof), but I want to point out that Kimura's method of proof does not work in the hyperbolic case. Hopf hypersurfaces with constant principal curvatures can be regarded as the most simple real hypersurfaces of $\mathbb{C}P^m$ and $\mathbb{C}H^m$. Therefore it is of interest to study the influence of the anisotropy of the ambient space $\mathbb{C}P^m$ or $\mathbb{C}H^m$ on these hypersurfaces. I will deal with this topic in the second part of this note.

1. Hypersurfaces with constant principal curvatures in real space forms

In real space forms (spaces of constant sectional curvature) the classification problem of hypersurfaces with constant principal curvatures is equivalent to the well-known classification problem of isoparametric hypersurfaces ([Ca], p. 178). These have already been classified in Euclidean space E^m by Levi-Civita [Le] ($m = 3$) and Segre [Se] ($m \geq 3$) and in real hyperbolic space $\mathbb{R}H^m$ by E. Cartan [Ca]. As well in E^m as in $\mathbb{R}H^m$ the isoparametric hypersurfaces are essentially the totally umbilical hypersurfaces and the tubes around totally geodesic submanifolds. In the sphere S^m, however, a complete classification has not been obtained until now (for essential results see [Ca], [No], [OT], [FKM], [Mü] and [DN] and the literature cited there).

For a profitable submanifold theory in general ambient spaces one has to require that the geometry of a submanifold is adapted somehow to the geometry of the ambient space. In the theory of real hypersurfaces in complex space forms (spaces of constant holomorphic sectional curvature) a suitable adapted class of submanifolds is formed by the Hopf hypersurfaces.

2. Hopf hypersurfaces

Let M be a real hypersurface of a Kähler manifold $\overline{M}$. We denote by J the complex structure of $\overline{M}$ and by TM (resp. $\perp M$) the tangent (resp. normal) bundle of M (in $\overline{M}$). The 1-dimensional foliation of M by the integral manifolds of the subbundle $J(\perp M)$ of TM will be called the *Hopf foliation* of M. We say that M is a *Hopf hypersurface* of $\overline{M}$ if the Hopf foliation of M is totally geodesic. In this sense S^{2m-1} is a Hopf hypersurface of $\mathbf{C}^m$; the Hopf foliation of S^{2m-1} consists exactly of the fibers of the classical Hopf map $S^{2m-1} \to \mathbf{C}P^{m-1}$. If M is orientable and ξ is a global unit normal field on M, then the vector field $U := -J\xi$ will be called the *Hopf vector field* on M (w.r.t. ξ). The following lemma is very useful for the study of Hopf hypersurfaces in Kähler manifolds.

Lemma. *Let M be an orientable real hypersurface of a Kähler manifold $\overline{M}$ and ξ a global unit normal field on M. Then M is a Hopf hypersurface of $\overline{M}$ if and only if the Hopf vector field U on M is a principal curvature vector of M everywhere.*

This lemma has been proved by Maeda [Ma] in the case of $\mathbf{C}P^m$, but Maeda's proof can be generalized for arbitrary Kähler manifolds without any difficulties.

Hopf hypersurfaces in non-Euclidean complex space forms possess an important property: the principal curvature function corresponding to the Hopf vector field is locally constant. Geometrically this implies that the integral curves of the Hopf vector field are circles in the ambient space (see [Be2], Chapter 5.2, for details and more results).

3. Hopf hypersurfaces with constant principal curvatures in non-Euclidean complex space forms

Firstly it should be remarked that in non-Euclidean complex space forms the notions of real hypersurfaces with constant principal curvatures and of isoparametric real hypersurfaces do not coincide. An example of an isoparametric real hypersurface of $\mathbf{C}P^m$ with non-constant principal curvatures can be found in [Wa1].

If not stated otherwise we denote in the following by M a connected real hypersurface with constant principal curvatures in a non-Euclidean complex space form $\overline{M}$. The number of distinct principal curvatures of M will be denoted by g. The first result concerning the classification problem of such hypersurfaces is due to Tashiro/Tachibana [TT], who proved that there are no totally umbilical real hypersurfaces in non-Euclidean complex space forms. Hence the case $g = 1$ cannot occur.

3.1 Classifications in complex projective spaces

We denote by $\mathbb{C}P^m$ the m-dimensional complex projective space of constant holomorphic sectional curvature 4.

The classifications for the cases $g \in \{2,3\}$ are completely known (Takagi: $g = 2$ [Ta2] and $g = 3$, $m \geq 3$ [Ta3]; Wang: $g = 3$, $m = 2$ [Wa2]):

Theorem 1. *Let M be a connected real hypersurface of $\mathbb{C}P^m$ $(m \geq 2)$ with two distinct constant principal curvatures. Then M is an open part of a geodesic hypersphere (distance sphere) in $\mathbb{C}P^m$.*

Theorem 2. *Let M be a connected real hypersurface of $\mathbb{C}P^m$ $(m \geq 2)$ with three distinct constant principal curvatures. Then M is holomorphic congruent to an open part of a tube around $\mathbb{C}P^k$ in $\mathbb{C}P^m$ for some $k \in \{1, \ldots, m-2\}$ or of a tube around the complex quadric Q^{m-1} in $\mathbb{C}P^m$.*

It should be noticed that in both theorems only Hopf hypersurfaces occur. Kimura [Ki] proved in 1986 that (expressing it in our notion) every connected Hopf hypersurface with constant principal curvatures in $\mathbb{C}P^m$ is an open part of a homogeneous real hypersurface of $\mathbb{C}P^m$. According to Takagi's [Ta1] classification of homogeneous real hypersurfaces of $\mathbb{C}P^m$ one therefore obtains the following classification.

Theorem 3. *Let M be a connected Hopf hypersurface of $\mathbb{C}P^m$ $(m \geq 2)$ with constant principal curvatures. Then M is holomorphic congruent to an open part of one of the following real hypersurfaces of $\mathbb{C}P^m$:*

(A) a tube of some radius $r \in]0, \frac{\pi}{2}[$ around the canonically (totally geodesic) embedded $\mathbb{C}P^k$ for some $k \in \{0 \ldots, m-1\}$,

(B) a tube of some radius $r \in]0, \frac{\pi}{4}[$ around the canonically embedded complex quadric $Q^{m-1} = SO(m+1)/SO(2) \times SO(m-1)$,

(C) a tube of some radius $r \in]0, \frac{\pi}{4}[$ around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^n$ in $\mathbb{C}P^m$, $m = 2n+1$,

(D) a tube of some radius $r \in]0, \frac{\pi}{4}[$ around the Plücker embedding of the complex Grassmann manifold $\mathbb{C}G_{2,3}$ in $\mathbb{C}P^9$,

(E) a tube of some radius $r \in]0, \frac{\pi}{4}[$ around the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^{15}$.

Remarks. a) A table of the principal curvatures of the model spaces (A)-(E) can be found in [Ta1].

b) The number g of distinct principal curvatures of the model spaces (A)-(E) satisfies always $g \in \{2, 3, 5\}$ ($g = 2$: type (A) for $k = 0$ and $k = m - 1$; $g = 3$: type (A) for $k \in \{1, \ldots, m-2\}$ and type (B)).

c) The tube of radius $r \in]0, \frac{\pi}{2}[$ around $\mathbf{C}P^k$ in $\mathbf{C}P^m$ is holomorphic congruent to the tube of radius $\frac{\pi}{2} - r$ around $\mathbf{C}P^{m-k-1}$ in $\mathbf{C}P^m$. Similarly, the tube of radius $r \in]0, \frac{\pi}{4}[$ around Q^{m-1} in $\mathbf{C}P^m$ can be regarded as the tube of radius $\frac{\pi}{4} - r$ around the canonically (totally geodesic) embedded m-dimensional real projective space $\mathbf{R}P^m$.

Problem 1. In order to obtain the classification of *all* real hypersurfaces of $\mathbf{C}P^m$ with constant principal curvatures it is of interest to know whether all of them are Hopf hypersurfaces.

3.2 Classifications in complex hyperbolic spaces

We denote by $\mathbf{C}H^m$ the m-dimensional complex hyperbolic space of constant holomorphic sectional curvature -4.

The corresponding result to theorem 1 has been obtained by Montiel [Mon].

Theorem 4. *Let M be a connected real hypersurface of $\mathbf{C}H^m$ $(m \geq 3)$ with two distinct constant principal curvatures. Then M is holomorphic congruent to an open part of one of the following real hypersurfaces of $\mathbf{C}H^m$: a geodesic hypersphere in $\mathbf{C}H^m$; a tube around $\mathbf{C}H^{m-1}$ in $\mathbf{C}H^m$; a tube of radius $ln(2+\sqrt{3})$ around $\mathbf{R}H^m$ in $\mathbf{C}H^m$; a horosphere in $\mathbf{C}H^m$.*

Problem 2. Is theorem 4 also valid for dimension $m = 2$?

The classification of Hopf hypersurfaces with constant principal curvatures in $\mathbf{C}H^m$ has been obtained recently by the author [Be1]:

Theorem 5. *Let M be a connected Hopf hypersurface of $\mathbf{C}H^m$ $(m \geq 2)$ with constant principal curvatures. Then M is holomorphic congruent to an open part of one of the following real hypersurfaces of $\mathbf{C}H^m$:*

(A) a tube of some radius $r \in \mathbf{R}_+$ around the canonically (totally geodesic) embedded $\mathbf{C}H^k$ for some $k \in \{0, \ldots, m-1\}$,

(B) a tube of some radius $r \in \mathbf{R}_+$ around the canonically (totally geodesic) embedded m-dimensional real hyperbolic space $\mathbf{R}H^m$,

(C) a horosphere in $\mathbf{C}H^m$.

Remarks. a) A table of the principal curvatures of the model spaces (A)-(C) can be found in [Be1].

b) The model spaces (A)-(C) always satisfy $g \in \{2, 3\}$.

c) The crucial point of the proof of theorem 5 is a *fundamental formula* for complex space forms which corresponds to the one given by E. Cartan [Ca] for real space forms. In the case of $\mathbf{C}H^m$ we use this formula to estimate the maximal number of distinct principal curvatures. An important role plays also Jacobi field theory which is used for calculating the shape operator of focal sets of Hopf hypersurfaces. It should be remarked that the whole proof of theorem 5 is "intrinsic", i.e. (in contrast to many other authors) we do not consider the Hopf map from anti-de Sitter space onto $\mathbf{C}H^m$.

Problem 3. (Compare with problem 1) Is every real hypersurface with constant principal curvatures in $\mathbf{C}H^m$ a Hopf hypersurface?

Problem 4. The classification of homogeneous real hypersurfaces of $\mathbf{C}H^m$ is not known until now. Since every homogeneous real hypersurface of $\mathbf{C}H^m$ has constant principal curvatures and all the spaces (A)-(C) are homogeneous real hypersurfaces of $\mathbf{C}H^m$, the following problem naturally arises: Is every homogeneous real hypersurface of $\mathbf{C}H^m$ a Hopf hypersurface? (An affirmative answer to problem 3 would be obviously affirmative to problem 4.)

4. Geometrical properties of Hopf hypersurfaces with constant principal curvatures

The Hopf hypersurfaces with constant principal curvatures can be regarded as the "simplest" real hypersurfaces of $\mathbf{C}P^m$ and $\mathbf{C}H^m$. Therefore it is of interest to study the influence of the anisotropy of $\mathbf{C}P^m$ and $\mathbf{C}H^m$ on the geometry of the model spaces occurring in theorems 3 and 5. We will state now some geometrical results concerning this topic (for details see [Be2]).

4.1 Geodesic hyperspheres are ordinary spheres with warped Hopf circles

Let M be a geodesic hypersphere (distance sphere) of radius r in $\overline{M} \in \{\mathbf{C}P^m, \mathbf{C}^m, \mathbf{C}H^m\}$, where $r \in]0, \frac{\pi}{2}[$ in the projective case and $r \in \mathbf{R}_+$ in both other cases. M is a Hopf hypersurface of $\overline{M}$ and the following properties are valid:

a) The Hopf foliation L of M consists of closed circles of the same perimeter.

b) The space of leaves M/L is a complex projective space and the canonical projection $\pi : M \to M/L$ is a Riemannian submersion.

c) The horizontal (w.r.t. π) geodesics of M are also closed circles of the same perimeter. The ratio between the perimeter of the Hopf circles and the perimeter of the horizontal circles is equal to

$$\begin{array}{ll} \cos(r) & , \text{ in the projective case} \\ 1 & , \text{ in the Euclidean case} \\ \cosh(r) & , \text{ in the hyperbolic case.} \end{array}$$

Summing up these facts we see that there exists always a "modified Hopf map" onto the complex projective space. But the anisotropy of $\mathbb{C}P^m$ and $\mathbb{C}H^m$ entails that the sphere M is warped in the direction of the Hopf circles.

Remark. Geodesic hyperspheres in $\mathbb{C}P^m$ and $\mathbb{C}H^m$ are Berger spheres (for $\mathbb{C}P^m$ see [We]).

4.2 Integrability of eigenbundles and curvature lines

In real space forms the eigenbundle corresponding to a constant principal curvature of a hypersurface is integrable [No]. This is no more valid if the ambient space is $\mathbb{C}P^m$ or $\mathbb{C}H^m$. Let M be one of the model spaces occurring in theorems 3 and 5. The Hopf vector field U of M is a principal curvature vector of M everywhere. The corresponding principal curvature α has multiplicity one except for the tube of radius $ln(2+\sqrt{3})$ around $\mathbb{R}H^m$ in $\mathbb{C}H^m$. For a principal curvature λ of M, $\lambda \neq \alpha$, we denote by T_λ the subbundle of TM consisting of all corresponding eigenspaces. For the exceptional model space mentioned above we allow $\lambda = \alpha$, but denote by T_λ the subbundle of TM consisting of all corresponding eigenspaces orthogonal to $\mathbb{R}U$. Then the following statements are valid for every such principal curvature λ of M:

a) T_λ is totally real or complex.

b) T_λ (resp. $T_\lambda \ominus \mathbb{R}U$) is integrable if and only if T_λ is totally real (resp. complex).

c) If T_λ is integrable, then each of its integral manifolds is totally geodesic in M and spherical (an extrinsic sphere) in the ambient space $\mathbb{C}P^m$ or $\mathbb{C}H^m$.

d) If $T_\lambda \ominus \mathbb{R}U$ is integrable, then each of its integral manifolds is totally geodesic in M and holomorphic congruent to a Hopf hypersurface with two distinct constant principal curvatures in $\mathbb{C}P^k$ or $\mathbb{C}H^k$, where $k := 1 + \dim_{\mathbb{C}} T_\lambda$.

e) If γ is a geodesic in M tangent to T_λ at one point, then γ is a curvature line, i.e. γ is tangent to T_λ at every point (even in the case of non-integrability of T_λ). These curvature lines are spherical curves (circles) in the ambient space $\mathbb{C}P^m$ or $\mathbb{C}H^m$.

4.3 Riemannian foliations on Hopf hypersurfaces

As Riemannian foliations (see e.g. [Mol], [To]) form a geometrically important class of foliations we ask under which conditions the Hopf foliation of a Hopf hypersurface is of that kind:

Theorem 6. *Let M be a connected Hopf hypersurface of $\mathbb{C}P^m$ or $\mathbb{C}H^m$ $(m \geq 2)$ on which the Hopf foliation is Riemannian. Then M is holomorphic congruent to an open part of one of the following real hypersurfaces:*

a) in case of $\mathbb{C}P^m$: a tube around $\mathbb{C}P^k$ for some $k \in \{0, \dots m-1\}$.

b) in case of $\mathbb{C}H^m$: a tube around $\mathbb{C}H^k$ for some $k \in \{0, \dots, m-1\}$ or a horosphere in $\mathbb{C}H^m$.

Let M be one of these model spaces and denote by L the Hopf foliation of M. Then it is well-known that the orbit space M/L can be equipped with a holomorphic structure and a Hermitian metric such that the canonical projection from M onto the Hermitian manifold M/L becomes a Riemannian submersion. In some important cases we describe M/L in the following table:

M	M/L
geodesic hypersphere of radius r in $\mathbb{C}P^m$	complex projective space $\mathbb{C}P^{m-1}(c)$ of constant holomorphic sectional curvature $c = 4/\sin^2(r)$
geodesic hypersphere of radius r in $\mathbb{C}H^m$	complex projective space $\mathbb{C}P^{m-1}(c)$ of constant holomorphic sectional curvature $c = 4/\sinh^2(r)$
horosphere in $\mathbb{C}H^m$	$\mathbb{C}^{m-1}$
tube of radius r around $\mathbb{C}H^{m-1}$	complex hyperbolic space $\mathbb{C}H^{m-1}(c)$ of constant holomorphic sectional curvature $c = -4/\cosh^2(r)$.

Remark. Even when the Hopf foliation L of one of the model spaces in theorems 3 and 5 is not Riemannian, the orbit space M/L can be constructed. It is easy to see that in the case of a tube M around the complex quadric Q^{m-1} in $\mathbb{C}P^m$ the space M/L is congruent to Q^{m-1} itself. If M is a tube around $\mathbb{R}H^m$ in $\mathbb{C}H^m$, then M/L is congruent to the non-compact dual space of the complex quadric Q^{m-1}, i.e. $M/L \simeq SO(1,m)/SO(1,1) \times SO(m-1)$.

4.4 Sasakian space forms realized as Hopf hypersurfaces in complex space forms

Sasakian structures play an important role in the study of odd-dimensional manifolds (for definition see e.g. [Bl]). In fact, there exists a strong relation between Kähler structures on even-dimensional manifolds and Sasakian structures (e.g. [Re]). For every real number c there exists a Sasakian space form of constant φ-sectional curvature c which is realized as a Hopf hypersurface with two distinct constant principal curvatures in a complex space form. More exactly: M is up to a homothetic change of the metric with factor λ a complete Sasakian space form of constant φ-sectional curvature c, where

M	λ	c
geodesic hypersphere of radius $r \in]0, \frac{\pi}{2}[$ in $\mathbb{C}P^m$	$\cot^2(r)$	$1 + 4\tan^2(r)$
geodesic hypersphere of radius 1 in $\mathbb{C}^m$	1	1
geodesic hypersphere of radius $r \in \mathbb{R}_+$ in $\mathbb{C}H^m$	$\coth^2(r)$	$1 - 4\tanh^2(r)$
horosphere in $\mathbb{C}H^m$	1	-3
tube of radius $r \in \mathbb{R}_+$ around $\mathbb{C}H^{m-1}$ in $\mathbb{C}H^m$	$\tanh^2(r)$	$1 - 4\coth^2(r)$

Moreover, all spaces except for the tubes around $\mathbb{C}H^{m-1}$ in $\mathbb{C}H^m$ (which have fundamental group isomorphic to $\mathbb{Z}$) are simply connected. In all cases the structural vector field of the Sasakian space form is exactly the Hopf vector field of the Hopf hypersurface.

Remark. We now consider the $(2m+1)$-dimensional anti-de Sitter space

$$H^{2m+1} := \{(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} \mid -z_0\bar{z}_0 + \sum_{i=1}^{m} z_i\bar{z}_i = -1\} \quad ,$$

which is well known as the bundle space of the usual Hopf map $\pi : H^{2m+1} \to \mathbb{C}H^m$. H^{2m+1} is a Lorentzian Hopf hypersurface of $\mathbb{C}^{m+1}$ with Hopf vector field as a distinguished time-like unit vector field on H^{2m+1} tangent to the fibers of π. By a canonical process we can switch H^{2m+1} into a Riemannian manifold (H^{2m+1}, g): the Hopf vector field becomes a (space-like) unit vector field, g coincides on the horizontal subbundle $\mathcal{H}$ of π with the original metric on H^{2m+1} and the Hopf vector field remains orthogonal to $\mathcal{H}$.

Theorem 7. *$(H^{2m+1}, 2g)$ is isometric to the tube of radius $ln(1+\sqrt{2})$ around $\mathbb{C}H^m$ in $\mathbb{C}H^{m+1}$. Moreover, (H^{2m+1}, g) is a Sasakian space form of constant φ-sectional curvature -7.*

Proof: The isometry is

$$(H^{2m+1}, 2g) \to M \quad , \quad z \mapsto [\sqrt{2}z, 1] \quad ,$$

where $[\sqrt{2}z, 1]$ is the image of $(\sqrt{2}z, 1) \in H^{2m+3}$ under the Hopf map $H^{2m+3} \to CH^{m+1}$. By the preceding results in 4.4 the other statement is clear. $\square$

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Mathematisches Institut
Universität zu Köln
Weyertal 86-90
5000 Köln 41
Federal Republic of Germany