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NATURALLY REDUCTIVE RIEMANNIAN HOMOGENEOUS SPACES AND REAL HYPERSURFACES IN COMPLEX AND QUATERNIONIC SPACE FORMS

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ABSTRACT. We prove that the η-umbilical real hypersurfaces in non-flat complex space forms and the Q-quasiumbilical real hypersurfaces in non-flat quaternionic space forms are equipped with a naturally reductive homogeneous structure. Moreover, we show that all simply connected, non-symmetric, three-dimensional naturally reductive Riemannian homogeneous spaces can be realized via standard models of η-umbilical real hypersurfaces in complex projective and hyperbolic spaces of complex dimension two.

KEYWORDS. Complex and quaternionic space forms, η-umbilical and Q-quasiumbilical real hypersurfaces, naturally reductive homogeneous spaces and structures.

MS CLASSIFICATION. 53B20, 53C30, 53C40, 53C55.

1. INTRODUCTION

In [1] W. Ambrose and I.M. Singer provided an infinitesimal characterization of Riemannian homogeneous spaces which extends that of E. Cartan for symmetric spaces. Using this theory, F. Tricerri and the second author [17] introduced the concept of homogeneous structures and used decomposition theory of spaces of tensors to characterize among the Riemannian homogeneous spaces the naturally reductive ones by a simple additional property of their homogeneous structures. The concept of (naturally reductive) homogeneous structures constitutes the methodical background in this note. We shall now summarize our results.

We prove that every η-umbilical real hypersurface in a non-Euclidean Kähler manifold of constant holomorphic sectional curvature can be equipped with a naturally reductive homogeneous structure (Theorem 1). From this we conclude that every

- geodesic hypersphere in a complex projective space or a complex hyperbolic space,
- horosphere in a complex hyperbolic space;
- universal covering space of a tube about a totally geodesic complex hyperbolic hyperplane in a complex hyperbolic space

This paper is in final form and no version of it will be submitted for publication elsewhere.

is a non-symmetric naturally reductive Riemannian homogeneous space (Corollary 1). For the geodesic hyperspheres this is already known from the work of W. Ziller in [20]. Every horosphere in a complex hyperbolic space is isometric to a Heisenberg group (see for instance [6]), which is also known to be naturally reductive (see [9]). However, it seems to us not to be known that the latter spaces are naturally reductive.

O. Kowalski [11] has provided a group-theoretical classification of all simply connected three-dimensional naturally reductive Riemannian homogeneous spaces. Using his classification it turns out (Theorem 2) that every non-symmetric, simply connected, three-dimensional naturally reductive Riemannian homogeneous space is isometric to one of the above examples (where, of course, the ambient space has to be of complex dimension two). So we see that these particular homogeneous spaces can be realized geometrically in a nice way.

We then study the quaternionic analogue and prove that every Q-quasiumbilical real hypersurface in a non-Euclidean quaternionic Kähler manifold of constant quaternionic sectional curvature can be equipped with a naturally reductive homogeneous structure (Theorem 3). This leads to the conclusion that every

- geodesic hypersphere in a quaternionic projective space or a quaternionic hyperbolic space;
- horosphere in a quaternionic hyperbolic space;
- tube about a totally geodesic quaternionic hyperbolic hyperplane in a quaternionic hyperbolic space

is a non-symmetric naturally reductive Riemannian homogeneous space.(Corollary 2). Also here the latter examples for naturally reductive spaces seem to be new for us. The geodesic hyperspheres have been treated by W. Ziller [20]. And every horosphere in a quaternionic hyperbolic space is isometric to a generalized Heisenberg group with three-dimensional center (see [6]), which is known to be naturally reductive (see [9]).

The article is organized in the following way. In Section 2 we recall the basic notion of homogeneous structure. Section 3 is divided into two parts. Firstly, we provide basic material about real hypersurfaces in complex space forms. In the second part we prove the above results concerning the η-umbilical real hypersurfaces. Finally, in Section 4 we study the quaternionic analogue.

2. Homogeneous structures

In this section we recall briefly the notion of homogeneous structure (see [17] for more details).

A homogeneous structure on a Riemannian manifold (M, g) is a tensor field T of type (1,2) such that

$$(1) \qquad g(T_X Y, Z) + g(Y, T_X Z) = 0,$$

$$(2) \quad (\nabla_W R)(X,Y)Z = T_W R(X,Y)Z - R(T_W X,Y)Z - R(X,T_W Y)Z - R(X,Y)T_W Z,$$

$$(3) \qquad (\nabla_X T)_Y Z = T_X T_Y Z - T_Y T_X Z - T_{T_X Y} Z$$

for all vector fields W, X, Y, Z on M. Here, ∇ is the Levi Civita connection and R is the Riemannian curvature tensor of (M, g). If we put $\tilde{\nabla} := \nabla - T$, then the conditions (1), (2) and (3) are equivalent to $\tilde{\nabla} g = 0, \tilde{\nabla} R = 0$ and $\tilde{\nabla} T = 0$, respectively.

A Riemannian manifold on which there exists a homogeneous structure is a locally homogeneous Riemannian manifold. W. Ambrose and I.M. Singer [1] have proved that a complete and simply connected Riemannian manifold is homogeneous if and only if it admits a homogeneous structure. *Naturally reductive Riemannian homogeneous spaces* are characterized by the additional property

$$(4) \qquad T_X X = 0$$

for all vector fields X [17, Chapter 6], that is, by means of a *naturally reductive homogeneous structure.*

3. Real hypersurfaces in complex space forms

a) Preliminaries

Let $\bar{M}$ be an m-dimensional ($m \geq 2$) Kähler manifold of constant holomorphic sectional curvature $c \in \mathbb{R} \setminus \{0\}$. The standard models for such non-Euclidean complex space forms are the complex projective space $\mathbb{C}P^m(c)$ (for $c > 0$) and the complex hyperbolic space $\mathbb{C}H^m(c)$ (for $c < 0$). We denote by g the Riemannian metric and by $\bar{\nabla}$ the Levi Civita connection of $\bar{M}$. J is the complex structure and Ω the corresponding Kähler form on $\bar{M}$ defined by $\Omega(X, Y) := g(X, JY)$. The Riemannian curvature tensor $\bar{R}$ of $\bar{M}$ is given by

$$\bar{R}(X,Y)Z = \frac{c}{4}(g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ).$$

Next, let M be an orientable real hypersurface in $\bar{M}$ and ξ a unit normal field on M. We also denote by g the induced Riemannian metric on M. ∇ is the Levi Civita connection and R the Riemannian curvature tensor of (M, g). Further, A denotes the shape operator of M with respect to ξ. We define a unit vector field U on M by

$$U := -J\xi$$

and denote the corresponding one-form by η, that is,

$$\eta(X) := g(X, U) \quad (= -\Omega(X, \xi)).$$

Let P be the skew-symmetric tensor field of type $(1,1)$ on M characterized by

$$JX = PX + \eta(X)\xi$$

for all vector fields X on M. PX is the tangential component of JX and we have

$$(5) \qquad P^2 X = -X + \eta(X)U.$$

For the readers convenience we formulate explicitely the fundamental equations for submanifolds adapted to our special situation (see for instance [19]) :

the *equation of Gauss :*

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX,Y)\xi;$$

the *equation of Weingarten :*

$$\bar{\nabla}_X \xi = -AX;$$

the *equation of Gauss of second order :*

$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}(g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX - g(PX,Z)PY - 2g(PX,Y)PZ) \\ &+ g(AY,Z)AX - g(AX,Z)AY; \end{aligned}$$

the *equation of Codazzi :*

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}(\eta(X)PY - \eta(Y)PX - 2g(PX,Y)U).$$

Evaluating the tangential part of both sides of $\bar{\nabla}_X(JY) = J\bar{\nabla}_X Y$ by means of the equations of Gauss and Weingarten yields

$$(\nabla_X P)Y = \eta(Y)AX - g(AX,Y)U. \tag{6}$$

Analogously, from $\bar{\nabla}_X(J\xi) = J\bar{\nabla}_X \xi$ we get

$$\nabla_X U = PAX, \tag{7}$$

and hence

$$(\nabla_X \eta)Y = g(PAX,Y). \tag{8}$$

b) η-umbilical real hypersurfaces and naturally reductive homogeneous structures

The Codazzi equation for a real hypersurface in a non-flat complex space form $\bar{M}$ is not trivial, that is, ∇A is not symmetric in its variables. As a consequence there are no umbilical real hypersurfaces in non-Euclidean complex space forms of complex dimension greater than one, a fact already observed by Y. Tashiro and S. Tachibana [16]. Further, the shape operator of a geodesic hypersphere in $\bar{M}$ is of the form

$$AX = \lambda X + \mu\eta(X)U \tag{9}$$

for some $\lambda, \mu \in \mathbf{R}$ (see below). This suggests the following definition [10] : An orientable real hypersurface M of $\bar{M}$ is called *η-umbilical* if its shape operator A is of the form (9) for some functions λ and μ on M.

Let M be a connected η-umbilical real hypersurface in $\bar{M}$. Then the functions λ and μ are constant and so M has two distinct constant principal curvatures λ and $\alpha := \lambda + \mu$

of multiplicities $2(m-1)$ and 1, respectively [10, Lemma 2.5]. The corresponding spaces of principal curvature vectors are $\mathcal{D}$ and $J(\perp M)(=\mathbb{R}U)$, where $\perp M$ is the normal bundle of M in $\bar{M}$ and $\mathcal{D}$ is the orthogonal complement of $J(\perp M)$ in the tangent bundle TM of M. The model spaces for η-umbilical real hypersurfaces in $\mathbb{C}P^m(c)$ and $\mathbb{C}H^m(c)$ and their principal curvatures are drawn up in the following Table 1 (see [2] for more details concerning the explicit computation of α and λ for these model spaces):

M	α	λ
geodesic hypersphere of radius $r \in]0,\pi/\sqrt{c}[$ in $\mathbb{C}P^m(c)$	$\sqrt{c}\cot(\sqrt{c}r)$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$
geodesic hypersphere of radius $r \in \mathbb{R}_+$ in $\mathbb{C}H^m(c)$	$\sqrt{-c}\coth(\sqrt{-c}r)$	$\frac{\sqrt{-c}}{2}\coth(\frac{\sqrt{-c}}{2}r)$
horosphere in $\mathbb{C}H^m(c)$	$\sqrt{-c}$	$\frac{\sqrt{-c}}{2}$
tube of radius $r \in \mathbb{R}_+$ about $\mathbb{C}H^{m-1}(c)$ in $\mathbb{C}H^m(c)$	$\sqrt{-c}\coth(\sqrt{-c}r)$	$\frac{\sqrt{-c}}{2}\tanh(\frac{\sqrt{-c}}{2}r)$

Table 1. η-umbilical real hypersurfaces in $\mathbb{C}P^m(c)$ and $\mathbb{C}H^m(c)$

Remarks. 1. It has been proved by M. Kon [10, Theorem 3.2] (for $c>0$) and S. Montiel and A. Romero [14, Corollary 5.3] (for $c<0$) that every η-umbilical real hypersurface in a non-flat complex space form is locally isometric to one of these model spaces.

2. Note that $\pi/\sqrt{c}$ is the injectivity radius of $\mathbb{C}P^m(c)$. The set of points at a distance of $\pi/\sqrt{c}$ from a fixed point in $\mathbb{C}P^m(c)$ is a totally geodesic embedding of $\mathbb{C}P^{m-1}(c)$ in $\mathbb{C}P^m(c)$. Thus a geodesic hypersphere of radius $r \in]0,\pi/\sqrt{c}[$ in $\mathbb{C}P^m(c)$ can also be regarded as a tube of radius $\pi/\sqrt{c}-r$ about $\mathbb{C}P^{m-1}(c)$ in $\mathbb{C}P^m(c)$ (see [4, p. 493] for more details).

3. In the above table $\mathbb{C}H^{m-1}(c)$ is the standard totally geodesic embedding of $\mathbb{C}H^{m-1}(c)$ in $\mathbb{C}H^m(c)$. Every tube M about $\mathbb{C}H^{m-1}(c)$ in $\mathbb{C}H^m(c)$ is an S^1-bundle over $\mathbb{C}H^{m-1}(c)$. As $\mathbb{C}H^{m-1}(c)$ is contractible to a point, M must be diffeomorphic to $\mathbb{C}^{m-1}\times S^1$. Hence the fundamental group of M is isomorphic to $\mathbb{Z}$. The other model spaces in Table 1 are simply connected : Every geodesic hypersphere is obviously diffeomorphic to an ordinary sphere, a horosphere in $\mathbb{C}H^m(c)$ is diffeomorphic to $\mathbb{R}^{2m-1}$.

Following the method of proof for geodesic hyperspheres used by F. Tricerri and the second author in [18] we have

Theorem 1. *Let M be an η-umbilical real hypersurface in a Kähler manifold of constant holomorphic sectional curvature $c \in \mathbb{R}\setminus\{0\}$. Then*

$$T_XY := \lambda(\eta(Y)PX - \eta(X)PY - g(PX,Y)U) \tag{10}$$

defines a naturally reductive homogeneous structure on M.

Proof. Equations (1) and (4) are consequences of the skew-symmetry of P. Next, inserting (9) into the Gauss equation of second order and using the fact that $\lambda\mu + c/4 = 0$ (which can be deduced easily from Table 1) we get

$$\begin{aligned} R(X,Y)Z =& (\lambda^2 + \frac{c}{4})(g(Y,Z)X - g(X,Z)Y) \\ &+ \frac{c}{4}(g(PY,Z)PX - g(PX,Z)PY - 2g(PX,Y)PZ \\ &\quad - g(Y,Z)\eta(X)U + g(X,Z)\eta(Y)U - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y). \end{aligned}$$

Using this expression for R and (6) - (10) a straightforward calculation shows that (2) is valid. Eventually, equation (3) follows by a simple computation using (5) - (10). □

Remark. Note that (10) is equivalent to

$$g(T_X Y, Z) = 3\lambda(\eta \wedge \Omega)(X, Y, Z),$$

and so $T_X X = 0$ trivially.

Every model space in Table 1 can be realized as a level set of a certain function and is therefore closed and hence complete : A geodesic hypersphere is a level set of a distance function to a point, a horosphere in $\mathbb{C}H^m(c)$ arises as a level set of a Busemann function on $\mathbb{C}H^m(c)$, and a tube about $\mathbb{C}H^{m-1}(c)$ in $\mathbb{C}H^m(c)$ is a level set of the distance function to $\mathbb{C}H^{m-1}(c)$ on $\mathbb{C}H^m(c)$. The completeness of these model spaces follows also from their homogeneity : Given any two points p and q on the model space M, the two-point homogeneity of the ambient space $\bar{M}$ implies that there is an isometry (not uniquely determined) f of $\bar{M}$ with $f(p) = q$ and $f_*\xi_p = \xi_q$. It is not hard to see that such an isometry maps M into itself and hence induces an isometry on M (see also [2, Section 6.2]). Combining now Theorem 1 and the above mentioned result of Ambrose and Singer we get

Corollary 1. *Every*
- *geodesic hypersphere in a complex projective space or a complex hyperbolic space;*
- *horosphere in a complex hyperbolic space;*
- *universal covering space of a tube about a totally geodesic complex hyperbolic hyperplane in a complex hyperbolic space*

is a (simply connected) naturally reductive Riemannian homogeneous space.

In [11] O. Kowalski gave a group-theoretical classification of all simply connected, three-dimensional naturally reductive Riemannian homogeneous spaces (see also [17]). We shall now see that the non-symmetric ones are precisely those spaces listed in Corollary 1 which are of dimension three.

Theorem 2. *Let N be a three-dimensional, simply connected, naturally reductive Riemannian homogeneous space. Then N is isometric to one of the following spaces :*
(I) a symmetric space (more precisely, to $S^3, \mathbb{E}^3, \mathbb{R}H^3, S^2 \times \mathbb{R}$ or $\mathbb{R}H^2 \times \mathbb{R}$);
(II) a geodesic hypersphere of some radius $r \in]0, \pi/\sqrt{c}[$ in $\mathbb{C}P^2(c)$ or a geodesic hypersphere of some radius $r \in \mathbb{R}_+$ in $\mathbb{C}H^2(c)$;

(III) the universal covering space of a tube of some radius $r \in \mathbb{R}_+$ about $\mathbb{C}H^1(c)$ in $\mathbb{C}H^2(c)$;

(IV) a horosphere in $\mathbb{C}H^2(c)$.

If N is not simply connected, it is locally isometric to one of these model spaces.

Proof. In the course of proving Theorem 1 we have already calculated an expression for the Riemannian curvature tensor of an η-umbilical real hypersurface M. From this expression we get for the Ricci tensor Ric of M

$$RicX = aX - \frac{c}{2}m\eta(X)U \tag{11}$$

with

$$a := 2(m-1)\lambda^2 + \frac{c}{2}m.$$

In particular, the Ricci tensor of M has the two distinct eigenvalues a and $2(m-1)\lambda^2$ of multiplicity $2(m-1)$ and 1, respectively.

Due to O. Kowalski [11] N is isometric to a symmetric space or to (G,g), where G is one of the following groups equipped with a suitable left-invariant metric g : the special unitary group $SU(2)$, the universal covering group $SL(2,\mathbb{R})^\sim$ of $SL(2,\mathbb{R})$, or the three-dimensional Heisenberg group H_3. Moreover, the Ricci tensor of (G,g) has two distinct eigenvalues a and b of multiplicity two and one, respectively, with

$$\begin{aligned} &b > 0 \text{ and } a + b > 0, \text{ if } G = SU(2); \\ &b > 0 \text{ and } a + b < 0, \text{ if } G = SL(2,\mathbb{R})^\sim; \\ &b > 0 \text{ and } a = -b, \quad \text{if } G = H_3. \end{aligned}$$

We assume that N is non-symmetric. Then N must be isometric to one of the above spaces (G,g). At first we assume that $G = SU(2)$ and put $c := a - b$. If $c > 0$, there exists precisely one $r \in]0, \pi/\sqrt{c}[$ such that $b = (c/2)\cot^2(\sqrt{c}r/2)$. By means of Table 1 and (11) a geodesic hypersphere of radius r in $\mathbb{C}P^2(c)$ has the two distinct Ricci roots a and b of multiplicity two and one, respectively. As a geodesic hypersphere in $\mathbb{C}P^2(c)$ is diffeomorphic to S^3 and hence simply connected, the uniqueness part of Kowalski's classification yields that $(SU(2),g)$ is isometric to a geodesic hypersphere of this particular radius r in $\mathbb{C}P^2(c)$. Analogously, if $c < 0$, we get that $(SU(2),g)$ is isometric to a geodesic hypersphere of radius $r \in \mathbb{R}_+$ in $\mathbb{C}H^2(c)$, where r is uniquely determined by $b = -(c/2)\coth^2(\sqrt{-c}r/2)$. Along the same line of argumentation we get that $(SL(2,\mathbb{R})^\sim, g)$ is isometric to the universal covering of the tube of radius r about $\mathbb{C}H^1(c)$ in $\mathbb{C}H^2(c)$, where c and r are given by $c := a - b$ and $b = -(c/2)\tanh^2(\sqrt{-c}r/2)$, and that (H_3, g) is isometric to a horosphere in $\mathbb{C}H^2(-2b)$. □

Remark. From Theorem 2 and the classification of all four-dimensional naturally reductive Riemannian homogeneous spaces in [12] we also get the following conclusion : *The non-symmetric, simply connected, four-dimensional naturally reductive Riemannian homogeneous spaces are precisely the Riemannian products $N \times \mathbb{R}$, where N is one of the model spaces (II), (III) or (IV) in Theorem 2.*

4. Real hypersurfaces in quaternionic space forms

We now study the analogous situation in non-flat quaternionic space forms.

a) Preliminaries

Let $\bar{M}$ be an m-dimensional ($m \geq 2$) quaternionic Kähler manifold of constant quaternionic sectional curvature $c \in \mathbb{R} \setminus \{0\}$. The standard models for such spaces are the quaternionic projective space $\mathbb{H}P^m(c)$ (for $c > 0$) and the quaternionic hyperbolic space $\mathbb{H}H^m(c)$ (for $c < 0$). Let g be the Riemannian metric, $\bar{\nabla}$ the Levi Civita connection and $\mathcal{J}$ the quaternionic Kähler structure of $\bar{M}$. The Riemannian curvature tensor $\bar{R}$ of $\bar{M}$ is locally of the form

$$\bar{R}(X,Y)Z = \frac{c}{4}\Big(g(Y,Z)X - g(X,Z)Y + \sum_{i=1}^{3}\big(g(J_iY,Z)J_iX - g(J_iX,Z)J_iY - 2g(J_iX,Y)J_iZ\big)\Big),$$

where J_1, J_2, J_3 is a canonical local basis of $\mathcal{J}$ (see [8] for more details).

Next, let M be an orientable real hypersurface in $\bar{M}$ and ξ a unit normal field on M. The induced Riemannian metric on M will also be denoted by g. We denote by A the shape operator of M with respect to ξ, by ∇ the Levi Civita connection and by R the Riemannian curvature tensor of (M,g). Let J_1, J_2, J_3 be a canonical local basis of $\mathcal{J}$ such that

$$U_i := -J_i\xi$$

is a vector field on an open subset V of $M (i = 1,2,3)$. The corresponding one-form will be denoted by η_i, that is,

$$\eta_i(X) = g(X, U_i).$$

Let P_i be the skew-symmetric tensor field of type (1,1) on V characterized by

$$J_iX = P_iX + \eta_i(X)\xi$$

for all vector fields X on V. P_iX is the tangential component of J_iX and we have (here, and henceforth, the index has to be taken modulo three)

$$(12) \quad \begin{cases} P_iU_i = 0, \quad P_iU_{i+1} = U_{i+2}, \quad P_iU_{i+2} = -U_{i+1}, \\ P_i^2X = -X + \eta_i(X)U_i, \\ P_iP_{i+1}X = P_{i+2}X + \eta_{i+1}(X)U_i, \quad P_iP_{i+2}X = -P_{i+1}X + \eta_{i+2}(X)U_i. \end{cases}$$

The equations of Gauss and Weingarten are like the ones in the complex case. The equation of Gauss of second order and the equation of Codazzi are locally of the form

$$R(X,Y)Z = \frac{c}{4}\Big(g(Y,Z)X - g(X,Z)Y + \sum_{i=1}^{3}\big(g(P_iY,Z)P_iX - g(P_iX,Z)P_iY - 2g(P_iX,Y)P_iZ\big)\Big)$$
$$+ g(AY,Z)AX - g(AX,Z)AY$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\sum_{i=1}^{3}(\eta_i(X)P_iY - \eta_i(Y)P_iX - 2g(P_iX,Y)U_i),$$

respectively. As $\mathcal{J}$ is parallel, there exist local one-forms q_1, q_2, q_3 on $\bar{M}$ such that

$$\bar{\nabla}_X J_i = q_{i+2}(X)J_{i+1} - q_{i+1}(X)J_{i+2} \tag{13}$$

for all vector fields X on $\bar{M}$. Evaluating the tangential part of $(\bar{\nabla}_X J_i)Y$ by means of (13) on the one hand, and by means of the equations of Gauss and Weingarten on the other hand, gives

$$(\nabla_X P_i)Y = \eta_i(Y)AX - g(AX,Y)U_i + q_{i+2}(X)P_{i+1}Y - q_{i+1}(X)P_{i+2}Y. \tag{14}$$

Also, from (13) and the equations of Gauss and Weingarten we get

$$\nabla_X U_i = q_{i+2}(X)U_{i+1} - q_{i+1}(X)U_{i+2} + P_iAX, \tag{15}$$

and hence

$$(\nabla_X \eta_i)Y = q_{i+2}(X)\eta_{i+1}(Y) - q_{i+1}(X)\eta_{i+2}(Y) + g(P_iAX,Y). \tag{16}$$

b) Q-quasiumbilical real hypersurfaces and naturally reductive homogeneous structures

¿From the Codazzi equation it can be deduced that there are no umbilical real hypersurfaces in non-flat quaternionic space forms of quaternionic dimension greater than one (see for example [5]). Thus a geodesic hypersphere in $\bar{M}$ is not umbilical. Its shape operator A is locally of the form

$$AX = \lambda X + \mu\sum_{i=1}^{3}\eta_i(X)U_i \tag{17}$$

for some $\lambda, \mu \in \mathbb{R}$ (see for instance [3]). Following C.S. Houh [7] we call a connected real hypersurface M in $\bar{M}$ *Q-quasiumbilical* if its shape operator is locally of the form (17) with some functions λ and μ on M. J.S. Pak [15, Theorem 4] proved that on every Q-quasiumbilical real hypersurface M in $\bar{M}$ the functions λ and μ are constant and

$$\lambda\mu + \frac{c}{4} = 0. \tag{18}$$

(Note that Pak used another sign convention.) Therefore every Q-quasiumbilical real hypersurface M in a non-flat quaternionic space form has two distinct principal curvatures, namely λ and $\alpha := \lambda + \mu$, with multiplicities $4(m-1)$ and 3, respectively. The corresponding spaces of principal curvature vectors are $\mathcal{D}$ and $\mathcal{J}(\perp M)(= \mathbb{R}U_1 \oplus \mathbb{R}U_2 \oplus \mathbb{R}U_3$ locally), where $\mathcal{D}$ is the orthogonal complement of $\mathcal{J}(\perp M) = \{J\xi | J \otimes \xi \in \mathcal{J} \otimes \perp M\}$ in TM. The standard models of Q-quasiumbilical real hypersurfaces in $\mathbb{H}P^m(c)$ and $\mathbb{H}H^m(c)$ are :

- a geodesic hypersphere of radius $r \in]0, \pi/\sqrt{c}[$ in $\mathbb{H}P^m(c)$;
- a geodesic hypersphere of radius $r \in \mathbb{R}_+$ in $\mathbb{H}H^m(c)$;
- a horosphere in $\mathbb{H}H^m(c)$;
- a tube of radius $r \in \mathbb{R}_+$ about the standard totally geodesic embedding of $\mathbb{H}H^{m-1}(c)$ in $\mathbb{H}H^m(c)$.

Every Q-quasiumbilical real hypersurface in $\bar{M}$ is locally congruent to one of the above model spaces (for $c > 0$ see [13, Theorem 5.7], for $c < 0$ see [3, Theorem 2]).

We now prove the quaternionic analogue of Theorem 1.

Theorem 3. *Let M be a Q-quasiumbilical real hypersurface in a quaternionic Kähler manifold of constant quaternionic sectional curvature $c \in \mathbb{R} \setminus \{0\}$. Then the tensor field T on M, which is locally given by*

$$
\begin{aligned}
T_X Y = &\lambda \sum_{i=1}^{3} (\eta_i(Y) P_i X - \eta_i(X) P_i Y - g(P_i X, Y) U_i) \\
&- \mu \sum_{i=1}^{3} (\eta_{i+1}(X)\eta_{i+2}(Y) - \eta_{i+2}(X)\eta_{i+1}(Y)) U_i,
\end{aligned}
\tag{19}
$$

is a naturally reductive homogeneous structure on M.

Proof. The proof is analogous to the one of Theorem 1, but needs some more effort. Equations (1) and (4) are consequences of the skew-symmetry of P_1, P_2 and P_3. To see the validity of (2) one may proceed as follows :

Step 1. Insert (17) into the equation of Gauss of second order and use (18) to obtain an expression for R.

Step 2. Insert (17) into (14), (15) and (16) and use these formulae to compute $(\nabla_W R)(X,Y)Z$ with R as calculated in Step 1.

Step 3. Compute the right-hand side of (2) with R as calculated in Step 1.

Step 4. The resulting expression involves terms of the form

$$(T_W P_i - P_i T_W)X \ , \ \eta_i(T_W X) \text{ and } T_W U_i,$$

which can be computed by means of the explicit expression of T in (19) and the formulae in (12).

Step 5. Insert these expressions into the one computed in Step 3. This gives precisely the formula derived in Step 2, whence (2) is shown.

The validity of (3) can be seen in this way : First compute the left-hand side of (3) from (19) using (14)-(16) and then the right-hand side of (3) from (19) using (12). Both sides coincide. □

Remark. The tensor field T defined in (19) satisfies

$$g(T_X Y, Z) = 3\lambda \sum_{i=1}^{3} (\eta_i \wedge \Omega_i)(X,Y,Z) - 6\mu(\eta_1 \wedge \eta_2 \wedge \eta_3)(X,Y,Z),$$

where $\Omega_i(X,Y) := g(X, J_iY)$.

Every geodesic hypersphere in $\mathbb{H}P^m(c)$ or $\mathbb{H}H^m(c)$ is diffeomorphic to S^{4m-1}, every horosphere in $\mathbb{H}H^m(c)$ is diffeomorphic to $\mathbb{R}^{4m-1}$, and every tube about $\mathbb{H}H^{m-1}(c)$ in $\mathbb{H}H^m(c)$ is diffeomorphic to $\mathbb{H}^{m-1} \times S^3$. Thus all the above model spaces for Q-quasiumbilical real hypersurfaces in $\mathbb{H}P^m(c)$ and $\mathbb{H}H^m(c)$ are simply connected. As in the complex case the completeness of these spaces can be deduced both from their homogeneity and from the fact that they arise as level sets of certain functions. So Theorem 3 combined with the result of Ambrose and Singer yields

Corollary 2. *Every*
- *geodesic hypersphere in a quaternionic projective space or a quaternionic hyperbolic space;*
- *horosphere in a quaternionic hyperbolic space;*
- *tube about a totally geodesic quaternionic hyperbolic hyperplane in a quaternionic hyperbolic space*

is a (simply connected) naturally reductive Riemannian homogeneous space.

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