Minimal Liouville surfaces in Euclidean spaces

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Abstract. We determine all simply connected holomorphic Liouville curves in $\mathbb{C}^n$ $(n \geq 3)$ and deduce from this the classification of all minimal Liouville surfaces in $\mathbb{R}^3$.

1. Introduction.

Let S be a two-dimensional Riemannian manifold. A local coordinate system (x,y) on S is called a *Liouville coordinate system* if the metric of S is of the form

$$ds^2 = (f(x) + g(y))(dx^2 + dy^2) \tag{1.1}$$

for suitable smooth functions f(x) and g(y). Also, S is a *Liouville surface* if it may be covered by a family of Liouville coordinate neighbourhoods. Rotational surfaces in $\mathbb{R}^3$ are examples of Liouville surfaces, the meridians and parallels of latitude are the coordinate lines of a Liouville coordinate neighbourhood. The local theory of Liouville surfaces has been investigated by, for instance, Liouville [12], Dini [8], Stäckel [13,14], Koenigs [11] and Walser [16], who discovered many interesting properties. We review some of them in Section 2, together with some of the more recent global developments due to Viesel [15] and Kiyohara [10].

In 1940, Alt [1] showed that any minimal Liouville surface in $\mathbb{R}^3$ is locally isometric to a rotational surface in $\mathbb{R}^3$. The simplest examples of minimal Liouville surfaces in $\mathbb{R}^3$, apart from the plane, are the catenoid and helicoid. These and their associates have the property that the one-parameter family of isometries of the surface extends to a one-parameter family of rigid motions of $\mathbb{R}^3$. Indeed, these are the only minimal surfaces with this property. Thus all other examples, which include Enneper's surface, are examples of Bonnet surfaces (see [5, p. 72-92]).

Alt's proof of his theorem has two main ingredients. Firstly, he uses the fact that a simply connected minimal surface S in $\mathbb{R}^3$ is the real part of a holomorphic curve $\varphi : S \to \mathbb{C}^3$ with isotropic differential and obtains an expression for the form φ must take if S is to be a Liouville surface in the induced metric. Secondly, he uses a criterion (due to

Darboux) to obtain conditions that must be satisfied by the Weierstraß representation of a minimal Liouville surface in $\mathbb{R}^3$. Putting these together, he obtains an explicit expression for the Weierstraß representation of any minimal Liouville surface and then deduces his theorem.

In this note we present an alternative proof of Alt's result. There are similarities, in that we also obtain conditions on a holomorphic curve φ which has a minimal Liouville surface as its real part, but we then use isotropy conditions to obtain an explicit expression for the Weierstraß representation of a minimal Liouville surface in $\mathbb{R}^3$. As a consequence of this description it is not hard to see that any minimal Liouville surface is an open subset of a minimally immersed complete Liouville $\mathbb{R}^2$ which admits a one-parameter group of isometries.

In Section 2 we give a brief history of the study of Liouville surfaces. In Section 3 we determine all simply connected holomorphic Liouville curves in $\mathbb{C}^n$, and then use this to prove Alt's result in Section 4. The arguments we give rely on elementary facts from linear algebra rather than the arguments involving differential equations and polynomials given by Alt. We also give, in Section 5, an example of a minimal Liouville surface in $\mathbb{R}^4$ which is not locally isometric to a surface of revolution.

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2. A brief history.

In this section we shall outline some features of Liouville surfaces to recall the importance of such surfaces in geometry.

a) *Integrability of the geodesic flow.* In 1839 C.G.J. Jacobi [9] succeeded in integrating the differential equations of the geodesics on a triaxial ellipsoid by use of a method developed by him and W.R. Hamilton, and which is generally known nowadays as Hamilton-Jacobi theory. The basic idea was to introduce suitable coordinates on the ellipsoid so that the Hamilton-Jacobi equation of the geodesics could be solved by separation of variables. Shortly after that J. Liouville [12] recognized that Jacobi's method could be applied to a larger class of surfaces. He proved that if the square of the line element ds of a surface S is given by

$$ds^2 = (f(x)+g(y))(dx^2+dy^2) , \tag{2.1}$$

where (x,y) is a coordinate system on S and f and g are functions depending on x and y only, respectively, then the Hamilton-Jacobi equation of the geodesics of S can be solved by separation of variables. To establish this, Liouville showed that

$$f(x)\sin^2(\alpha)+g(y)\cos^2(\alpha) \equiv \text{const}, \tag{2.2}$$

where α is the angle between some geodesic and the lines $y \equiv$ const. Surfaces which may be covered by a family of coordinate neighbourhoods such that ds^2 is of the form (2.1) are now known as Liouville surfaces. Rotational surfaces and quadrics in $\mathbb{R}^3$ are examples of Liouville surfaces. In 1889 P. Stäckel [13] proved that Liouville surfaces are characterized among surfaces by the property that the Hamilton-Jacobi equation of the geodesics can be solved by separation of variables.

Equation (2.2) is precisely the condition for

$$\frac{1}{f(x)+g(y)}\left(g(y)dx^2+f(x)dy^2\right)$$

to be a first integral for the geodesic flow of S. This shows that Liouville surfaces admit a quadratic first integral for the geodesic flow (apart from the trivial one, namely the energy function). It was proved by G. Darboux [7, p. 33] that this property essentially characterizes Liouville surfaces. M.G. Koenigs [11] found all surfaces admitting several quadratic first integrals for the geodesic flow. Recently K. Kiyohara [10] determined the diffeomorphism classes of all two-dimensional compact Riemannian manifolds admitting a quadratic first integral for the geodesic flow which is not a constant multiple of the energy function and which does not come from a local Killing vector field (note that this is Kiyohara's definition of a compact Liouville surface). These classes are given by the sphere S^2, the real projective plane $\mathbb{R}P^2$, the torus T^2 and the Klein bottle K^2. Moreover, he characterizes all Liouville surfaces which are diffeomorphic to S^2 or conformally isomorphic to T^2 or K^2.

b) *Geodesic maps.* E. Beltrami [2] proved in 1865 that the only surfaces admitting a geodesic map into the plane are those of constant curvature. In the same article he posed the problem of determining all pairs of surfaces admitting locally a geodesic and non-homothetical diffeomorphism from one to the other. Four years later this problem was solved by U. Dini [8]. He proved that two surfaces can be mapped geodesically and non-homothetically onto each other if and only if their first fundamental forms, say ds_1^2 and ds_2^2 respectively, are of the form

$$ds_1^2 = (f(x)-g(y))(dx^2+dy^2) \quad \text{and} \quad ds_2^2 = \left(\frac{1}{g(y)}-\frac{1}{f(x)}\right)\left(\frac{dx^2}{f(x)}+\frac{dy^2}{g(y)}\right)$$

with respect to a suitable coordinate system (x,y). One can readily see, using an elementary coordinate transformation, that ds_2^2 is the metric of a Liouville surface. Thus the solution of Beltrami's problem is given by suitable pairs of Liouville surfaces.

c) *Geometric characterizations of Liouville coordinates.* There are two interesting geometric characterizations of Liouville coordinates. Firstly, in [8] Dini proved that a coordinate system (x,y) on a surface is Liouville if and only if it is isothermal and has coordinate lines consisting of geodesic ellipses and hyperbolas. Recall that a geodesic ellipse (resp. hyperbola) is a curve such that the sum (resp. difference) of the geodesic distances to two given geodesically non-parallel curves is constant. On the triaxial ellipsoid, for example, the geodesic ellipses and hyperbolas of the standard Liouville coordinate net are given by the intersections with the confocal hyperboloids (see e.g. [17]).

A second geometric characterization of Liouville coordinates is due to K. Zwirner [17], see also [4]. A coordinate system (x,y) on a surface is Liouville if and only if it satisfies Ivory's diagonal property, namely that for any sufficiently small rectangle along the coordinate lines of x and y, the two geodesic diagonals are of the same length.

d) *Critical points.* The critical points on a Liouville surface with line element ds satisfying (2.1) are defined to be the points where $f+g = 0$. Geometric properties of closed curves consisting of critical points have been studied by P. Stäckel [14]. On analytic Liouville surfaces the set of critical points has been investigated by H. Viesel [15]. In particular, he gives an argument to show that in the analytic case there are at most four critical points on a Liouville surface. H. Walser [16] studied the behaviour of Liouville coordinates in the neighborhood of critical points.

3. Holomorphic Liouville curves in $\mathbb{C}^n$.

We first recall that a conformal minimal immersion of a simply connected Riemann surface S into $\mathbb{R}^n$ ($n \geq 3$) is the real part of a holomorphic curve in $\mathbb{C}^n$ with isotropic differential. In this section we find all holomorphic Liouville curves in $\mathbb{C}^n$, and then in the next section we consider consequences of the isotropy condition.

Let S be a metric Riemann surface and $\varphi : S \to \mathbb{C}^n$ be a holomorphic isometric immersion. If (x,y) is a Liouville coordinate system on S then the corresponding complex coordinate $z = x+iy$ will be called a *complex Liouville coordinate.* Note that if z is a complex Liouville coordinate then so is $i^r z$ for r=1,2,3.

Theorem 3.1. *Let $\varphi : S \to \mathbb{C}^n$ be a holomorphic isometric immersion of a metric Riemann surface S. Then a complex coordinate z on S is a complex Liouville coordinate if and only if, up to a translation of $\mathbb{C}^n$, the map φ can be written in the form*

$$\varphi = \varphi_+ + \varphi_0 + \varphi_- , \tag{3.1}$$

where

$$\varphi_+(z) = \sum_{p=1}^{r}\left(\alpha_p \exp(\omega_p z) + \beta_p \exp(-\omega_p z)\right)u_p \quad ,$$

$$\varphi_-(z) = \sum_{p=r+1}^{r+s}\left(\alpha_p \exp(i\omega_p z) + \beta_p \exp(-i\omega_p z)\right)u_p \quad ,$$

$$\varphi_0(z) = \sum_{p=r+s+1}^{n} z(\alpha_p + \beta_p z)u_p \quad .$$

Here, r,s *are non-negative integers with* $r+s \le n$, $\omega_1,\dots,\omega_{r+s}$ *are positive real numbers,* $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n$ *are complex numbers and* $u_1,\dots,u_n$ *is a unitary basis of* $\mathbb{C}^n$.

We prove the theorem by writing the Liouville condition (1.1) in terms of a differential equation for φ (see Proposition 3.2 below), which is easily seen to have (3.1) as its general solution. In fact, the basis $u_1,\dots,u_n$ of $\mathbb{C}^n$ in Theorem 3.1 is a basis of eigenvectors of the Hermitian matrix A of Proposition 3.2, while $\omega_1^2,\dots,\omega_r^2,-\omega_{r+1}^2,\dots,-\omega_{r+s}^2$ are the non-zero eigenvalues of A.

We recall that a holomorphic curve $\varphi : S \to \mathbb{C}^n$ is said to be *linearly full* if $\varphi(S)$ is not contained in an affine hyperplane of $\mathbb{C}^n$, or equivalently, if $\partial\varphi/\partial z,\dots,\partial^n\varphi/\partial z^n$ are linearly independent almost everywhere on S.

Proposition 3.2. *A complex coordinate* z *on* S *is a complex Liouville coordinate if and only if there is an* $n \times n$ *Hermitian matrix* A *such that*

$$\varphi_{zzz} = A\varphi_z \; . \tag{3.2}$$

Proof. The induced metric on S is given by $ds^2 = |\varphi_z|^2|dz|^2$, so z is a complex Liouville coordinate if and only if

$$(|\varphi_z|^2)_{zz} = (|\varphi_z|^2)_{\bar z\bar z} \quad , \tag{3.3}$$

or equivalently,

$$(\varphi_z)^*\varphi_{zzz} = (\varphi_{zzz})^*\varphi_z \quad , \tag{3.4}$$

where * denotes the adjoint. Let F be the $n \times n$ matrix-valued function whose k-th column is $\partial^k\varphi/\partial z^k$. Repeated differentiation of (3.4) with respect to both z and $\bar z$ shows that

$$F^*F_{zz} = (F_{zz})^*F \; . \tag{3.5}$$

Thus, as (3.4) is just one entry in the matrix equation (3.5), we see that (3.4) and (3.5) are equivalent. We may assume without loss of generality that φ is linearly full (otherwise replace $\mathbb{C}^n$ by the vector subspace spanned by the image of φ_z). Then F is a non-singular matrix except possibly at isolated points, and it now follows that z is a complex Liouville coordinate if and only if

$$F_{zz}F^{-1} = (F_{zz}F^{-1})^* \tag{3.6}$$

almost everywhere.

Assume then that z is a complex Liouville coordinate. Since the left hand side of (3.6) is holomorphic and the right hand side is anti-holomorphic, it follows that the matrix valued function $A := F_{zz}F^{-1}$ is a constant Hermitian matrix. Hence

$$F_{zz} = AF , \tag{3.7}$$

and taking the first column of (3.7) gives (3.2). (Note that (3.7) must hold everywhere.)

Conversely, if (3.2) holds for some Hermitian matrix A, then repeated differentiation with respect to z shows that (3.7), and hence (3.6), holds. Hence z is a complex Liouville coordinate, and Proposition 3.2 is proved. □

It follows from Theorem 3.1 that, if S is simply connected and a Liouville surface, then S is an open subset of $\mathbb{C}$ and $\varphi : S \to \mathbb{C}^n$ may be extended to a holomorphic curve defined on the whole of $\mathbb{C}$. In the next section we will be interested in knowing when S is locally isometric to a surface admitting a one-parameter family of isometries. To this end we first consider the corresponding problem for holomorphic curves in $\mathbb{C}^n$.

Proposition 3.3. *Let* $\varphi : \mathbb{C} \to \mathbb{C}^n$ $(n \geq 3)$ *be an immersion such that the standard coordinate z on* $\mathbb{C}$ *is a complex Liouville coordinate, and suppose that* φ *is linearly full. Then the metric induced on* $\mathbb{C}$ *admits a one-parameter family of isometries if and only if each of the following conditions holds:*

(i) $rs = 0$,

(ii) $\alpha_p\beta_p = 0$ *for* $p=1,\ldots,r+s$,

(iii) $\beta_p = 0$ *for* $p=r+s+1,\ldots,n$.

Proof. The metric induced on $\mathbb{C}$ is given by $ds^2 = |\varphi_z|^2|dz|^2$. Assume that (i), (ii) and (iii) hold with, say, $s = 0$. Then the one-parameter family, indexed by the real variable t, given by $z \mapsto z+it$ is a group of isometries. If $r = 0$ but $s \neq 0$, then the required family is given by $z \mapsto z+t$.

Conversely, any one-parameter family of isometries must consist of conformal transformations of $\mathbb{C}$ and so be of the form $\tau(t) : z \mapsto a(t)z + b(t)$, $t \in I$, for suitable continuous complex functions $a(t)$ and $b(t)$ defined on an open interval I of real numbers with $0 \in I$ and $\tau(t)$ equal to the identity map if and only if $t = 0$. By the Calabi rigidity theorem [6] the family extends to a unique one-parameter family of holomorphic isometries

$$v \mapsto B(t)v + c(t)$$

of $\mathbb{C}^n$, where $B(t) \in U(n)$ and $c(t) \in \mathbb{C}^n$. Thus

$$\varphi(a(t)z + b(t)) = B(t)\varphi(z) + c(t) \quad , \quad t \in I \,. \tag{3.8}$$

Since φ is assumed to be linearly full, the eigenspaces of the matrix A of Proposition 3.2 are all of dimension at most two. Hence r,s are not both zero. It then follows from (3.8) and (3.1) that $a(t) = 1$ for all $t \in I$ and that the eigenspaces of A are stable under $B(t)$. For simplicity, we first consider the case where the eigenspaces are all of dimension one. Also, using the fact that $b(t) = 0$ if and only if $t = 0$, we see that $\alpha_p\beta_p = 0$ for $p=1,\ldots,r+s$ (and that $c(t)$ is unitarily orthogonal to u_p). Further, if $r \neq 0$, then $b(t)$ is imaginary, while if $s \neq 0$ then $b(t)$ is real. Hence $rs = 0$. Lastly, if $r+s < n$ (so $r+s+1 = n$ by the assumption that the dimension of the eigenspaces of A are of dimension one) we see that $\beta_n = 0$.

The case where A has at least one eigenspace of dimension two is similar but requires the expression for φ to be modified. For instance, if the eigenspace of A corresponding to the eigenvalue $\omega_p^2 = \omega_q^2$ $(p \neq q)$ has dimension two then, in (3.1),

$$(\alpha_p\exp(\omega_p z)+\beta_p\exp(-\omega_p z))u_p + (\alpha_q\exp(\omega_q z)+\beta_q\exp(-\omega_q z))u_q$$

should be replaced by

$$\tilde{\alpha}_p\exp(\omega_p z)\tilde{u}_p + \tilde{\beta}_q\exp(-\omega_q z)\tilde{u}_q,$$

where $\tilde{u}_p,\tilde{u}_q$ are suitable unit vectors with the same span as u_p,u_q. It then follows that $\tilde{u}_p$ is orthogonal to $\tilde{u}_q$ and the proof proceeds as before. □

Remark 3.4. In the case where $n = 2$ the above proposition still holds if one of r,s is non-zero. In the case $r = s = 0$ the curve is, up to a translation, of the form

$$\varphi(z) = (\alpha_1 z + \beta_1 z^2, \alpha_2 z + \beta_2 z^2)$$

and φ is linearly full if and only if $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. In this case the induced metric always admits a one-parameter group of isometries.

Remark 3.5. We note from the preceding proof that the one-parameter family of isometries of S is the restriction of a one-parameter family (g_t) of isometries of $\mathbb{C}^n$ which, in the linearly full case, is of the form

$$g_t\Big(\sum_{p=1}^{n}\gamma_p u_p\Big) = \begin{cases} \displaystyle\sum_{p=1}^{n}\gamma_p\exp(i\Theta_p(t))u_p \,, & \text{if A is non-singular} \,, \\ \displaystyle\sum_{p=1}^{n-1}\gamma_p\exp(i\Theta_p(t))u_p + (\gamma_n + \alpha_n b(t))u_n \,, & \text{if A is singular} \,. \end{cases}$$

Here, we have put $\Theta_p(t)$ equal to $-i\omega_p b(t)$ if $\beta_p=0$ and $s=0$, to $i\omega_p b(t)$ if $\alpha_p=0$ and $s=0$, to $\omega_p b(t)$ if $\beta_p=0$ and $r=0$, and to $-\omega_p b(t)$, if $\alpha_p=0$ and $r=0$.

4. Minimal Liouville surfaces in $\mathbb{R}^3$.

Recall that if $f : S \to \mathbb{R}^n$ is a conformal minimal immersion of a simply connected Riemann surface S then $f = \operatorname{Re} \varphi$, where $\varphi : S \to \mathbb{C}^n$ is a holomorphic curve with isotropic differential $\alpha = d\varphi$. In this case the metrics induced on S by f and φ are equal. We also have

$$f = \operatorname{Re} \int \alpha \,, \tag{4.1}$$

which gives the Weierstraß representation of f.

Let $\varphi : S \to \mathbb{C}^n$ be a holomorphic isometric immersion of a metric Riemann surface S and let z be a complex coordinate on S. To simplify the notation let us write $Y = \varphi_z$. Then $\alpha = Ydz$, and it follows from Proposition 3.2 that z is a complex Liouville coordinate on S and φ has isotropic differential if and only if there exists a Hermitian $n \times n$ matrix A such that Y is a solution of

$$Y_{zz} = AY \,, \quad Y \cdot Y = 0 \,, \tag{4.2}$$

where $\cdot$ denotes the complex bilinear extension to $\mathbb{C}^n$ of the standard inner product on $\mathbb{R}^n$.

In this section we find all solutions to (4.2) in the case $n = 3$, and obtain the following theorem.

Theorem 4.1. (Alt [1]) *Let $f : S \to \mathbb{R}^3$ be a linearly full isometric minimal immersion of a simply connected surface S. Then S is a Liouville surface if and only if $f = \operatorname{Re} \int \alpha$, where the one-form α on S is given in terms of a suitable complex Liouville coordinate z on S by*

$$\alpha = c\left(r\exp(\lambda_1 z)v_1 + i\sqrt{2}\exp(\lambda_2 z)v_2 + \tfrac{1}{r}\exp(\lambda_3 z)v_3\right)dz \tag{4.3}$$

for some unitary basis v_1,v_2,v_3 of $\mathbb{C}^3$ with $v_3 = \bar{v}_1$, $v_2 = \bar{v}_2$, some complex number $c \neq 0$ and some real numbers $r \neq 0$, $\lambda_1,\lambda_2,\lambda_3$ with $\lambda_1 > 0$, $\lambda_1 \neq \lambda_2$ and $\lambda_1+\lambda_3 = 2\lambda_2$.

In particular, since $|\alpha|^2$ is a function of Re z only we have the following corollary.

Corollary 4.2. (Alt [1]) *A minimal Liouville surface in $\mathbb{R}^3$ is locally isometric to a rotational surface in $\mathbb{R}^3$.*

Remark 4.3. The Weierstraß representation of a conformal minimal immersion into $\mathbb{R}^3$ is given in terms of a suitable holomorphic differential w and meromorphic function g by f = Re $\int\alpha$, where

$$\alpha = \left(1-g^2, i(1+g^2), 2g\right)w . \tag{4.4}$$

The one-form α of (4.3) can be written in this form as follows. Let a = log(r) and let ζ be the complex coordinate on S given by $\zeta = -((\lambda_1 - \lambda_2)z + a)$. Also, let e_1, e_2, e_3 be the orthonormal basis of $\mathbb{R}^3$ obtained by setting

$$v_1 = (i/\sqrt{2})(e_1 + ie_2) \quad , \quad v_2 = e_3 .$$

Then the one-form α of (4.3) becomes

$$\alpha = \tilde{c}\exp(\lambda\zeta)\left((1-\exp(2\zeta))e_1 + i(1+\exp(2\zeta))e_2 + 2\exp(\zeta)e_3\right)d\zeta ,$$

where $\lambda = \lambda_1/(\lambda_2 - \lambda_1)$ and $\tilde{c}$ is a non-zero complex constant. This is of the form of (4.4) with $g = \exp(\zeta)$ and $w = \tilde{c}\exp(\lambda\zeta)d\zeta$. A further coordinate change, namely $\xi = \exp(\zeta)$, gives α in the form

$$\alpha = \tilde{c}\xi^{\mu}\left((1-\xi^2)e_1 + i(1+\xi^2)e_2 + 2\xi e_3\right)d\xi ,$$

where $\mu = (2\lambda_1 - \lambda_2)/(\lambda_2 - \lambda_1)$. As noted in Bianchi [3, p. 379] this is exactly the form of the Weierstraß representation for a non-planar minimal surface which is locally isometric to a rotational surface. For example, taking $\lambda_2 = 0$ gives the family of minimal surfaces associated to the helicoid. Similarly, taking $\lambda_p = p$, p=1,2,3, gives the associated family of Enneper's surface.

Remark 4.4. It follows from Remark 3.5 that, with the exception of the helicoid and its associates, each minimal Liouville surface in $\mathbb{R}^3$ is a Bonnet surface, that is to say a surface in $\mathbb{R}^3$ which is a member of a one-parameter family of surfaces in $\mathbb{R}^3$ with the same metric and mean curvature, no two of which are congruent in $\mathbb{R}^3$.

We will now prove Theorem 4.1 by obtaining all linearly full solutions to the system (4.2) in the case n = 3. We do this by first writing down all linearly full solutions of $Y_{zz} = AY$, where A is a Hermitian 3×3 matrix, and then determining which solutions are isotropic. The proof involves some case by case arguments. However, it turns out in the case n = 3, that if (4.2) admits a linearly full isotropic solution then this imposes strong restrictions on the matrix A. For instance, as will be seen, A and $\bar{A}$ have to commute. A direct proof of this result, which would further simplify the argument, has eluded us.

From now on we assume n = 3. We first find all solutions to (4.2) in the "generic case" where the Hermitian 3×3 matrix A has distinct non-zero eigenvalues. The solutions of $Y_{zz} = AY$ are then of the form

$$Y = Y_+ + Y_- \ , \tag{4.5}$$

where

$$Y_+(z) = \sum_{p=1}^{r}\big(\alpha_p\exp(\omega_p z) + \beta_p\exp(-\omega_p z)\big)u_p \ ,$$

$$Y_-(z) = \sum_{p=r+1}^{3}\big(\alpha_p\exp(i\omega_p z) + \beta_p\exp(-i\omega_p z)\big)u_p \ .$$

Here, $0 \le r \le 3$; $\omega_1 > \ldots > \omega_r > 0$; $\omega_{r+1} > \ldots > \omega_3 > 0$; $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3$ are complex numbers; and u_1,u_2,u_3 is a unitary basis of $\mathbb{C}^3$. The eigenvalues of A are $\omega_1^2,\ldots,\omega_r^2,-\omega_{r+1}^2,\ldots,-\omega_3^2$.

The method of proof of Theorem 4.1 is indicated by Proposition 3.3. In the following two lemmas we show that the conditions (i) and (ii) of that proposition are satisfied by any linearly full isotropic curve Y of the form (4.5).

Lemma 4.5. *If* Y *is given by (4.5) and is both isotropic and linearly full, then* $r = 0$ *or* $r = 3$.

Proof. Assume that $r = 1$ or 2. Then the coefficients of $\exp(\pm 2\omega_1 z)$, $\exp(\pm 2i\omega_{r+1}z)$ and $\exp((\pm\omega_1 \pm i\omega_{r+1})z)$ in the expression obtained from (4.5) for $Y\cdot Y$ must all be zero. This gives the following:

$$\alpha_1^2 u_1\cdot u_1 = 0 \ , \ \beta_1^2 u_1\cdot u_1 = 0 \ , \ \alpha_{r+1}^2 u_{r+1}\cdot u_{r+1} = 0 \ , \ \beta_{r+1}^2 u_{r+1}\cdot u_{r+1} = 0 \ ,$$

$$\alpha_1\alpha_{r+1}u_1\cdot u_{r+1} = 0 \ , \ \beta_1\beta_{r+1}u_1\cdot u_{r+1} = 0 \ , \ \alpha_1\beta_{r+1}u_1\cdot u_{r+1} = 0 \ , \ \alpha_{r+1}\beta_1 u_1\cdot u_{r+1} = 0.$$

Since Y is assumed to be linearly full, α_1 and β_1 are not both zero, and neither are α_{r+1} and β_{r+1}. The above equations then show that u_1 and u_{r+1} span a two-dimensional isotropic subspace of $\mathbb{C}^3$, which is impossible. □

Lemma 4.5 shows that, replacing z by iz if necessary, we may assume that Y is of the form of (4.5) with $r = 3$ and $Y_- = 0$. That is to say, $Y = Y_+$ and $r = 3$.

Lemma 4.6. *If* $Y = Y_+$ *with* $r = 3$ *and if* Y *is linearly full and isotropic then* $\alpha_p\beta_p = 0$ *for* $p=1,2,3$.

Proof. By replacing z by -z if necessary, we may assume that $\alpha_1 \neq 0$. Then, consideration of the highest power, $\exp(2\omega_1 z)$, in the expression for $Y\cdot Y$ shows that

$$u_1\cdot u_1 = 0 \ . \tag{4.6}$$

The case $\alpha_2 \neq 0$. Considering the next highest power $\exp((\omega_1+\omega_2)z)$ we have that

$$u_1 \cdot u_2 = 0 \quad . \tag{4.7}$$

But then, from (4.6) and (4.7) we see that, without loss of generality we may assume that

$$u_3 = \bar{u}_1 \quad \text{and} \quad u_2 = \bar{u}_2 \quad . \tag{4.8}$$

Then

$$\begin{aligned} 0 = Y \cdot Y = &\left(\alpha_1 \exp(\omega_1 z) + \beta_1 \exp(-\omega_1 z)\right)\left(\alpha_3 \exp(\omega_3 z) + \beta_3 \exp(-\omega_3 z)\right) \\ &+ \left(\alpha_2 \exp(\omega_2 z) + \beta_2 \exp(-\omega_2 z)\right)^2 . \end{aligned} \tag{4.9}$$

The constant term on the right hand side of (4.9) must be zero, so it follows, since $\alpha_2 \neq 0$, that $\beta_2 = 0$. Consideration of the negative powers, $\exp(-(\omega_1-\omega_3)z)$ and $\exp(-(\omega_1+\omega_3)z)$, shows that $\beta_1 = 0$. Consideration of the positive powers, $\exp((\omega_1+\omega_3)z)$, $\exp((\omega_1-\omega_3)z)$ and $\exp(2\omega_2 z)$, shows that $\alpha_3\beta_3 = 0$.

The case $\alpha_2 = 0$. Here, the only positive exponents of $\exp(z)$ in $Y \cdot Y$ are $\omega_1+\omega_3$, $\omega_1-\omega_2$, $\omega_1-\omega_3$ and $2\omega_3$ with coefficients $\alpha_1\alpha_3 u_1 \cdot u_3$, $\alpha_1\beta_2 u_1 \cdot u_2$, $\alpha_1\beta_3 u_1 \cdot u_3$ and $\alpha_3^2 u_1 \cdot u_3$ respectively. Since $\omega_1+\omega_3$ is strictly the largest exponent, we have $\alpha_3 u_1 \cdot u_3 = 0$. It now follows that

$$u_1 \cdot u_3 = 0 \quad , \tag{4.10}$$

because if $\alpha_3 = 0$ then consideration of the coefficient of $\exp((\omega_1-\omega_3)z)$ shows that (4.10) still holds. Then, as before, it follows from (4.6) and (4.10) that we may assume that

$$u_2 = \bar{u}_1 \quad \text{and} \quad u_3 = \bar{u}_3 \quad . \tag{4.11}$$

Thus

$$Y \cdot Y = \left(\alpha_1 \exp(\omega_1 z) + \beta_1 \exp(-\omega_1 z)\right)\beta_2 \exp(-\omega_2 z) + \left(\alpha_3 \exp(\omega_3 z) + \beta_3 \exp(-\omega_3 z)\right)^2 ,$$

and consideration of the coefficients of the negative powers of $\exp(z)$ shows that $\beta_1 = \beta_3 = 0$. This completes the proof of Lemma 4.6. □

It now follows in the case where A has three distinct non-zero eigenvalues that Y may be written in the form

$$Y = \gamma_1 \exp(\lambda_1 z) v_1 + \gamma_2 \exp(\lambda_2 z) v_2 + \gamma_3 \exp(\lambda_3 z) v_3 \, , \tag{4.12}$$

where $\lambda_1 > \lambda_2 > \lambda_3$, $\lambda_1 > 0$, $\gamma_1, \gamma_2, \gamma_3$ are non-zero complex numbers and v_1, v_2, v_3 form a unitary basis of $\mathbb{C}^3$. Then, using (4.12),

$$0 = Y \cdot Y = \sum_{i,j=1}^{3} \gamma_i \gamma_j \exp((\lambda_i+\lambda_j)z) v_i \cdot v_j \quad . \tag{4.13}$$

However, since $\lambda_1 > \lambda_2 > \lambda_3$ we see that

$$v_1 \cdot v_1 = v_1 \cdot v_2 = v_2 \cdot v_3 = v_3 \cdot v_3 = 0 \, , \tag{4.14}$$

so, without loss of generality we may assume that $v_3 = \overline{v}_1$ and $v_2 = \overline{v}_2$. Then (4.13) reduces to

$$0 = \gamma_2^2 \exp(2\lambda_2 z) + 2\gamma_1\gamma_3 \exp((\lambda_1+\lambda_3)z) .$$

Theorem 4.1 now follows in the generic case, after possibly applying a suitable allowable coordinate change of the form $w = i^k z$, for $k=1,2,3$.

We now outline the arguments required to prove the theorem in the non-generic cases where either A is singular or has a repeated eigenvalue.

Assume A is not singular but has a repeated non-zero eigenvalue. In this case we easily see that all eigenvalues may be assumed to be positive. Hence we may write Y as follows:

$$Y = \gamma_1 \exp(\omega_1 z) v_1 + \gamma_2 \exp(-\omega_1 z) v_2 + \big(\alpha_3 \exp(\omega_3 z) + \beta_3 \exp(-\omega_3 z)\big) v_3 ,$$

with ω_1, ω_3 distinct positive real numbers, γ_1, γ_2 non-zero complex numbers and v_3 unitarily orthogonal to both v_1 and v_2. Consideration of the coefficients of $\exp(\pm(\omega_1+\omega_3)z)$ and $\exp(\pm 2\omega_3 z)$ in $Y \cdot Y = 0$ shows that if $\alpha_3\beta_3 \neq 0$ then $v_3 \cdot v_p = 0$ for $p=1,2,3$, which is impossible. Hence $\alpha_3\beta_3 = 0$ and in a similar manner to the generic case one may show that Y may be written in the required form with $\lambda_2 = -\lambda_1$ and $\lambda_3 = -3\lambda_1$. (Pictures of an example of this type appear at the end of this section.)

Assume A is singular. In this case it is easily seen that Y may be written in the form of (4.3) with either $\lambda_3 = 0$, or $\lambda_2 = 0$ (which gives the one-parameter family of surfaces associated with the helicoid).

On the next page are two pictures of the surface which is the image of the immersion f given by (4.3) with $c = \sqrt{2}$, $r = 1$, $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = -3$. This image (which is a minimally immersed cylinder of total curvature equal to -8π) has reflectional symmetry in the origin (corresponding to $z \mapsto z + i\pi$) and a symmetry given by rotation through π about the second coordinate axis (corresponding to $z \mapsto \bar{z}$). We would like to thank Cherry Kearton for his help in producing these pictures.

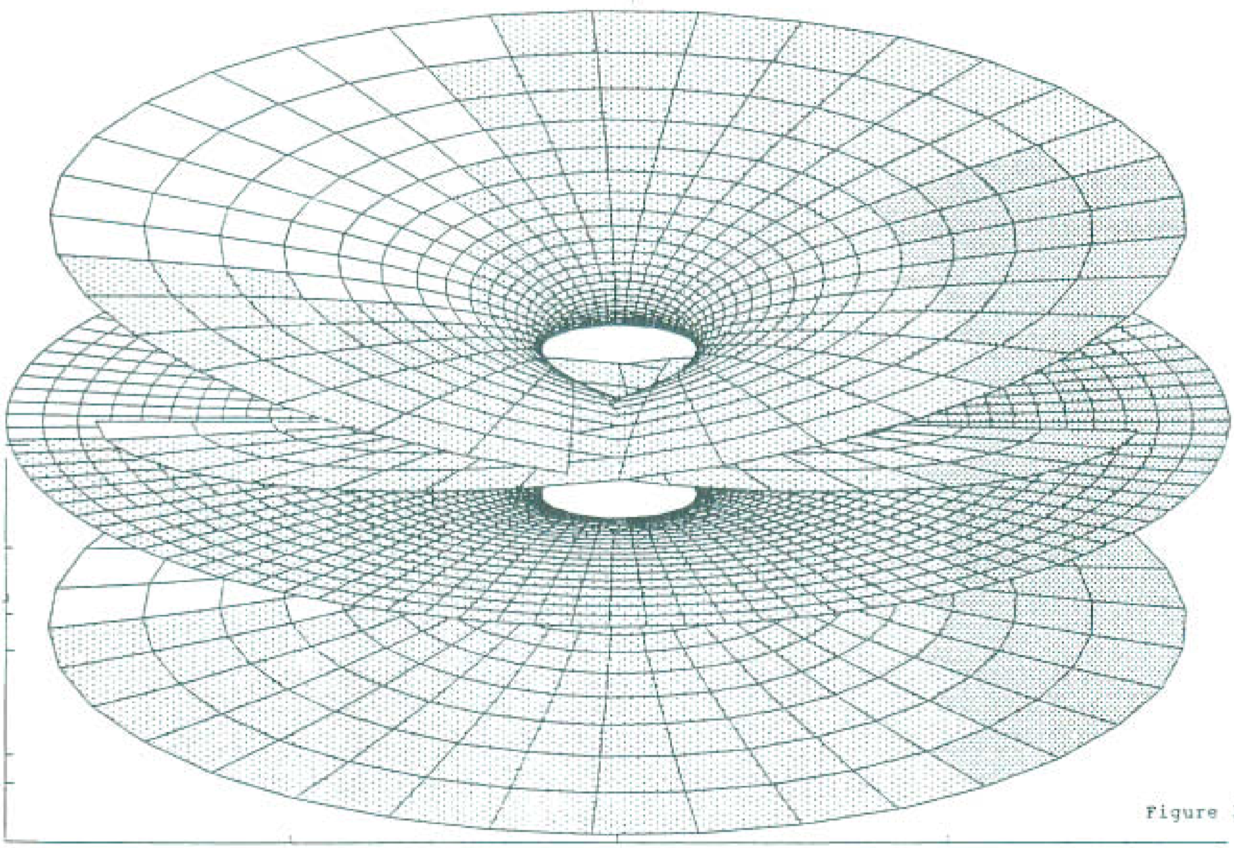

Figure 1

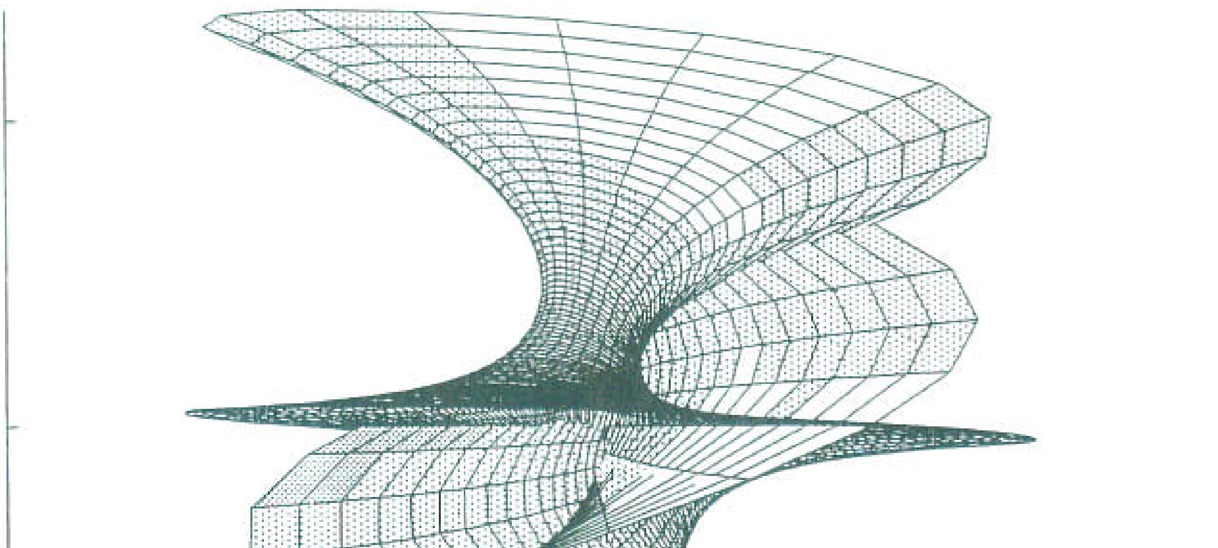

Figure 2

5. A remark concerning the four-dimensional case.

It is not true in general that a minimal Liouville surface in $\mathbb{R}^n$ admits a one-parameter group of isometries. This is easily seen by considering the minimal immersion $f = \mathrm{Re}\,\varphi$: $\mathbb{C} \to \mathbb{R}^4$ determined by

$$\varphi_z = \exp(z)v_1 + \exp(-z)\bar{v}_1 + \exp(iz)v_2 - \exp(-iz)\bar{v}_2 ,$$

where $v_1 = (1/\sqrt{2})(e_1 + ie_2)$, $v_2 = (1/\sqrt{2})(e_3 + ie_4)$ and $e_1,\ldots,e_4$ is the standard basis of $\mathbb{R}^4$. In this particular example, φ_z satisfies (3.2) with the corresponding Hermitian matrix A having eigenvalues of different signs. It appears to be the case that if S is a minimal Liouville surface in $\mathbb{R}^4$ which is the real part of a holomorphic curve in $\mathbb{C}^4$ whose corresponding Hermitian matrix A is semi-definite, then S is locally isometric to a rotational surface. However, our discussion of this case is neither short nor illuminating!

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