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19 May 2011

London Taught Course Center
Fall 2012

Integrability

Integrability is a wide subject that comprises many deep ideas and that can be applied to very diverse physical systems. This course will give an overview of some of the ideas behind integrability, and of the way they are developed in applications to four areas of theoretical physics: finite dynamical systems, classical field theory, quantum chains, and quantum field theory. In each case, slightly different techniques are involved, but one of the goal of this course is to emphasise the conceptual similarity between them. In particular, I will try to show how the basic concepts in integrable dynamical systems, such as action-angle variables, conserved quantities, and the simplification of the set of phase space trajectories that ensues, are realised in other contexts. I will develop simple, standard examples in order to illustrate the concepts. Naturally, the course will be divided into four parts, corresponding to the four areas of theoretical physics.

Since the course is only 10 hours long, and since I'd like to be able to emphasise the concepts and their similarities in various areas of applications of integrability, instead of the technicalities, it will be helpful if the students can do some reading before each lecture (the appropriate reading will be suggested in due time). It will also be helpful if the students can make sure they are somewhat comfortable with the important concepts in classical Hamiltonian mechanics, complex analysis, quantum mechanics and quantum field theory before the course starts.

It is important to note that the choice of topics and the way they are presented here reflect mainly the strong time constraints and my personal take on the subject of integrability, rather than "the correct" overview of integrability.

Physical systems and integrability

In physics, what is usually understood as a *system* is a topological space of states M (usually a manifold, although often infinite-dimensional), and an evolution map, a bijective map $U_t : M \to M$ parametrised by the time $t \in \mathbb{R}$. Every state is a mathematical object which encodes observable predictions, and the evolution map determines how the predictions change in time. In various physical applications, the space of states may be very different, hence also the evolution equation may look different. In classical dynamical systems, the space of states is the phase space: a symplectic manifold (hence with a non-degenerate Poisson structure). In classical field theory, it is a space of functions (which we will mainly take to be on $\mathbb{R}$). In quantum mechanics and in quantum field theory, it is a Hilbert space (but in both cases with somewhat different structure).

In most, if not all, physical cases, the evolution equation is such that time evolution of any given state is continuous: the map $t \mapsto U_t(p)$, for any given $p \in M$, is a continuous map (at least in a neighbourhood of $t = 0$). However, the map U_t from M to M, for fixed time t, is in general very complicated. In particular, although it may almost everywhere continuous, this continuity is almost nowhere uniform in time. For instance, in many cases, two nearby states map to states that are very far apart, and that become exponentially further apart as t is increased - this is chaos. Integrability is essentially the opposite of chaos: the map U_t is as nice as it can be. It possesses infinitely many invariant submanifolds that foliate M, parametrised by as many

continuous parameters as there are "degrees of freedom", and on these submanifolds, states that start nearby stay nearby uniformly in time. This is what is explicitly seen in classical dynamical systems, and it seems to form a common underlying principle at the basis of integrability in general. Let us see how this works in the four physical situations that I talked about.

Integrable classical dynamical systems

For this part, I follow Chapter 2 of [1] (but many other references exist). This part of the course will be covered in approx. 1 or 1+1/2 hour, hence I'll ask the students to read Chapter 2 in advance.

Consider a classical system: a symplectic $2n$-dimensional manifold which we'll take to be spanned by the usual coordinates $p_i \in \mathbb{R}$, $q_i \in \mathbb{R} : i = 1, \ldots, n$ with canonical Poisson brackets, along with a Hamiltonian H generating the Hamiltonian flow $\dot{F} = \{H, F\}$ where F is any function on the manifold.

The basic, old definition of integrability is that a system is integrable (Liouville integrable) if there exist n independent conserved quantities $F_i : i = 1, \ldots, n$, $\{H, F_i\} = 0$ that are in involution, $\{F_i, F_j\} = 0 \ \forall \ i, j$ (these naturally include the Hamiltonian H, since having more than n conserved quantities would imply that the Poisson bracket is degenerate). By Liouville theorem, this implies that the dynamics can be solved by "quadrature": it is possible to solve it by making a canonical change of coordinates $(p, q) \to (F, \psi)$. Clearly, after this change of coordinates, along with $\dot{F}_i = 0$ we find $\dot{\psi}_i = \partial H / \partial F_i$ which is a constant, so the equations of motion are trivial to solve. The change of coordinates is obtained by constructing a function $S(F, q)$ whose differential is $dS = \sum_j \psi_j dF_j + p_j dq_j$; such a function exists (this differential is exact) thanks to the involution property of the conserved charges. The exactness of the differential implies that (F, ψ) are canonical coordinates.

In this quadrature solution, the only implicit aspect is the form of the coordinates ψ_j. These can in principle be written explicitly in terms of the original coordinates by constructing $S = \int_{q_0}^{q} \sum_i p_i(f, q') dq'_i$ where $p_i(f, q')$ is obtained by inverting $F_i(p, q) = f_i = const.$, and then by using $\psi_j = \partial S / \partial F_j$.

We see that the principles according to which integrability occurs and the ideas leading to coordinates with simple evolution equations are very nice and relatively simple, but that the explicit solution, in practice, will require some more "dirty" work: some potentially complicated algebraic manipulations with the conserved quantities. In integrability, the problem of solving a given system is often divided into two parts: the "direct" problem whereby we obtain variables with simple dynamics, and the "inverse" problem whereby we try to reconstruct the original variables and their dynamics in terms of the new ones. The inverse problem is the most complicated one.

But before discussing other physical systems, I must describe a bit more of the general classical dynamics situation. An argument of Arnold goes as follows: for any F_i there is an associated Hamiltonian flow, with time t_i. Since the F_i are in involution, then any function on M can be evolved simultaneously under all times t_i to form a function of the t_is in a neighbourhood of the origin (in times). If the submanifolds M_f corresponding to $F_i = f_i$ are connected and compact, then we have a surjective, locally free and transitive action of $\mathbb{R}^n$ on M_f, with stablisers $\mathbb{Z}_n$, so that $M_f \cong \mathbb{R}^n / \mathbb{Z}_n$: the invariant submanifolds are tori.

Hence we see indeed that we have (at least in the compact, connected case) nice invariant

submanifolds, parametrised by as many parameters as there are degrees of freedom (n). Since these are tori, there are n cycles, and we may define n angle variables along these cycles. The canonical conjugates, called the action variables, are functions of the F_is, hence indeed two nearby states on any given torus are uniformly nearby under evolution. These are the *action-angle* variables: consequences of the nice structure of the invariant submanifolds.

This provides the general structure of integrability, but an important question is: how do we find the n independent conserved quantities in involution? A wealth of techniques have been developed in the context of integrability essentially in order to solve this problem, and classify its solutions. In fact, often one starts with a description that immediately implies the existence of these conserved quantities, and often that contain some part of the direct or inverse problem. I'll discuss the Lax pairs: a pair of n by n matrices L and W with a Hamiltonian time evolution $\dot{L} = [W, L]$, the associated classical r-matrix, and how conserved quantities naturally emerge.

Integrable classical field theory

For this part, I follow Chapter 13 of [1]. This part of the course will be covered in approx. 2+1/2 hours.

In this case, the space of states is a space of (doublets of) functions on a manifold B (which we will refer to as the base space). In order to go forward with such an infinite-dimensional space of states, we need one additional principle: that of locality. Locality pertains to the evolution equation, and in a sense links the topology in the space of states to that on the base space. Loosely speaking, it says that not only an infinitesimal evolution $U_{\delta t}(p)$ gives a state $p + \delta p$ that is near to p (topology of the space of states), but also, for any neighbourhood $N \subset B$ (topology on the base space), $(\delta p)(x)$ may only depend on the restriction $p|_N$ for every $x \in N$. It seems that this principle, along with integrability, singles out one-dimensional base spaces as being very special. Indeed, the usual, well-developed theory of integrable models of local classical field theory seems to be restricted to one dimension, except for free field models[1]. Here, we will consider $B = \mathbb{R}$.

An attempt at generalising Liouville integrability for local field theories is to define integrable models as those that possess infinitely many *local conserved charges* (quantities that are integrals of local densities). This alone is unsatisfactory, for two reasons: first, this infinitude would need to be better specified; second, more importantly, the quadrature solution is simply unworkable in infinite dimension. Yet it does seem to be a characteristic of integrable field theories.

However, the most powerful techniques to study integrable field theories are based on generalisations of Lax pairs. Instead of generalising them to infinite dimensional matrices, they stay finite dimensional but depend on an additional "spectral" parameter. Further, thanks to the presence of the base space B, the pairs may acquire a geometrical character and becomes a zero-curvature formulation. Essentially, we have a flat connection on $B \times \mathbb{R}$ (where $\mathbb{R}$ represents the time) parametrised by a spectral parameter λ and an arbitrary function u on $B \times \mathbb{R}$, and the condition of flatness for all values of λ is equivalent to the integrable equation of motion for u, and to the presence of infinitely many conserved quantities (the fact that they are local charges is verified in all known models). The actual solution of the problem proceeds as before in two stages: in the direct problem, the linear problem of parallel transport of a vector Ψ with this flat connection is seen as a scattering problem for waves propagating along B. This

[1]There are higher-dimensional integrable models, and there is research activity in this area, but the usual development has to be changed, and much more still needs to be done.

gives rise to time-dependent scattering data, which turn out to evolve much like action-angle variables. Then, in the inverse problem, there is a process by which this map is inverted: these time-evolved scattering data give back a function u that solves the equations of motion. In this way, explicit solitonic solutions can be constructed, but also much more general solutions. Essentially, this *inverse scatterering method* is a generalisation of the Fourier transform, and the scattering data give rise to simple invariant subspaces for the evolution equations where trajectories stay uniformly close (this includes, for instance, the space of k-soliton solutions). I will work out the example of the sine-Gordon theory. Note that there are other techniques, besides the classical inverse scattering method, to extract solutions from generalisations of Lax pairs.

Integrable quantum chains

This part of the course is based on (small) parts of [2, 3, 4]. It will be covered in approx. 3 hours.

A quantum chain is a quantum mechanical system where the Hilbert space is seen as a tensor product $\mathcal{H} = \otimes_{j=-m}^{m} V_j$ of (usually isomorphic) vector spaces V_j (the "sites"), let's say of dimension d. Here again there are some fundamental problems in trying to generalise Liouville integrability: for any diagonalisable operator H (like the Hamiltonian), there are d^m mutually commuting operators that commute with H: just diagonalise H to see this! Hence the presence of conserved quantities in involution is certainly not enough for integrability. Again, the central principle is that of locality, which makes full sense in the thermodynamic limit $m \to \infty$. What makes $\otimes_{j=-\infty}^{\infty} V_j$ a "chain" is the locality of the Hamiltonian as a charge: it is a sum of terms each of which act like the identity on all but a finite, fixed number of sites. This implies a topology on the chain, from which we may define local linear operators (from the notion of local operator, there is then a natural topology on the space of quantum states that makes sense directly in the infinite-m limit). Then, we may ask for the presence of infinitely many local conserved charges as before. This indeed seems to be a characteristics of integrable quantum chains.

Again, the explicit solution of the model is hard to obtain from a quadrature based on the conserved quantities. But again, there are other techniques that are available. The modern way is to use another generalisation of Lax pairs; the central role is played more precisely by the generalisation of the r-matrix (the quantum R-matrix) and the Yang-Baxter equations. The direct problem is that of the construction of the eigenstates of the Hamiltonian. This is possible to do from the R matrix; it may also be seen as the formulation of an ansatz for these eigenstates, known as the Bethe ansatz. In this ansatz, only a small number of parameters need to be adjusted: the number of degrees of freedom in the state, interpreted as the "exact quasi-particles" of the model. The fact that this ansatz works is characteristics of the fact that there are indeed "well-behaved" invariant subspaces parametrised by as many parameters as there are degrees of freedom, and in this ansatz one can extract what looks like the equivalent of the action-angle variables. The inverse problem, called the quantum inverse scattering problem, is translated into the construction of the local operators on the Bethe ansatz basis, and the evaluation of their correlation functions. This is more complicated, but I'll say a little bit about it. I will work out the example of the Heisenberg spin chain.

Integrable quantum field theory

This part of the course is based on (small) parts of my notes that can be found on

http://www.mth.kcl.ac.uk/˜bdoyon/notesIQFT08.pdf (but see also [5, 6]). It will be covered in approx. 3 hours.

Quantum field theory (QFT) is the theory that describes emerging behaviours of lattice models near to critical points. It can also be seen, in a more intuitive but less accurate fashion, as the quantum version of classical field theory. Hence, the ideas and methods of integrability pertaining to quantum chains and to classical field theory tell us a lot about what to do in QFT.

In QFT, there is already a natural notion of locality: quantum mechanical independence at space-like distances. In different ways, this is naturally the counter part of locality in quantum chain, and of locality in classical field theory. Hence, we may wish to define Liouville integrability as the existence of infinitely many local conserved charges in involution.

It is important to note that in local relativistic QFT, the diagonalisation of the Hamiltonian is a trivial task: the states are the asymptotic states, with the relativistic spectrum. However, this does not tell us too much of the physics: the scattering of the particles and the correlation functions of local operators.

What is very surprising, however, is that in relativistic QFT, one can extract very strong conditions on the scattering matrix just from the presence of infinitely many conserved charges (instead of only being a "side" observation). Indeed, they imply that the scattering matrix of the QFT model simplifies drastically: the multi-particle scattering is factorised into two-particle scattering, whereby the set of momenta of the particles are conserved. This, along with unitarity and crossing symmetry, reduces so much the set of possible scattering matrices, for a given particle spectrum, that it is often possible to determine the scattering matrix exactly. In a sense, this is the solution of the direct problem: the invariant subspaces are the states with a given number of particles and given associated set of momenta, and the action-angle variables can be extracted from these states. Hence, in integrable QFT, the simplified canonical coordinates are those associated to the physical particles of the model.

The inverse problem, as usual, is much more complicated, and essential in order to extract most of the physical information. It amounts to calculating correlation functions of local fields. This can be done using the "form factor program", essentially a quantisation of the classical inverse scattering method. I'll be describing the quantum sine-Gordon model (in its simplest regime).

References

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