King's College London

University Of London

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Candidate No: .... **Desk No:** ....

BSc and MSci Examination

5CCM131B Introduction to Dynamical Systems for Joint Honours

Summer 2009

Time Allowed: Two Hours

This paper consists of two sections, Section A and Section B.
Section A contributes half the total marks for the paper.
Answer all questions in Section A.
All questions in Section B carry equal marks, but if more than two are attempted, then only the best two will count.

NO calculators are permitted.

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A 1. Find a solution of the initial value problem

$$\frac{dx}{dt} = t\, x^{1/2}, \qquad x(0) = 0$$

other than the trivial solution $x(t) = 0\ \forall t$. Comment on the existence of two solutions with $x(0) = 0$ in the light of Picard's theorem.

[20 marks]

A 2. Consider a dynamical system described by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) = x(x-2)(x-a)$$

where a is a non-negative number, i.e. $a \geq 0$.

(a) State for which values of a the system is structurally stable and give reasons for your answer.

(b) In what follows, assume $a = 1$.

(i) Identify the fixed points, characterize their stability, and give the invariant open sets of the dynamics.

(ii) Sketch the phase portrait.

(iii) Draw qualitative solution curves for the initial conditions $x(0) = 0$, $x(0) = 1/2$, and $x(0) = 3/2$.

[20 marks]

A 3. Consider the first order dynamical system described by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) = x(x-1)(x-2)$$

(a) Show that, for the initial condition $x(0) = x_0 = 3$, the motion is terminating by deriving a finite upper bound for the time needed to reach $x = +\infty$.

(b) Find the asymptotic form of the above equation of motion for $x \gg 2$.

(c) Solve the asymptotic equation of motion assuming the initial condition is $x(0) = x_0 \gg 2$. Using *this* solution, analyse whether $x = +\infty$ can be reached in finite time.

[35 marks]

A 4. Consider the second order dynamical system given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(x,y) = x^2 + y^2 - 2$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f_2(x,y) = x^2 - y^2$$

(a) Find the null-clines and sketch their location in the $x - y$ plane.

(b) Find the fixed points.

(c) Compute the Jacobian of the system and evaluate it at the fixed point with positive coordinates.

[25 marks]

B 5. A particle of mass m is moving on a straight line under the influence of a force generated by a potential V, but also experiences friction, so that the equation of motion is

$$m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}x}{\mathrm{d}t} = -V'(x)$$

with $\gamma > 0$.

(a) Use this equation of motion to show that energy is a monotonically non-increasing function of time in this system.

(b) Rewrite the equation of motion as a system of two first order equations by setting $x = x_1$ and $\dot{x} = x_2$.

(c) State the conditions necessary to have a stationary point of the dynamics. Let $x^\star$ be the coordinate of such a point. Calculate the Jacobian J of the system dynamics in $x^\star$, and show that the eigenvalues are given by

$$\lambda_{1,2} = -\frac{\gamma}{m} \pm \sqrt{\frac{\gamma^2}{m^2} - \frac{V''(x^\star)}{m}}$$

(d) Assume x^* is a maximum of the potential $V(x)$. Give the Jordan canonical form J^*.

(e) The Jordan canonical form J^* is related to J via a similarity transformation $J^* = P^{-1}JP$, with a suitable non-singular matrix P. You are not asked to find P.

Now we pass to the transformed variables $\begin{pmatrix} X \\ Y \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$ and consider the equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X \\ Y \end{pmatrix} = J^* \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Solve these equations and thereby investigate the nature of the fixed point.

[50 marks]

B 6. A particle of unit mass moves on a straight line under the influence of a force $F(x)$ derivable from the potential

$$V(x) = -a\ (x^2 - 1)^2\ ,$$

where a is a positive constant.

(a) Sketch the potential. Find the equilibrium points of the system and characterize them according to their stability.

(b) Compute the period τ of small oscillations about the stable equilibrium point of $V(x)$.

(c) At $t = 0$ the particle is released from $x(0) = 0$ with velocity u_0 in the positive x-direction. Show that, in order to escape to $x = +\infty$, the initial velocity of the particle must satisfy

$$u_0 > u_{\text{min}} = \sqrt{2a}\ .$$

More specifically show, in case u_0 does satisfy this bound, that the velocity of the particle, as it escapes to infinity, has the following x-dependence

$$u(x) = \sqrt{2\Big[\frac{1}{2}(u_0^2 - u_{\text{min}}^2) - V(x)\Big]}\ ,$$

and verify that under this condition $u(x) > 0$ for all $x > 0$.

(d) Write down the Hamiltonian for this system and derive the equation for the phase curves, in particular that of the separatrix.

[50 marks]

B 7. The motion of a particle of unit mass along the x axis is governed by the Hamiltonian

$$H(x,p) = \frac{1}{2}p^2 - x^2 e^{-x^2}$$

(a) Write down Hamilton's equations of motion for this system. Find the fixed points and classify them as either elliptic or hyperbolic.

(b) Derive the equation for the phase curves of the system, in particular that of the separatrix, and sketch the phase portrait.

(c) The particle is released from rest at $x = 1/2$. Explain briefly, in the context of your diagram, why the subsequent motion is oscillatory.

(d) For the initial condition given in (c), find the maximum speed of the particle and find an expression for the period τ of one complete oscillation in terms of an integral (You are not required to evaluate this integral).

[50 marks]