On the Classification of Surface Homeomorphisms

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Introduction.

The purpose of this article is to give a brief and elementary approach to the classification of mapping-classes for a surface with negative Euler characteristic. Work of Thurston ([9], [10], [7]) has brought out a deep analogy with the classical structure of the modular group $SL_2(\mathbb{Z})$ acting on the upper half plane $\mathcal{H}$ by constructing a completion of $\mathcal{L}$, the space of simple loops in a base surface viewed modulo isotopy. This serves as the boundary sphere for a proper geometric action of the mapping-class group on Teichmüller space, a natural analogue of $\mathcal{H}$. The theory has also been treated in this setting using techniques of extremal quasiconformal mappings by Bers [1]; for a detailed summary, which enlarges on the relationship between the present work and that cited above, the reader is referred to the article [3]. The book [Abikoff] presents an outline of the requisite Teichmüller theory as part of a selfcontained account to the approach of [Bers1].

Our method is a purely combinatorial one: the mapping-classes of infinite order are distinguishable by their operation on a certain connected graph $\mathcal{K}$, which describes the permutation action on the set of simple loops $\mathcal{L}$. By application of results on automorphisms of trees due to J. Tits, we show that the mapping-classes which are not reducible to mappings of any proper sub-surface are precisely those with hyperbolic $\mathcal{K}$-action, preserving a doubly-infinite geodesic line in $\mathcal{K}$. The structure of reducible homeomorphisms is described in §3, and a technique for construction of hyperbolic elements is given in the last section; this extends an original method of Thurston.

It should be noted that our treatment in this paper studies only the topological aspects of Thurston's theorem. The beautiful geometric form underlying his classification is not accessible without use of analysis.

1. A simplicial action for mapping-class groups.

We fix a reference surface S with g handles and n boundary components. The pair (g, n) is the *topological type* of S. We shall restrict attention always to the types with negative Euler characteristic, so that $2g - 2 + n > 0$.

The *mapping classes* of S are the isotopy classes, relative to the boundary ∂S, of homeomorphisms of S which fix pointwise each component of ∂S; they form a group which we denote by $\Gamma = \Gamma(S)$.

The mapping class group $\Gamma(S)$ has a natural permutation action on the set $\mathcal{L}(S)$ of free isotopy classes of simple loops in S. We have introduced in previous work [4] a structure of simplicial complex based on $\mathcal{L}$, which renders the Γ-action more accessible in some respects. It is closely related to a discrete action of $\Gamma(S)$ on the Teichmüller moduli space of marked hyperbolic structures on the base surface S which has received much attention over the years.

A *partition* of S is a non-empty subset $\lambda \subset \mathcal{L}$ such that any pair of loop classes in Λ have disjoint representatives. It is often helpful to view a partition as a way of dissecting S into subsurfaces modulo isotopy. Performing successive dissections corresponds to a partial ordering by inclusion on partitions; this provides a structure of abstract *simplicial complex* on the set $\mathcal{T}(S)$ of all partitions. For our purposes, it is more natural to restrict attention to the subcomplex of partitions that contain no class of loops enclosing a single boundary component; henceforth we tacitly make this restriction. The k-simplex of $\mathcal{T}(S)$ which represents a partition Λ with $k+1$ loops is denoted σ_Λ.

The topological structure of the surface S/Λ, obtained by cutting up S along a disjoint set of loops representing the partition Λ, is conveniently summarised in the *partition graph* K_Λ, which is defined as follows. Assign a vertex with label (g', n') to each connected component of S/Λ with type (g', n'), and put an edge between vertices (possibly co-incident) for each loop connecting those components in S.

Fig. 1: Partition graphs in genus 2.

By convention, we omit any $(0, 3)$ labelling of a vertex.

Each simple loop ℓ in S has an associated *twist map* τ_ℓ (usually called a *Dehn twist*), which is a self-homeomorphism of S fixing ℓ pointwise: it is obtained by cutting S along ℓ, applying a full 360° twist in a clockwise direction to a small annular neighbourhood of one bank of the cut and then sewing the surface back together. In view of the fact that the homotopy class of ℓ determines the isotopy class of τ_ℓ, and since twists about disjoint loops commute, we may assign to each partition Λ a free abelian *twist subgroup* of Γ, denoted $Tw(\Lambda)$, generated by the twists about loops in Λ.

For a proper inclusion of surfaces $S_0 \subseteq S$, with $S \setminus S_0$ containing no discs or annuli, we define the *relative partition complex* $\mathcal{T}(S, S_0)$ as the subcomplex of $\mathcal{T}(S)$ comprising partitions with all loops carried in S_0. One sees immediately that $cT(S, S_0)$ is isomorphic to the join of the complexes $\mathcal{T}(S')$, with S' running through all distinct components of S_0.

Example. If $S_0 = S \setminus \Lambda$ for some partition Λ, then $\mathcal{T}(S, S_0)$ is the *link* of σ_Λ in $\mathcal{T}(S)$.

The framework which we use to study $\Gamma(S)$ is the 1-skeleton of the first barycentric subdivision of $\mathcal{T}(S)$. This admits a direct definition, without prior reference to $\mathcal{T}(S)$, as the simplicial complex $\mathcal{K}(S)$ whose *vertices* are the partitions Λ, with an *edge* between Λ and Λ' corresponding to a proper inclusion $\Lambda \subset \Lambda'$. There is a *grading* by positive integers defined on the vertices of $\mathcal{K}$ by the rule $\Lambda \mapsto \mathrm{card}(\Lambda)$. It follows from consideration of

Euler characteristic for dissections of a genus g surface that the maximum value of this grading is $3g-3+n$.

Proposition 1.1 *If S is not of type $(1,1)$, $\mathcal{K}(S)$ is a connected 1-complex on which $\Gamma(S)$ operates by simplicial automorphisms that preserve the grading. The quotient is a finite combinatorial graph $\mathcal{X}(S)$.*

Proof. The first statement follows from the fact, proved in [5] (see also [Har6],[Iv]), that $\mathcal{T}(S)$ is connected. Briefly, one can apply the theorem of Dehn and Lickorish that $\Gamma(S)$ is generated by the set of Dehn twists about a finite collection of loops in S [6], to construct a path in $\mathcal{K}$ between any given pair of vertices which represent single non-dividing loops ℓ, $\gamma(\ell)$ ($\gamma \in \Gamma$) respectively; from this the path connectedness follows easily. The permutation action on classes of loops induces naturally a simplicial action on the barycentric subdivision of $\mathcal{T}$, which in turn gives rise to the action on $\mathcal{K}$. Since elements of Γ preserve the grading and adjacent vertices must have distinct grade, no inversion of an edge can occur.

As there are only finitely many different ways to dissect a surface of finite type up to topological equivalence, the remaining assertions follow from the simplicial nature of the Γ-action. □

Examples. (a) If S has type $(1,1)$, then vertices of $\mathcal{K}$ correspond bijectively to the points in $\mathbb{Q} \cup \{\infty\} \subseteq \partial\mathcal{H}$, acted upon by the classical modular group, $SL_2(\mathbb{Z})/\pm Id$. The quotient is a single point.

(b) If S is a closed surface of genus 2, then $\mathcal{X}(S)$ takes the form indicated below.

Figure 1. $S_{2,0}$ and $\mathcal{X}_{2,0}$

Note. In these two cases the action of Γ on $\mathcal{K}$ is not effective. The canonical elliptic and hyperelliptic involutions fix the entire graphs in the respective cases. In case (b), this involution amounts to rotation of the model surface $S_{2,0}$ through angle π about the axis indicated in Figure 1.

PROPOSITION 1.2 *If S has type (g, n) with $3g + n > 6$, then $\Gamma(S)$ acts effectively on $\mathcal{K}(S)$.*

Proof. For any surface $S = S_{g,n}$ with $3g + n > 6$, if a homeomorphism h fixes up to isotopy all simple loops in a given maximal partition, then h must preserve (modulo isotopy) the space S' obtained by cutting S apart along the loops. The components of S' are spheres with three boundary curves, and it is well known that their self-homeomorphisms are generated up to isotopy by Dehn twists about the boundary loops (see for instance Birman's book [2]). Now each Dehn twist τ_ℓ acts effectively on $\mathcal{K}$ – the set of loops intersecting ℓ nontrivially are permuted non-trivially by τ_ℓ. It follows, in view of the independence of the various twists, that the full group $\Gamma(S)$ acts effectively. □

2. Groups operating on graphs.

The graph $\mathcal{K}(S)$ is equipped with a $\Gamma(S)$-invariant metric $d(\ ,\ )$ determined by assigning the standard Euclidean metric of length 1 on each edge. A *geodesic* in $\mathcal{K}$ is a combinatorial path not containing any segments of the form ee^{-1} with e an edge. When restricted to the *vertex set* $V(\mathcal{K})$, d is an integer-valued function with $d(v, w)$ the number of edges in a shortest geodesic linking v with w.

We begin by outlining some elementary results about automorphisms of trees. For more details the reader may refer to Serre [8] or Tits [11]. Throughout we restrict attention to automorphisms which do not invert any edge.

PROPOSITION 2.1 *Let G be a group operating without inversions on a tree T. The following conditions are equivalent:*

- *(i) there is a vertex $v \in T$ with the orbit Gv bounded;*
- *(ii) there is a vertex of T fixed by G.*

Proof. *(ii)* $\Longrightarrow$ *(i)* is trivial. To show *(i)* $\Longrightarrow$ *(ii)*, embed the set Gv in a bounded subtree T_0 of T by connecting each pair of vertices gv, $g'v$ with a geodesic. Now T_0 is G-invariant, and an induction argument on the diameter of T_0 shows that some vertex (or edge) must be fixed by G. □

The key fact about automorphisms of trees is the following theorem due to J. Tits.

THEOREM 2.2 *If γ operates without inversion on a tree T, then precisely one of two exclusive possibilities occurs:*

- *(i) there is a vertex fixed by γ;*
- *(ii) there is a geodesic line in T on which γ acts as a translation.*

If case *(ii)* prevails, the line is termed an *axis* of γ, in analogy with terminology from hyperbolic geometry, and the automorphism γ is called *hyperbolic*.

NOTE. The proof of the theorem [11, 8] proceeds by considering the set of pairs of vertices v, $\gamma(v)$ in T with $d(v, \gamma(v))$ minimal.

This result will be applied to characterise the elements of Γ by their action on the universal covering tree $T = \widetilde{\mathcal{K}}$ of $\mathcal{K}$. Of course, the nature of an element $\gamma \in \Gamma$ is not necessarily reflected in the action of an individual lift $\widetilde{\gamma}$. We introduce the following terminology for mapping-classes.

Definition. An element $\gamma \in \Gamma$ is

(i) *reducible* if it fixes a vertex of $\mathcal{K}$,

(ii) *irreducible* if it fixes no vertex of $\mathcal{K}$.

In contrast with other approaches to classification, we shall deal separately with finite order elements, and an element will be called

(iii) *elliptic* if it has finite order.

The corresponding subsets of $\Gamma^* = \Gamma/[Id]$ are denoted Γ_R, Γ_I and Γ_E.

As a (virtually immediate) consequence of Proposition 2.1, we have the following dichotomy for the lifted action on T of a mapping class.

THEOREM 2.3 *A mapping-class lifts to a hyperbolic automorphism of T if and only if it has no bounded orbit in $V(\mathcal{K})$.*

Proof. We employ the standard construction of the universal covering $T = \widetilde{\mathcal{K}}$ as homotopy classes of path in $\mathcal{K}$ with chosen vertex v as base point : of course, geodesic segments in $\mathcal{K}$ represent these path classes. Clearly, reducible mapping classes have a lift which fixes some vertex, since if we choose a base point $v \in \mathcal{K}$ fixed by γ then γ of $\mathcal{K}$ lifts to an automorphism $\widetilde{\gamma}$ of T with fixed point the trivial path class and, conversely, the projection of an automorphism of T which fixes a vertex $\widetilde{v}$ is reducible fixing v.

If, on the other hand, a mapping class γ has a lift $\widetilde{gamma}$ which is hyperbolic, preserving an axis A in T, then the projection of A to $\mathcal{K}$ is *either* an infinite geodesic axis in $\mathcal{K}$ *or* a closed simplicial loop, given by a finite sequence of vertices on which γ acts as a cyclic permutation. But in the latter case the $\widetilde{\gamma}$-orbit of some lifted vertex in T is bounded, so that Proposition 2.1 then implies that γ is either reducible or elliptic. Therefore the projection of an invariant axis in T must be an infinite geodesic in $\mathcal{K}$. □

The classification of mapping-classes now ensues.

THEOREM 2.4 $\Gamma^* = \Gamma_I \cup \Gamma_{II} \cup \Gamma_E$, *with* $I \cap II = \emptyset$, $II \cap III = \emptyset$.

Proof. By Tits' theorem, $I \cap II$ is empty. $II \cap III = \emptyset$ by the preceding result. □

Remarks.

1) An elliptic element may be reducible or may leave invariant a loop in $\mathcal{K}$. The latter

type is termed *irreducible.*

2) For a hyperbolic element, *every* lift to T has an invariant axis. However, there does not appear to be any unique projection in $\mathcal{K}$ of minimal translation length.

In the next section we refine the classification somewhat by further analysing the class I of reducible maps.

3. Reducible mapping-classes.

The standard example of a reducible homeomorphism is a Dehn twist about a simple loop ℓ. More complicated examples arise from products of twists around loops in a partition, composed with a symmetry of the partition graph arising from a permutation of loops which respects the topological structure of the partition. Another type of example comes from selection of an irreducible homeomorphism of a partitioned subsurface; this can be extended to a homeomorphism of S in various ways, for instance, by fixing all complementary parts or by twisting along the partition loops.

We shall now show, by analysis of the fixed set in $\mathcal{T}$, that a reducible homeomorphism falls into one or other of these typical patterns.

Theorem 3.1 *If γ is a reducible mapping-class, then either it fixes a maximal partition or there are subsurfaces $S_h,\ S_e,\ S_p$ with union S such that*

(i) $\mathcal{T}(S, S_p)$ is the fixed set of γ in $\mathcal{T}(S)$,

(ii) the restriction of γ to $\mathcal{T}(S, S_e)$ $\big(\mathcal{T}(S, S_h)\big)$ is elliptic (hyperbolic respectively).

Proof. Let $\sigma_\Lambda \in \mathcal{T}$ be a simplex of $\mathcal{T}$ of maximal dimension in $\mathrm{Fix}(\gamma)$. It follows that if ℓ is any closed loop in $S \setminus \Lambda$, distinct from Λ, then ℓ is not left fixed by γ, nor is ℓ part of any γ-invariant partition of S. This implies that on any part of $S \setminus \Lambda$ that is not a 3-holed sphere, the action of γ is hyperbolic or irreducibly elliptic.

If Λ is a maximal partition, then γ belongs to the (split) extension

$$1 \longrightarrow \mathrm{Tw}(\Lambda) \longrightarrow \mathrm{Stab}(\Lambda) \longrightarrow \mathrm{Aut}(\mathcal{K}_\Lambda) \longrightarrow 1\,,$$

because each component of $S \setminus \Lambda$ has no non-trivial homeomorphisms apart from the twists about bounding loops, and any homeomorphism preserving Λ must induce an automorphism of the associated graph $\mathcal{K}_\Lambda$. In this case one could say $S_p = S$.

If Λ is not a maximal partition, then $S \setminus \Lambda$ is expressible as a disjoint union $s_1 \cup S_2$, with $S_1 = \bigcup\{S' \mid X(S') < -1\}$ and S_2 a union of three-holed spheres. On S_2, γ acts as permutations and boundary twists, so we set $S_2 = S_p$. It remains to decompose S_1 into the two purely hyperbolic and irreducibly elliptic parts. Now γ is hyperbolic if and only if every $\langle\gamma\rangle$-orbit in $\mathcal{K}(S, S_1)$ is unbounded. Consequently, the elliptic subsurface $S_e \subseteq S_1$ may be characterised as the largest subsurface $S' \subseteq S_1$ such that the restriction of γ to $\mathcal{K}(S, S')$ is a bounded orbit. $\square$

Remarks.

1) S_h is the smallest subsurface on which γ is hyperbolic,

2) The decomposition may be regarded as a kind of analogue of the Jordan canonical form for matrices.

3) Elements with $S_h \neq \emptyset$ are termed *pseudo-loxodromic* by Bers [1].

The structure of the full stability group of the simplex σ_Λ of $\mathcal{T}$ can be analysed as follows. Recall that $\mathrm{Tw}(\Lambda)$ is the twist subgroup of the partition.

Proposition 3.2. The stability group of σ_Λ is the normaliser in $\Gamma(S)$ *of* $\mathrm{Tw}(\Lambda)$.

Proof. If f is a homeomorphism of S, then $f \circ \tau_\ell \circ f^{-1}$ is, up to isotopy, the twist about $f(\ell)$. Therefore any homeomorphism which normalises $\mathrm{Tw}(\Lambda)$ must permute the loops defining Λ, and conversely. □

We remark that the stabiliser of a simplex σ' in the barycentric subdivision is obtained by taking the kernel of the epimorphism from the stability group of the carrier simplex $\sigma_\Lambda \subset \mathcal{T}$ onto the automorphism group of the associated partition graph $\mathcal{K}_\Lambda$. This group $\mathrm{Stab}(\sigma')$ centralises the subgroup $\mathrm{Tw}(\Lambda)$.

4. Construction of hyperbolic elements.

At first sight, it is by no means obvious that hyperbolic surface homeomorphisms exist, in the higher genus case at least. The purpose of this section is to describe a simple method for constructing them from reducible ones, based on the original examples given by Thurston [9].

We shall need a simple test for determination of the axis of a hyperbolic automorphism of a tree.

Lemma 4.1 Let α be an automorphism of a tree T having no fixed edge and let (v_1, v_2) be a pair of vertices representing a directed edge. If the geodesic from v_1 to $\alpha(v_1)$ contains only one from the pair v_2, $\alpha(v_2)$, then it is a segment of the axis of α.

Proof. Without loss we may assume that the geodesic $\mathbf{a}$ from v_1 to $\alpha(v_1)$ passes through v_2 – otherwise argue with α^{-1}. Consider the union $\mathbf{a} \cup \alpha(\mathbf{a})$; it must be the geodesic between v_1 and $\alpha^2(v_1)$, since otherwise one would have

$$d(v_2, \alpha(v_2)) < d(v_1, \alpha(v_1))\,.$$

Similarly we find that the union A of all α-iterates of $\mathbf{a}$ is geodesic, and α operates on it by translation; hence A is the axis of α. □

Note. We continue to assume that α operates without inversion of any edge of T.

A further elementary fact about automorphisms of trees concerns the product $\beta \circ \alpha$ of two elements with fixed sets $\mathcal{A}, \mathcal{B} \in T$.

Lemma 4.2 *If α, β and $\beta\alpha$ all have non-empty fixed sets, then they have a common fixpoint in T.* (Cf. Serre,[11] prop. 26, p.90)

Proof. The fixed sets $\mathcal{A}$, $\mathcal{B}$ of α, β are subtrees of T. If they are disjoint, let $\mathbf{a}$ be the geodesic between a pair of points $a \in \mathcal{A}$, $b \in \mathcal{B}$. Then, applying β to $\mathbf{a}$, we obtain a path $\mathbf{a} \cup \beta(\mathbf{a})$ joining a to $\beta(a)$, which passes through $\mathcal{B}$. After cancellation of null-homotopic parts, this path is a geodesic $\mathbf{a}'$ meeting $\mathcal{B}$ in a single point b' which is the mid-point. Now $\beta\alpha(a) = \beta(a)$ and $\beta\alpha$ has a fixpoint. It follows that $\beta\alpha$ must fix the midpoint b' of the geodesic $\mathbf{a}'$, so clearly $\alpha = \beta^{-1} \cdot \beta\alpha$ will also fix b', contradicting the assumption. □

We now proceed to show that a product of reducible homeomorphisms is usually hyperbolic.

Theorem 4.3 *Let $\alpha, \beta \in \Gamma$ be reducible mapping-classes with disjoint fixed sets in $\mathcal{T}(S)$. Then $\beta\alpha$ is either hyperbolic or elliptic.*

Proof. By lemma 4.2, if we choose $\widetilde{\alpha}, \widetilde{\beta} \in \mathrm{Aut}(\widetilde{\mathcal{K}})$ covering α, β with non-empty fixed sets $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{B}}$, the product $\widetilde{\beta}\widetilde{\alpha} = \widetilde{\gamma}$ must be hyperbolic since $\mathrm{Fix}(\widetilde{\alpha}) \cap \mathrm{Fix}(\widetilde{\beta}) = \emptyset$. In fact, the axis $A(\widetilde{\gamma})$ connects $\widetilde{\mathcal{A}}$ to $\widetilde{\mathcal{B}}$. It remains to check the compositions $\lambda \circ \widetilde{\gamma}$, for all cover transformations λ of $\widetilde{\mathcal{K}} \to \mathcal{K}$. Without loss, we may assume λ and $\widetilde{\gamma}$ have axes passing through a fixed reference point $\widetilde{a} \in \widetilde{A}$. There are two cases to consider.

Case (i): The axes of λ and of $\widetilde{\gamma}$ meet in the single point $\widetilde{a}$. Let x be the vertex preceding $\widetilde{\alpha}$ on the oriented $\widetilde{\gamma}$-axis. We consider the geodesic from $\widetilde{a}$ to $\lambda\widetilde{\gamma}(\widetilde{a}$,

Figure 2.

which must contain the geodesic joining $A(\widetilde{\gamma})$ and $A(\lambda\widetilde{\gamma}\lambda^{-1})$. It follows from distance arguments that $\lambda\widetilde{\gamma}(x)$ lies between $\widetilde{a}$ and $\lambda\widetilde{\gamma}(\widetilde{a})$. Hence, by Lemma 4.1, $\lambda\widetilde{\gamma}$ has axis containing $\widetilde{a}$.

Case (ii): The axes have common edges. Here too, we find that the geodesic from $\widetilde{a}$ to $\lambda\widetilde{\gamma}(\widetilde{a})$ extends to an axis. If for instance the sense of translation for λ is opposite to that of $\widetilde{\gamma}$, then one can represent the situation by the diagram below.

Figure 3.

Here we denote by $(\widetilde{a}, y)$ the intersection $A(\widetilde{\gamma}) \cap A(\lambda)$. Note that if the length d is zero, then y and $\widetilde{\gamma}(\widetilde{a})$ cannot coincide, since that would imply that $A(\widetilde{\gamma})$ projects to a loop in $\mathcal{K}$, contradicting our assumption that $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Thus $\beta\alpha$ is either hyperbolic or irreducible elliptic. $\square$

NOTES.

1) An important special case, due to Thurston [10], is obtained by setting α, β to be the twists τ_ℓ, τ_m around loops which are non-adjacent in $\mathcal{K}(S)$. Masur [] has shown, using the trajectory structure of the Jenkins-Strebel quadratic differentials associated to the loops ℓ, m in S, that the extremal dilations for the hyperbolic mapping classes $\tau_e ll^n \tau_m^{-n}$ $(n = 1, 2, \ldots)$ are given by

$$\frac{2 + n^2 i^2 + i\sqrt{n^2 i^2 + 4}}{2}$$

where i is the intersection number of ℓ with m. These elements are employed in his proof that there is a Teichmüllerspace axis whose projection is dense in the quotient moduli space and this in turn plays an important part in Ivanov's proof [] of Royden's Theorem that the automorphism group of the Teichmüller space $T(S)$ is $\Gamma(S)$ if S is closed surface of genus $g \geq 2$.

2) An analogous procedure on appropriate subsurfaces produces pseudo-hyperbolic elements with prescribed canonical type.

3) Presumably not all hyperbolic elements arise in this fashion. Also, since the constructed maps all have even translation distance on $K(S)$, one can expect that these elements are not primitive.

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