

# THOMPSON'S GROUP AS A TEICHMÜLLER MAPPING CLASS GROUP

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## INTRODUCTION

An important part of the general problem of understanding complex analytic phenomena in one dimension is the question of moduli: find an appropriately defined marking and parametrisation for the model object (e.g. a type of Riemann surface or dynamical system). This is tantamount to construction of a suitable Teichmüller space. Co-existent with this problem is that of finding the modular group of this space, in order to make possible construction of a precise quotient space which represents the classifying space of shapes for the given (topologically marked) model: for instance this is the (genus  $g$ ) mapping class group in the case of a closed Riemann surface  $X_g$ .

The Thompson groups are important discrete groups which were first discovered by R.J Thompson as an ingredient in the study of solvable and unsolvable problems in logic. Thompson's group  $F$  can be defined in many different ways; perhaps the simplest is as the group of all piecewise linear self-homeomorphisms of the unit interval which are differentiable at all but finitely many dyadic rational points in  $[0, 1]$  and whose slope in each linear piece is a power of 2.

In this paper we give two examples where Thompson's  $F$  group acts as a Teichmüller mapping class group. In the first example, the underlying surface  $\Omega$  is the complement in the complex plane of the standard middle-thirds Cantor set and the Teichmüller space is  $T_0(\Omega)$ , the space of asymptotically conformal deformations of  $\Omega$ . Thompson's group turns out to be isomorphic to the group of homotopy classes of asymptotically conformal, quasiconformal self-maps of  $\Omega$  that commute up to homotopy with complex conjugation  $j(z) = \bar{z}$ .

In the second example  $F$  acts on the dynamical Teichmüller space  $AT(\Omega, e)$  consisting of all quasiconformal deformations of the dynamical system  $e$  into other uniformly asymptotically conformal systems  $(\tilde{\Omega}, \tilde{e})$ , factored by asymptotic conformal equivalence. The action is obtained by letting elements of  $F$  act on the dual Riemann surface  $\Omega^*$  with its dual dynamical system  $e^*$ .

### 1. DOMAINS OF BOUNDED GEOMETRIC TYPE.

Let  $\Lambda_0$  be the standard middle-thirds Cantor set,  $\Omega_0 = \mathbb{C} \setminus \Lambda_0$  and  $\Omega = f(\mathbb{C} \setminus \Lambda_0)$  for some quasiconformal homeomorphism  $f$  of  $\mathbb{C}$ . Such a plane domain  $\Omega$  will be called a domain of *bounded geometric type*.

It is sometimes convenient to think of  $\Omega$  as having been constructed from sewing together a single disk  $D_0$  with a marked central point and an infinite countable

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family of Riemann surfaces, each of which is a three-holed sphere usually referred to as a *pair of pants* or *Y-piece*. We single out one *Y-piece*  $Y$ , called the *top piece*, and label the remaining *Y-pieces* by finite words in 0's and 1's. Thus, for each  $n \geq 1$ , and for each  $w \in \{0, 1\}^n$ , we have a *Y-piece*  $Y_w$ . The collection  $\{Y_w\}$  is assumed to be of bounded type, which means that the extremal lengths of the three families of curves on each  $Y_w$  homotopic to its three boundary components are bounded below by  $1/M$  and above by  $M$ , where  $M$  is independent of  $w$ . We also *mark* the *Y-pieces*, by labelling the boundary components  $\gamma_w^0$ ,  $\gamma_w^1$  and  $\gamma_w^\infty$ . We call  $\gamma_w^\infty$  the *waist* of  $Y_w$  and  $\gamma_w^0$ ,  $\gamma_w^1$  the left and right *cuffs* of  $Y_w$ , respectively. The pasting is done in the following way. We think of the disc  $D_0$  as  $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$  with  $\infty$  as its marked central point. The top *Y-piece*  $Y$  is sewn along its waist curve to  $D_0$  along the curve  $\{|z| = 1\}$ . Then the waist curve  $\gamma_0^\infty$  of  $Y_0$  is sewn to the left cuff  $\gamma_0$  of  $Y$ , while the waist curve  $\gamma_1^\infty$  of  $Y_1$  is sewn to the right cuff  $\gamma_1$  of  $Y$ . Proceeding inductively, for each word  $w = i_1 i_2 \cdots i_n i_{n+1} \in \{0, 1\}^{n+1}$  of level  $n + 1$ , the waist curve  $\gamma_w^\infty$  of  $Y_w$  is sewn to the cuff  $\gamma_{w^*}^{i_{n+1}}$  of  $Y_{w^*}$  along its contour, where  $w^* = i_1 i_2 \cdots i_n$ . All sewings of boundary curves are performed with maps of the circles which have uniformly bounded distortion. The resulting surface  $\Omega$  is a planar Riemann surface, i.e. has genus zero (see Figure 1).

**Lemma 1.** *The Riemann surface  $\Omega$  can be conformally embedded in the Riemann sphere, and the embedded domain  $\Omega' \subset \overline{\mathbb{C}}$  is unique up to post-composition by a*

FIGURE 1. A binary-tree Riemann surface and its pants decomposition

complex affine map if we normalize it so that the point  $\infty$  in  $D_0$  is mapped to  $\infty \in \bar{\mathbb{C}}$ . Moreover, the complement  $\Lambda = \bar{\mathbb{C}} \setminus \Omega'$  is a Cantor set.

*Proof.* Since  $\Omega$  is planar, the first assertion follows from Koebe's uniformization theorem. Let  $C$  be a connected component of  $\Lambda$ . We can find an increasing sequence of words  $w_k \in \{0, 1\}^k$  with the property that the images of the corresponding sequence of waist curves  $\gamma_{w_k}^\infty$  under the embedding are nested in the complex plane and the bounded components of their complements all contain  $C$  in their interior. But each  $\gamma_{w_k}^\infty$  has a *collar*, which is an annulus of definite modulus going around  $\gamma_{w_k}^\infty$ , and such collars are pairwise disjoint. Since the sum of these moduli is infinite,  $C$  must reduce to a point. Hence  $\Lambda$  is totally disconnected. The same argument also shows that no point of  $\Lambda$  is isolated. It can also be refined to show that a classical theorem of Painlevé holds true here, namely,  $\Lambda$  is holomorphically removable. Hence, to prove the second assertion, suppose  $\Omega''$  is the image of another embedding of  $\Omega$  in  $\bar{\mathbb{C}}$ . Then we have a conformal homeomorphism  $\Omega' \rightarrow \Omega''$ , and since  $\Lambda$  is a removable set, this homeomorphism extends to a conformal mapping of the Riemann sphere fixing infinity, which must be complex affine.  $\square$

It follows from the uniformization theorem that  $\Omega$  carries a Poincaré metric  $\rho(z)|dz|$ , the unique complete Riemannian metric of constant curvature equal to  $-4$ . Every waist curve of every pair of pants  $Y_w$  (apart from that of  $Y$  which bounds  $D_0$ ) is homotopic to a unique closed  $\rho$ -geodesic  $\alpha_w$ . From now on we shall reserve the name *waist curve* for such geodesic loops. The totality  $\mathcal{W}$  of all waist curves  $\{\alpha_w : w \in \cup_{n=0}^\infty \{0, 1\}^n\}$  cut  $\Omega$  into a new family of pairs of pants  $\tilde{Y}_w$ , each isotopic to  $Y_w$ . These waist curves have a natural hierarchy, the top waist curve

FIGURE 2. A domain of bounded type and its waist curves

being the boundary of  $D_0$ , the next level being the two waist curves homotopic to the top boundaries of  $Y_0$  and  $Y_1$ , and so on.

**1.1. A dynamical example.** The complement  $\Omega_0$  of the standard middle-thirds Cantor set,  $\mathbb{C} \setminus \Lambda_0$ , can be created dynamically in the following similar way. For  $D_0$  one takes the set  $\{z \in \mathbb{C} : |z - 1/2| > (1/2) + (1/9)\}$ . Then let

$$(1) \quad e_0(z) = 3z \quad \text{and} \quad e_1(z) = 3z - 2,$$

and

$$(2) \quad e = \begin{cases} e_0(z) & \text{if } z \in e_0^{-1}(D_0^c) \\ e_1(z) & \text{if } z \in e_1^{-1}(D_0^c). \end{cases}$$

If  $D_0^c$  denotes the complement of  $D_0$  in  $\mathbb{C}$ , then the top pair of pants is  $P_0 = D_0^c \setminus (e_0^{-1}(D_0^c) \cup e_1^{-1}(D_0^c))$ , and all of the other pants that make up  $\mathbb{C} \setminus \Lambda_0$  are generated as images under backward powers  $e^{-n}(P_0)$ ,  $n \geq 1$ . Moreover, the middle-thirds Cantor set  $\Lambda_0$  is equal to

$$\bigcap_{k=1}^{\infty} e^{-k}(D_0^c),$$

and the hierarchy of waist curves is generated by the backward powers  $e^{-n}$  applied to  $\alpha_0 = \{z : |z - 1/2| = (1/2) + (1/9)\}$ . We write  $j$  for complex conjugation,  $j(z) = \bar{z}$ , and observe that this dynamical system has symmetry with respect to  $j$  in the sense that  $e \circ j = j \circ e$ . Note also that the sets  $e^{-n}(\alpha_0)$  constitute a standard family of waist curves that separate  $\mathbb{C} \setminus D_0$  into its standard pants decomposition, and such that all waist curves and all pairs of pants cut out by them are invariant with respect to  $j$ . The Cantor set  $\Lambda_0$  lies on the fixed point locus of  $j$ , namely, the real axis, and of course the points of  $\Lambda_0$  inherit the natural (L-R) ordering.

The set  $\mathcal{W}$  of waist curves, the involution  $j$ , the standard pants decomposition of  $\mathbb{C} \setminus \Lambda_0$ , and the natural ordering on  $\Lambda_0$  are carried by any quasiconformal homeomorphism  $f$  of  $\mathbb{C}$  to  $\Omega = f(\mathbb{C} \setminus \Lambda_0)$  and to  $\Lambda = f(\Lambda_0)$ .

We denote by  $\Omega_n$  the level- $n$  surface formed as the union of all the Y-pieces  $e^{-k}(D_0)$ ,  $k = 0, 1, \dots, n$ . Then  $\Omega_n$  is a surface of genus zero with boundary consisting of the point  $\infty$  and the  $2^n$  waist curves at level  $n$  which are the connected components of  $e^{-n}(\partial D_0)$ . Thus we obtain  $\Omega$  as the increasing union of the subsurfaces  $\Omega_n$ .

## 2. TEICHMÜLLER SPACE, ASYMPTOTIC TEICHMÜLLER SPACE AND TEICHMÜLLER SPACE WITH ASYMPTOTIC EQUIVALENCE

The *Teichmüller space*  $T(R)$  of a Riemann surface  $R$  is the space of all quasiconformal mappings  $f$  from the fixed surface  $R$  to a variable Riemann surface  $f(R)$  factored by an equivalence relation. Two maps  $f_0$  and  $f_1$  are equivalent if there is a conformal map  $c : f_0(R) \rightarrow f_1(R)$  and an isotopy  $g_t$  from  $c \circ f_0(R)$  to  $f_1(R)$  such that  $g_0 = c \circ f_0$ ,  $g_1 = f_1$  and  $g_t(p) = c \circ f_0(p) = f_1(p)$  for every point  $p$  in the ideal boundary of  $R$  and for every  $t$  with  $0 \leq t \leq 1$ . When  $R$  is the standard Cantor surface  $\Omega_0$ , this condition is equivalent to saying that the isotopy  $g_t$  fixes all points  $p$  in the Cantor set  $\Lambda_0$ .

The *Teichmüller space with asymptotic conformal equivalence*  $AT(R)$  (or *co-asymptotic Teichmüller space*) is defined in exactly the same way except that the conformal map  $c$  in the definition is allowed to be *asymptotically conformal*. By definition,  $c : R \rightarrow S$  is asymptotically conformal if it is quasiconformal and if for every  $\epsilon > 0$  there

exists a compact subset  $E$  of  $R$  such that  $K_z(c) < 1 + \epsilon$  for  $z \in R \setminus E$ .

Finally, the *asymptotic Teichmüller space*  $T_0(R)$  is the closed subspace of  $T(R)$  consisting of asymptotically conformal classes. A class is *asymptotically conformal* if for every  $\epsilon > 0$  there is a representative  $f$  of the class and a compact subset  $E$  of  $R$  such that  $K_z(f) < 1 + \epsilon$  for  $z \in R \setminus E$ . For a proof that the set of such classes is closed in  $T(R)$ , see [?]. It should be noted that a typical mapping in an asymptotic conformality class will not necessarily be asymptotically conformal; for instance there are Teichmüller extremal maps - of constant dilatation almost everywhere - in many cases. However, by employing representative mappings whose Beltrami coefficients are harmonic and vanishing at infinity, it turns out that for any Riemann surface  $R$ , each asymptotically conformal class in  $T(R)$  contains an asymptotically conformal representative mapping. Moreover, these spaces  $T(R)$ ,  $AT(R)$  and  $T_0(R)$  are both complete metric spaces and complex manifolds with charts given by the Bers embedding, see [?]. It is also important for our purposes to note that the conformally natural Douady-Earle extension can be used to show each of these three spaces is contractible, [?]. The proof uses the following lemma of Earle, Markovic and Saric, [?]

**Lemma 2.** *The Douady-Earle extension  $f$  of an asymptotically conformal mapping class is  $(1 + \epsilon)$ -biLipshitz off compact sets. That is, given any  $\epsilon > 0$ , there exists a compact subset of the Riemann surface  $R$  such that for  $z$  and  $w$  in  $R$  and outside this compact subset,*

$$(1 + \epsilon)^{-1}d(z, w) \leq d(f(z), f(w)) \leq (1 + \epsilon)d(z, w),$$

where  $d$  is the Poincaré metric on  $R$ .

The group of homotopy classes of quasiconformal self-maps of  $R$  is called the mapping class group,  $MCG(R)$ . Any  $h$  representing an element of  $MCG(R)$  induces an automorphism  $\beta_h$  of  $T(R)$  by the recipe  $\beta_h([f]) = [f \circ h^{-1}]$ , and  $[h] \mapsto \beta_h$  is a homomorphism from  $MCG(R)$  to  $Aut(T(R))$ . The map  $h \mapsto \beta_h$  induces by restriction an action of  $MCG_0(R)$  on  $T_0(R)$ , where  $MCG_0(R)$  is the subgroup of  $MCG(R)$  consisting of those mapping classes that have asymptotically conformal representatives.

### 3. THOMPSON'S GROUP AND THE PURE MAPPING CLASS GROUP

We begin by describing two models for Thompson's  $F$ -group. In the first model  $F_1$  for  $F$ , the group is viewed as a group of piecewise linear homeomorphisms of the unit interval. (For an exposition of the range of definitions and properties of Thompson's groups see [?] and [?].) Select two copies of the unit interval  $I_D$  and  $I_R$ , which we will call the domain interval and the range interval. Corresponding to any integer  $n > 0$ , subdivide the domain interval and the range interval into subintervals by drawing  $n$  hash marks in the respective intervals according to the following rules:

1. the first hash mark is placed at the midpoint, dividing the interval into two subintervals;
2. the second hash mark is placed at the midpoint of one of the two subintervals, so that the original interval is now divided into one interval of length  $1/2$  and two intervals of length  $1/4$ ;
3. by induction, the  $k$ -th hash mark is placed at the midpoint of one of the  $k$  subintervals obtained from the placement of the  $(k - 1)$ -st hash mark.

FIGURE 3. A pair of unit intervals subdivided into 5 subintervals representing an element of  $F_1$

Since both  $I_D$  and  $I_R$  are now divided into  $n+1$  intervals, there is a unique piecewise linear, increasing mapping of  $I$  to itself that maps the  $j$ -th subinterval of  $I_D$  linearly onto the  $j$ -th subinterval of  $I_R$ .

In the first model, Thompson's group is the group  $F_1$  of all homeomorphisms obtained in this manner, with the group operation given by composition.

In the second model  $F_2$ , the group is viewed as the set of equivalence classes of pairs of trees embedded in the plane. We let  $(D, R)$  be such a pair,  $R$  stands for "range" and  $D$  stands for "domain." By definition each tree  $R$  and  $D$  consists of the same number of carets, and there is a top caret which we label by  $\infty$  and call the "anchor" or the "point at infinity." The endpoints or leaves of  $R$  and  $S$  can then be labelled from left to right, and since  $R$  and  $S$  have the same number of carets, they have the same number of leaves. Two pairs  $(D_1, R_1)$  and  $(D_2, R_2)$  are equivalent by an elementary step if  $(D_1, R_1)$  is obtained from  $(D_2, R_2)$  by the addition of a caret at the  $k$ -th position of  $D_2$  and at the  $k$ -th position of  $R_2$ , or vice-versa. Following Greenberg [?], we call such an addition a "birth." Clearly, the

FIGURE 4. A pair of labelled trees representing an element of  $F_2$ 

process of performing an arbitrary finite number of births induces an equivalence relation on pairs of trees.

Composition of two equivalence classes  $h_1$  and  $h_2$ , represented by  $(D_1, R_1)$  and  $(D_2, R_2)$ , respectively, is achieved by first doing a sufficient number of births in both pairs of trees so that the tree representing the range of  $h_1$  matches precisely the tree representing the domain of  $h_2$ . When the range of  $h_1$  matches the domain tree of  $h_2$ , the composition  $h_2 \circ h_1$  is represented by the pair of trees  $(D_1, R_2)$ . One shows easily that this composition respects the equivalence relation defining the elements of  $F_2$ . Also each tree induces a dyadic partition of the unit interval, and the dyadic partitions associated to  $D$  and  $R$  determine a piecewise affine homeomorphism  $\iota(h)$  of the unit interval corresponding to the element  $h$  represented by  $(D, R)$  in  $F_2$ . The map  $h \mapsto \iota(h)$  is an isomorphism from  $F_2$  onto  $F_1$ . We refer to [?] and [?] for more details.

In this section,  $R$  will denote  $\Omega_0$ , the complement of the middle-thirds Cantor set, which we view as generated dynamically by the mappings  $\{e^n\}_{n=1}^\infty$  given in the previous section. Since any quasiconformal self-map defined on  $\Omega_0$  extends to a quasiconformal self-map of  $\mathbb{C}$ , there is a natural homomorphism  $\alpha$  from  $MCG(\Omega_0)$  into the group of homeomorphisms of the Cantor set  $\Lambda$  defined as follows:- given a mapping  $h$  representing an element  $[h]$  of  $MCG(\Omega_0)$ , let  $\tilde{h}$  be its unique continuous extension to  $\Lambda_0$ ; then  $\alpha([h])$  is defined to be the restriction of  $\tilde{h}$  to  $\Lambda_0$ .

Let  $MCG_0(\Omega_0)$  be the subgroup of  $MCG(\Omega_0)$  consisting of asymptotically conformal mapping classes and consider the subgroup  $G$  of elements  $g \in MCG_0(\Omega_0)$

satisfying two conditions:- (i) the extension  $\alpha(g)$  is an order preserving self-mapping of  $\Lambda_0$ ;

(ii)  $g$  preserves the set of homotopy classes of waist curves.

Let  $B$  be the subgroup of  $G$  consisting of those elements that are represented by mappings  $b$  for which there is a level  $n$  such that  $b$  is equal to the identity in  $\Omega \setminus \Omega_n$ .

**Theorem 1.** *The homomorphism  $\alpha$  induces a short exact sequence of groups:-*

$$(3) \quad 0 \longrightarrow B \xrightarrow{\subset} G \xrightarrow{\alpha} F \longrightarrow 0.$$

*The image  $\alpha(G)$  is a subgroup of the group of homeomorphisms of  $\Lambda_0$  isomorphic to the Thompson  $F$ -group, and the kernel is the subgroup  $B$ , which is the union of pure mapping class groups on subsurfaces  $\Omega_n$  of  $\Omega_0$ .*

*Proof.* We construct an isomorphism  $\vartheta$  from  $\alpha(G)$  onto  $F_2$ . By the definition of our subgroup  $G \subset MCG_0(R)$ ,  $\alpha(g)$  is an order-preserving homeomorphism of  $\Lambda_0$  and the quasiconformal map  $g$  has a vanishing Beltrami coefficient. From Lemma ?? the assumption that  $g$  has vanishing Beltrami coefficient implies that the Douady-Earle extension  $\tilde{g}$  for the boundary mapping representing the class  $g$  is a  $(1 + \epsilon)$ -hyperbolic isometry outside compact subsets of  $\Omega$ . But a simple lemma in hyperbolic trigonometry of rightangled hexagons – see the appendix to this chapter – implies that the hyperbolic length of any nonwaist loop is at least twice that of the waist curves. Therefore, there exists a level  $n$  such that the  $2^n$  waist curves at level  $n$  on  $\Omega$  are mapped by  $\tilde{g}$  to curves homotopic to another set of  $2^n$  waist curves that separate  $\infty$  from  $\Lambda_0$ . The part of  $\Omega$  lying above the  $2^n$  waist curves at level  $n$  is retractible in an obvious way to a binary tree with anchor at  $\infty$  and with  $2^n$  leaves: we consider the symmetry of  $\Omega \subset \mathbb{C}$  induced by complex conjugation, which divides  $\Omega$  into a union of hyperbolic regular right hexagons, and use the dual tree of hyperbolic line segments connecting all neighbouring barycentres of the set of hexagons above level  $n$ . We let this tree constitute the domain tree  $D$  of a map  $h$  in the model  $F_2$ . The range tree  $R$  is obtained in the same way from the corresponding  $2^n$  waist curves in the image of  $\tilde{g}$ .

Now we define  $\vartheta : \alpha(G) \rightarrow F_2$  by the rule which sends  $\alpha(g)$  to the pair  $(D, R)$ . It is obvious that  $\vartheta$  is a group homomorphism. It is one-to-one because the pair  $(D, R)$  determines the image of every point in the Cantor set  $\Lambda_0$ . To see that  $\vartheta$  is surjective, we must find quasiconformal self-maps  $g_0$  and  $g_1$  of  $\Omega$  which are asymptotically conformal and which induce order-preserving homeomorphisms of  $\Lambda$  whose images are the generators  $x_0$  and  $x_1$  of  $F_2$ . Thus we must show the existence of two maps  $g_1$  and  $g_2$  illustrated by the following pictures (Figure 5).

The preceding discussion implies that  $\alpha$  maps onto  $F_2$ , a group isomorphic to  $F$ . Moreover, if the asymptotically conformal map  $g$  is in the kernel of  $\alpha$ , then  $g$  induces the identity on  $\Lambda$ , and so there is a level  $n$  such that all of the homotopy classes of waist curves at that level are preserved by  $g$  and, moreover, on each part of  $\Omega \setminus \Omega_n$  below this level,  $g$  is homotopic to the identity. Also we note that  $g$  cannot involve a twist around any of these curves because such a twist would transform the homotopy class of a closed loop transversal to that waist curve to a homotopy class with much greater hyperbolic length, contradicting the  $1 + \epsilon$ -quasiconformality. This implies there is a level  $n$  such that  $g$  is homotopic to a mapping which is the identity outside  $\Omega_n$ .  $\square$

FIGURE 5. Two pairs of pairs of Riemann surfaces corresponding to the generators  $x_0$  and  $x_1$

Write  $j(z) = \bar{z}$  for complex conjugation and let  $G_0$  be the subgroup of mapping classes in  $G$  that commute up to homotopy with  $j$ . Then a quasiconformal self-mapping  $h$  of  $\Omega_0$  represents an element of  $G_0$  if

1. the mapping class of  $h$  is asymptotically conformal,
2. the unique continuous extension  $\tilde{h}$  of  $h$  to  $\Lambda_0$  is order-preserving on  $\Lambda_0$ ,
3.  $h \circ j$  is isotopic to  $j \circ h$  by an isotopy that fixes the points of  $\Lambda_0$ .

**Theorem 2.** *The restriction of  $\alpha$  to  $G_0$  is an isomorphism of  $G_0$  onto  $F_2$ .*

*Proof.* Let  $h$  be a self-map of  $\Omega_n$  that fixes the point at  $\infty$ , fixes the homotopy classes of the  $2^n$  waist curves at level  $n$  and commutes up to homotopy with  $j$ . Then  $h$  is given by a mapping of the hexagonal decomposition of  $\Omega_n$  whose boundary values are prescribed by the finite tree mapping; hence it is determined uniquely up to homotopy.  $\square$

Recall now the complex and Teichmüller-metric structures carried by the space  $T_0(\Omega_0)$ : for a proof that for any  $R$ ,  $T_0(R)$  is a complex submanifold of the complex manifold  $T(R)$ , see [?].

**Theorem 3.** *Thompson's  $F$ -group acts discretely as a group of isometric, biholomorphic automorphisms of the complex manifold  $T_0(\Omega_0)$  with trivial isotropy groups.*

*Proof.* It is a consequence of the definitions that the mapping class group acts as a group of automorphisms and isometries of  $T(R)$ . It is also clear that those elements of the mapping class group which have asymptotically conformal representatives preserve the submanifold  $T_0(R)$  of  $T(R)$ . If these mappings also commute with  $j$  up to homotopy, then by the previous theorem they form a subgroup isomorphic to  $F_2$ .

To prove that this action is free of fixed points, we note first that given a point  $t \in T_0(\Omega_0)$ , any mapping  $h$  of a finite subsurface  $S \subset \Omega_0$  representing the action of a (non trivial) element of  $F_2$  determines a nontrivial element of the pure mapping class group for  $S$ , since it preserves the individual waist curves which are boundary components; the group  $F_2$  therefore acts discretely and without fixed points, since the group of pure mapping classes does so on the space  $T(S)$ .  $\square$

### Appendix: A lemma in hyperbolic geometry

Fixing attention on the surface  $\Omega_0$  with marking induced by the chosen waist curve decomposition with side-seams added to each Y-piece, we mention an alternative approach to showing that the orbit under the  $F_2$ -action of this particular marked surface is discrete. On a finite tree-like subsurface which carries the action of a given element  $h$ , some waist curve  $\gamma$  of this hyperbolic surface is sent to a closed nonwaist curve  $h(\gamma)$ . We claim that this homotopy class will have extremal hyperbolic length on  $t(\Omega_0)$  at least  $\ell(\gamma)$  multiplied by a factor  $\delta > 1$ , by application of the following Lemma which is also important in the proof of Theorem ?:

**Lemma 3.** *Let  $P$  be a hyperbolic right-angled hexagon with two sets of three alternate lengths equal to  $a$ ,  $b$  respectively. Denote by  $l$  the length of one (hence any) perpendicular bisector of two opposite sides. Then  $l > a + b$ .*

*Proof.* Divide the hexagon up, using the three perpendicular bisectors of opposite pairs of sides, into six congruent hexagons with angles  $\pi/2, \pi/2, \pi/2$  and  $\pi/3$ . The desired inequality is then an elementary consequence of results from hyperbolic trigonometry on side lengths of these Lambert quadrilaterals. See [?], Theorem 7.17.1.  $\square$

Our assertion now follows from the observation that any closed geodesic nonwaist loop passes through two or more pairs of pants and must have length at least  $2l$ . Waist curves have length  $2a$ , say, and the perpendicular distance between two of them (length of the seams of a pants Y-piece) is  $2b$ .

But this fact implies that the Teichmüller-distance between the points  $t = [\Omega_0]$  and  $h(t)$ , which is the logarithm of the dilatation of this mapping  $h$ , is at least  $\log \delta > 0$ , by Teichmüller's lemma.

4. THE DYNAMICAL TEICHMÜLLER SPACE  $AT(\Omega, e)$ 

In this section we let  $e$  be the dynamical system described in formulas (??) and (??) and we consider a space of deformations of  $e$  into other *uniformly asymptotically conformal* dynamical systems whose Cantor repellers  $\Lambda$  have variable fine-scale structure.

The system  $e$  is uniformly asymptotically conformal in the sense that arbitrarily long compositions of the backwards branches  $e_0^{-1}$  and  $e_1^{-1}$  of  $e$  are nearly conformal, [?]. Moreover, these compositions act almost isometrically near the repelling set  $\Lambda_0$  in the following sense:-

For every  $\epsilon > 0$ , there is an open neighborhood  $U$  of the Cantor set  $\Lambda$  such that, for all positive integers  $n$  and for all  $z$  in  $e^{-n}(U) \setminus \Lambda_0$ ,

$$(1 + \epsilon)^{-1} \leq \frac{\rho(e^n(z))|(e^n)'(z)|}{\rho(z)} \leq (1 + \epsilon).$$

By integration this inequality implies that for all  $z$  and  $w$  in  $U \setminus \Lambda$ ,

$$(1 + \epsilon)^{-1} d_P(z, w) \leq d_P(e^{-n}(z), e^{-n}(w)) \leq (1 + \epsilon) d_P(z, w).$$

Here,  $e^{-n}(z)$  and  $e^{-n}(w)$  may be assumed to travel along the same backwards paths of inverse branches of  $e$ .

We shall say that a dynamical system that has this property is *uniformly asymptotically isometric*.

**DEFINITION.**  $AT(\Omega, e)$  is a quotient space of the set of all quasiconformal maps  $f$  mapping  $\Omega$  to a variable plane domain  $f(\Omega)$  with the property that  $f \circ e \circ f^{-1}$  is uniformly asymptotically conformal. The quotient space is determined by Teichmüller equivalence, that is,  $f_0$  is equivalent to  $f_1$  if they are equivalent as elements of the complex manifold  $AT(\Omega)$ .

**Lemma 4.**  $AT(\Omega, e)$  is a closed, connected subspace of  $AT(\Omega)$ .

*Proof.* Suppose  $f_k$  is a sequence of mappings representing elements of  $AT(\Omega)$  and each  $e_k = f_k \circ e \circ (f_k)^{-1}$  is a uniformly asymptotically conformal dynamical system on the domain  $f_k(\Omega)$  and  $f_k$  converges to  $f$  in the sense that the Beltrami coefficient  $\mu_k$  of  $f_k$  converges in the  $L_\infty$ -norm to the Beltrami coefficient  $\mu$  of  $f$ . Then up to a bounded factor the Beltrami coefficient of  $e_k^n$  is equal to

$$\sigma_k(z) = \mu_k(e_k^n(z)) \frac{\overline{(e_k^n)'(z)}}{(e_k^n)'(z)} - \mu_k(z)$$

and by hypothesis  $|\sigma_k(z)| \leq \epsilon$  for all  $z$  in  $e^{-n}(U)$ . By taking the limit as  $k \rightarrow \infty$  we obtain the same inequality with  $\mu_k$  replaced by  $\mu$ .

The space is connected because if  $f^\mu \circ e \circ (f^\mu)^{-1}$  is asymptotically conformal, then  $f^{t\mu} \circ e \circ (f^{t\mu})^{-1}$  is a curve of asymptotically conformal dynamical systems which for  $t = 0$  is  $e$  and for  $t = 1$  is  $f^\mu \circ e \circ (f^\mu)^{-1}$ .  $\square$

## 5. DUAL RIEMANN SURFACES AND DUAL CANTOR SETS

In the next section we will show how  $F$  acts as a subgroup of the group of automorphisms the closed subspace  $AT(\Omega, e)$  of  $AT(\Omega)$ . In order to describe the action of  $F$  we need to define dual dynamical system and Riemann surface  $(\Omega, e)^* =$

$(\Omega^*, e^*)$ . This construction mimics the construction of dual Cantor sets in dimension 1 studied by Sullivan in [?].

We begin by constructing  $\Omega_0^*$ , that is, the Riemann surface dual to the complement of the standard middle-thirds Cantor set with its dynamical system given in formulas (??) and (??). Let  $\mathcal{W}$  be a set of waist curves on  $\Omega_0$  generated by  $e$ . If we put the top waist curve in  $\mathcal{W}$  equal to  $\gamma_0 = \{z : |z - 1/2| = (1/2) + (1/9)\}$ , then all of the other curves in  $\mathcal{W}$  are images by compositions of finite length of words in the maps  $e_0^{-1}$  and  $e_1^{-1}$ . Cutting  $\mathbb{C} \setminus \Lambda_0$  along all of the curves of  $\mathcal{W}$  one obtains an infinite set of pairs of pants  $P_j$  and the punctured disc which is the exterior of  $\gamma_0$  in  $\mathbb{C}$ . These pairs of pants are divided into levels according to their distance from this punctured disc. The top level, level zero, is comprised of a single pair of pants. The  $n$ -th level is comprised of the  $2^n$  pairs of pants contiguous to the  $2^{n-1}$  pairs of pants at the previous level. The points of each waist curve of  $\mathcal{W}$  bounding the pairs of pants are uniquely identified with the points of any other waist curve of  $\mathcal{W}$  by some composition of the maps  $e_0, e_1, e_0^{-1}$  and  $e_1^{-1}$ .

The *dual Riemann surface*  $\Omega_0^*$  is obtained by sewing together the top curve and all of the pairs of pants according to the following rules:

1. the top disc remains at the top and the top pair of pants is affixed to it in exactly the same way that it was affixed for  $\Omega_0$ ,
2. the  $2^n$  pairs of pants at level  $n$  on  $\Omega_0$  remain at level  $n$  on  $\Omega_0^*$ ,
3. the top waist curve of each pair of pants at level  $n$  is sewn to one of the two bottom waist curves each pair of pants at level  $n - 1$  according the unique identification generated by  $e_0, e_1, e_0^{-1}$  and  $e_1^{-1}$ .

$\Omega_0^*$  has a natural anticonformal involution  $j^*$  which is the common extension of the natural involution of each individual pair of pants  $P_k$ . Therefore,  $\Omega_0^*$  can be embedded in the plane in such a way that its complement  $\Lambda_0^*$  lies on the real axis and  $j^*$  coincides with complex conjugation. In this way one sees that the points of  $\Lambda_0^*$  have a natural order induced by the order of the real axis and, except for the top disc and the top pair of pants of  $\Omega_0^*$ , all of the remaining pairs of pants of  $\Omega_0^*$  split into two parts, the right side and the left side.

Having constructed  $\Omega_0^*$ , we construct a quasiconformal expanding map  $e^* = e_0^* \vee e_1^*$ , where  $e_1^*$  and  $e_0^*$  map the right and left sides of  $\Omega_0^*$  over  $\Omega_0^*$  in the obvious way.

**Lemma 5.** *The asymptotic structure of the dual  $(\Omega_0^*, e^*)$  depends only on the asymptotic structure of  $(\Omega_0, e)$ . That is, if  $h$  is an asymptotically conformal map of  $\Omega_0$  conjugating  $e$  to  $h \circ e \circ h^{-1}$ , then there is an induced asymptotically conformal map  $h^*$  mapping  $\Omega_0^*$  to  $h^*(\Omega_0^*)$  and conjugating  $e^*$  to  $h^* \circ e^* \circ (h^*)^{-1}$ .*

*Proof.* □

**Lemma 6.**  $(\Omega, e)^{**}$  is isomorphic to  $(\Omega, e)$ .

## 6. THE ACTION OF $F$ ON $AT(\Omega, e)$ .

We let  $F$  act on the dual Riemann surface  $\Omega_0^*$  in the same way it acts on  $\Omega_0$  in section 3. The elements of  $F$  are represented by asymptotically conformal self-maps of  $\Omega_0^*$  that commute up to homotopy with the involution  $j^*$ .

**Theorem 4.** *Suppose  $h$  is an asymptotically conformal self-map of  $\Omega_0^*$  representing an element of  $F$ . Then  $(h \circ e^* \circ h^{-1})^*$  is a uniformly asymptotically conformal dynamical system acting on  $\Omega$ .*

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