HRI Introductory Lectures on Mapping Class Groups.

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Chapter 1.

This chapter covers introductory material very roughly, aiming to be sufficient to enable us to give meaningful algebraic and geometric definitions of these groups and their basic properties without the usual notational overload. It assumes some *basic facts on topology of surfaces, including covering spaces and the geometric action of the fundamental group; some familiarity with the three special models of two dimensional geometry, the complex plane, sphere and disc, is also very desirable. We develop the process of extracting an algebraic presentation of a discrete groups from a geometric action, and two prominent examples are described, which are familiar from elementary courses on complex analysis. These are the torus, with fundamental group isomorphic to the two dimensional Euclidean lattice* $\mathbb{Z}+\mathbb{Z}$*, and the modular surface, which is an orbifold quotient of the upper half plane by the modular group* $SL(2,\mathbb{Z})$*; in this way we encounter both abelian and hyperbolic surface groups in their natural geometric habitat. Riemann surface automorphisms will be discussed briefly, using both geometric and algebraic descriptions of how they act.*

(1.1) Introduction. We begin with a discussion of several alternative definitions of the mapping class groups. Let X be a closed orientable surface of genus g, perhaps with additional structure involving $r \geq 0$ holes and/or $n \geq 0$ marked points: often one concentrates on the case with no holes or points, known as the *closed surface case*. The set of homeomorphisms of X onto itself is taken, together with the equivalence relation of homotopy; it is an (elementary but slightly lengthy) exercise to verify that this produces a group, $MCG(X)$, by checking the axioms using several elementary facts about combining and reversing homotopies.

There is a more elaborate version of the homotopy equivalence relation, which brings in the extra decoration data: this involves homotopies relative to a fixed finite set of extra decoration on the surface, which fix pointwise any holes (boundary curves) and/or points, the latter either with branching number or puncture. This gives rise to the more elaborate mapping class groups $\Gamma^r_{g,n}$. Their importance is twofold: as a natural extension of the groups of the first type, they constitute a broadening of the class, while secondly, they arise naturally as a tool for understanding the subgroup structures arising from surface inclusions and coverings, possibly ramified, of one surface by another.

A competing definition uses smooth representatives for the mappings concerned and the concept of isotopy: an*isotopy* between two paths (or mappings) is a path in the space of paths (or mappings) from one to the other.

Also, we need to discuss the important algebraic version of a mapping class, which depends on the initial elementary observation that a surface homeomorphism $f : X \to X$ induces an isomomorphism $\theta = \theta_f : \pi_1(X, x_0) \to \pi_1(X, x_0')$ between the fundamental groups of X based at the points x_0 and $x_0' = f(x_0)$. Since it is trivial to construct a null-homotopic homeomorphism of the surface X to itself which moves x_0' to x_0, we may in fact view θ as an *induced automorphism* of $\pi_1(X)$, albeit non-canonical. There is a further elementary fact to take in, relating to homotopy on the surface: a surface homeomorphism f_0 is homotopic to the identity if there is a continuous map $H : I \times X \to X$ with $H(0, .) = f_0(.)$ and $H(1, .) = \mathrm{Id}$, and this implies that the induced automorphism θ_0 is *inner*, given by the rule (defined for closed paths γ in X based at x_0) that

$$\gamma \mapsto \partial H,$$

the boundary loop given by tracing out the image of the homotopy H.

The Nielsen theorem, which provides a precise link with these induced automorphisms of $\pi(X)$ modulo inner ones, will be discussed later (see Lecture 2).

Writing $\mathrm{MCG}(X) = \Gamma^r_{g,n}$: usually $r = 0$ in these notes. Special genus $(g = 0, 1)$ examples are discussed next and the following coincidences: $\Gamma_{0,4} = \Gamma_{1,0} \cong \Gamma_{1,1}$; $\Gamma_{0,5} = \Gamma_{1,2}/\langle \mathbf{j} \rangle$ with $\mathbf{j}$ the elliptic sheet interchange involution.

Here, for motivational purposes, we mention just a few reasons for studying these groups:- low dimensional topology and geometry, abstract geometric group theory, moduli of Riemann surfaces, string theory and QFT.

(1.2) Group-theoretic properties. A fundamental distinction exists between the mapping class groups and other geometrically defined groups like finite co-volume lattices in Lie groups, with which they nevertheless share some key algebraic properties. For instance the similarly defined groups $\mathrm{Out}(F_m)$ are quite different from the $\Gamma_{g,n}$ provided $m \geq 2$: in fact $\mathrm{Out}F_2 \cong GL_2(\mathbf{Z})$ and $\Gamma_{1,1} \cong \Gamma_1$ are identical. Also, although there are obvious inclusions (and non-split exact homology sequences) interconnecting the groups $\Gamma_{g,n}$ and $\Gamma_{g,m}$ with $m \neq n$, there are rather few natural relationships between the closed surface mapping class groups for distinct genera. This makes them intriguing objects from the standpoint of geometric group theory. It was realised gradually during the decade after 1970 that they share many properties with lattices in linear groups, although they are in fact not arithmetic groups and no faithful linear representation is known except in the very special torus case. On the other hand, it emerged during the 1990's following the work of Witten and others on topological and conformal field theories in two dimensions that there are projective representations: this amounts to the existence of so-called modular functors.

Properties shared with lattices in Lie groups include the following:

- Finite presentation; various finiteness properties of torsion subgroups, residual finiteness, torsion-free subgroups of finite index, cohomological dimension (virtual) is $4g - 5$ (or $4g - 4 + n$ if $n > 0$).

- Aside from Γ_1, these groups are NOT arithmetic: in fact no faithful linear representation is known.
- Analogous to rigidity theorems for lattices in Lie groups and for such groups as $Out(F_n)$, with F_n a free group of rank $n \geq 2$, one can prove there are usually no morphisms between mapping class groups.

(1.3) Complex structures on a torus. The special case $g = 1 \Rightarrow \Gamma_1 \cong SL_2(\mathbf{Z})$ was discussed in a bit of detail, including the important concept of a *homotopy-marking* together with action of the MCG on marked tori and the relation with action of the homogeneous modular group by fractional linear mappings of $\mathcal{U} = \{Im(\tau) > 0\}$.

(1.4) Higher genus. If $g \geq 2$, then we can define the Torelli subgroup as kernel of homomorphism to Aut $H_1(X, \mathbf{Z})$ – thus, it is a normal subgroup with infinite index and torsion-free. Hence the MCGroup cannot be arithmetic with rank≥ 2, by the Finiteness Theorem of Margulis, which rules out such normal subgroups.

Rank 1 arithmetic is ruled out also, incidentally: one way to see this is to show that the structure of maximal abelian subgroups is wrong (see J. Harer's excellent survey article in *Theory of Moduli*, Springer LN in Math. vol 1337).

(1.5) Examples of mapping classes. Dehn twists defined: as an example, go back to genus 1, where they are represented as shift matrices , eg. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

Exercise (1.1) Verify the *braid relation* $ABA = BAB$.
Exercise (1.2) Use the geometric connection with a standard Dirichlet fundamental domain in $\mathcal{U}$ for Γ_1 to identify the fractional linear group elements corresponding to the elements A, B, AB and BA.
[Note: these are the Order 3 torsion elements associated with the interior corner points of the tesselation.]

A more basic reason why mapping class groups are not isomorphic to rank 1 lattices is suggested in the following exercise for those who know a little about hyperbolic space isometries and lattices (or come back to it when you do!).
Exercise (1.3). The elements of infinite order in a hyperbolic lattice group satisfy the property that two of them commute if and only if they have the same fixed point set on the sphere at ∞. Show that this property cannot be true for the mapping class groups when the surface has genus $g > 1$.

Chapter 2.

This covers hyperbolic geometry of surfaces: metric, isometries, geodesic arcs and loops, translation lengths, shape coordinates for a hyperbolic surface; the Teichmüller space of marked hyperbolic structures. Nielsen's work on mapping classes (outline).

(2.1) Siblings: geometric and algebraic MC groups. Began by discussing briefly Nielsen's Theorem on representing geometric automorphisms of

π_1 by surface homeomorphisms, which completes the characterisations of the MCGroups.
[Omitted to mention the parallel theory fo the groups AutF_n and OutF_n for the free group F_n of rank $n \geq 2$: this is interesting because there is an analogous action of that group on the space of marked metric graphs which are homotopy equivalent to the wedge of n circles. This is called *outer space* (Culler-Shalen), and has much in common with the Teichmüller space.]

(2.2) Hyperbolic plane geometry. Recall here the basics of hyperbolic plane geometry, including definition of the infinitesimal and global forms of the Poincaré metric on $\mathcal{U}$ and types of hyperbolic isometry.

Exercise (2.1) Verify that the unit disc model of hyperbolic plane geometry, denoted $\mathcal{D} = \{|z| < 1\}$, with the infinitesimal metric form $ds_h = 2|dz|/(1-|z|^2)$ has as (direct) isometry group the Möbius transformations

$$S(z) = \frac{az+b}{cz+d} \quad \text{with } \bar{b} = c,\ \bar{a} = d, \text{ and } |a|^2 - |b|^2 = 1.$$

Exercise (2.2) Show that the function

$$\delta(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - z_1\overline{z_2}} \right|$$

is invariant under all disc automorphisms – transformations as in exercise (2.1) – and verify that $\delta < 1$ for all pairs of points in $\mathcal{D}$.

Note that one can show then that as $z_1 \to z_2$, $\delta(z_1, z_2)$ tends to the hyperbolic length element ds_h. This gives a nice formula for the global hyperbolic metric distance from z_1 to z_2 as

$$d(z_1, z_2) = \frac{1}{2} \ln\Big(\frac{1+\delta}{1-\delta}\Big).$$

Exercise (2.3) Show that the action of the isometry group Aut$(\mathcal{U})$ (or equally, Aut$(\mathcal{D})$) is transitive on the set of ordered triples of distinct points of the boundary circle. Interpret this set as the unit tangent bundle of $\mathcal{U}$, the set of all unit tangent vectors to the points of the h-plane.
(2.3) Representation spaces of π in Isom$\mathcal{U}$. The precursor of our formulation in this section is the Fricke spaces of Fuchsian groups, defined in the classic text *Vorlesungen üeber die Theorie der Automorphen Funktionen* by R. Fricke & F. Klein. We begin with a choice of Fuchsian group Γ with $X = \mathcal{U}/\Gamma$. Then $R_0(\Gamma, G)$ denotes the space of type-preserving faithful representations. It was proved by A. Weil [Annals, 1962, 1964] that the component containing the identity representation consists entirely of representations with discrete image. In fact, this space has two isomorphic components corresponding to the two choices of orientation possible, interchanged by the anticonformal self-mapping $z \mapsto -\bar{z}$ of $\mathcal{U}$.
[*Note* (for later discussion): The representation spaces into *any Lie group G* are interesting: they correspond to flat connections on G-bundles over surfaces. I

might have mentioned Goldman's results on euler numbers for flat $G-$bundles on a surface, which includes the case of Picard/Jacobi varieties ($G = U(1)$) and quasifuchsian space ($G = PSL_2(\mathbf{C})$), and also Hitchin's Theorem on the Teichmüller components for any real semisimple Lie group G which contains a copy of $PSL_2(\mathbf{R})$.]

The Teichmüller space of hyperbolic structures is a quotient space of $R_0(\Gamma, G)$ by the adjoint (conjugation) action of $G = PSL_2(\mathbf{R})$. Notice that this is a smooth fibration – even real analytic.

(2.4) Decompositions into pants and Fenchel-Nielsen coordinates. The approach by Fricke and Klein to classification of Fuchsian groups used decompositions into what are now called HNN-extensions and amalgamated free products of simpler groups with three generators, known as triangle groups, corresponding to geometric dissections of the surface X into pieces with three boundary components (perhaps including a puncture or conne point); in Thurston's evocative terminology, this is now called a *pants decomposition* of the surface/orbifold.

The data necessary for a complete pants decomposition is a maximal set of homotopically distinct simple loops in X, that is, we begin by choosing a sequence of loops which divide the surface (at least locally) into topologically simpler pieces, and continue until no further choice is possible. This situation is reached when any further choice is forced to be a loop in the homotopy class of one chosen earlier: this implies that the pieces must all be spheres with three boundary components, *pants* in Thurstonese.

Exercise (2.4) Use basic surface theory (eg. Euler characteristic) to show that the number of loops in a maximal decomposition of $X_{g,n}$ is $3g - 3 + n$. How many combinatorially different decompositions are there in genus 2 and 3?

Now we invoke a fundamental fact about Riemannian hyperbolic geometry: in the homotopy class of any given simple loop there is a unique closed geodesic, and distinct disjoint loops have disjoint geodesic representatives. This can be seen most clearly in the universal covering: try the following

Exercise (2.5) If $X = \mathcal{U}/\Gamma$, show by the path-lifting and homotopy theory of covering spaces that each simple loop lifts to an infinite path in $\mathcal{U}$ which determines two points on the boundary circle, fixed points of the covering transformation in the Fuchsian group $\Gamma \cong \pi_1(X)$ which corresponds to the loop α. Deduce that there is a unique geodesic which connects these two boundary points. Why does this imply the above statement?

Clearly, therefore, the business of manufacturing a hyperbolic surface structure on X amounts to the construction of all possible hyperbolic pants shapes and then assembly of such structures on the pieces with the correct sizes fitting together where necessary. The tailoring pattern is fixed by the decomposition and the choice of h-length of each loop, together with the *twist parameters*, one for each loop. Teichmüller space has $6g - 6 + 2n$ dimensions:

Theorem. *F-N coordinates determined by any choice of pants decomposition give bijective mapping to* $\mathcal{U}^{3g-3+n} \subset \mathbf{C}^{3g-3+n}$.

Note. Real analytic but not holomorphic in any direct way. Refer to Wolpert's work on the WP-metric and almost complex structure in F-N coordinates (à la Kerckhoff).

(2.5) Action of $\Gamma_{g,n}$. Proper discontinuity of the action of the mapping class group on $T_{g,n}$ now ensues – finite stabilisers are given by the Hurwitz theorem on finiteness of any group of (compact Riemann surface) automorphisms: here we need to insert the statement/proof of this theorem, using Macbeath's Theorem (in Proc. Glasgow Math. Assoc.**5**, 1960).

To prove proper discontinuity (discrete orbits) it is sufficient to define an invariant metric, eg. the Weil-Petersson infinitesimal (Kahler) metric on $T_{g,n}$(*vide infra*).

Quotient spaces are defined by these two parallel discrete actions:

$$R_0(\Gamma) \longrightarrow R_0(\Gamma)/\mathrm{Aut}(\Gamma) = \mathcal{D}(\Gamma)$$

$$T(\Gamma) \longrightarrow T(\Gamma)/\mathrm{Out}(\Gamma) = \mathcal{M}(\Gamma)$$

which shows that
THEOREM. *The moduli space $\mathcal{M}_{g,n}$ is an orbifold: i.e. only singularities are of the form $\mathbf{R}^N/H$, with $H \leq SO(N)$, finite subgroups.*
(2.6) Twisting a hyperbolic structure. Arbitrary length F-N twists operate on T_g, given in terms of a pants-partition: move one side (the rhs, say) from zero position (in an upper half-plane picture with imaginary axis as chosen lift of loop) to new position with opposite (left) side fixed, to obtain a new hyperbolic structure.
Note: These are a precursor of Thurston's *earthquakes*, which are the same kind of hyperbolic slide mapping but defined for *geodesic laminations*, which are much more complicated unions of disjoint simple geodesics in X.

Section 3.

Thurston's classification of surface homeomorphisms up to isotopy. Relationship with Teichmüller space and Brouwer's Fixed Point Theorem. The boundary sphere of geodesic laminations and Dehn-Thurston coordinates.
(3.1) Thurston's boundary. A second space on which MCG acts is closely related to the first: it is the *Thurston boundary sphere* of $T_{g,n}$, constructed as a means to classify isometries (elements of the Teichmüller modular group). Note that the boundary of the F-N image will not do: there is no natural action in these coordinates except that of the subgroup stabilising the surface pants partition, which is very small compared to $\Gamma_{g,n}$ in group-theoretic terms.
EXERCISE (3.1) Show that this subgroup is necessarily a virtually abelian group isomorphic to a Euclidean lattice in $\mathbf{R}^{3g-3+n}$.

Later we shall meet other parametrisations of T_g, with their boundaries.

(3.2) Classification of genus 1 mapping classes. First recall the three types occurring in genus 1: the modular group Γ_1 contains examples of each type as we saw earlier, including the hyperbolic isometry given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ generating an *Anosov automorphism* of the (square) torus.

EXERCISE (3.2) (a) Show that there are only finitely many conjugacy classes of elliptic and parabolic isometries in $SL_2(\mathbf{Z})$.
(b) Show that there are infinitely many conjugacy classes of hyperbolic isometries in $SL_2(\mathbf{Z})$.

NOTE. The rational boundary boundary points p/q) of $\mathcal{U}$ correspond bijectively to simple loops on the torus winding p (respectively q) times around in the A (resp. B) direction.

EXERCISE (3.3) What loop corresponds to the point ∞?

EXERCISE (3.4) Construct the *Farey tesselation* in $\mathcal{U}$, by insering the infinite hyperbolic geodesic line connecting two points, p/q and r/s, precisely when $ps - qr = 1$. Include $\infty = 1/0$ as part of this. Show that such a pair or boundary points represents two simple geodesic loops in the torus which intersect in one point.

Note. This Farey configuration of lines serves also to provide an augmented pattern for the action of the modular group on the boundary of the Teichmüller space of marked complex structures on the torus; this is important in Minsky's proof of the ending lamination conjecture for the case of punctured tori (Annals of Math., 2000).

The intuition necessary to prepare for Thurston's general version comes from consideration of the real boundary points which are fixed by some hyperbolic isometry $f \in \Gamma_1$ corresponding to an Anosov automorphism of the torus. Each of these has two real fixed points, one attracting and one repelling, and they are quadratic irrationals, roots of the polynomial equation with integral coefficients obtained from $f(z) = z$.

(3.3) Classification in genus two or more. Nielsen's sterling efforts on the same problem on closed surfaces with $g > 1$ over 20 years produced awkward-to-state but interesting results – see work of J. Gilman. His classification involved too many ingredients for us to give a clearcut summary, but did begin to distinguish three fundamentally different types of surface mapping.

Thurston solved the problem completely by inventing a new compactification of $T_{g,n}$ to which the action of MCG extends continuously.

For this, he invokes an intuition on the result of iterating a homeomorphism f on a geodesic loop α; result of $\lim_{n\to\infty} f^n(\alpha) := f \circ \ldots \circ f(\alpha)$ should foliate the surface in some sense, rendering almost all of X into flat Euclidean planar pieces, but with a finite set of *finite-pronged singularities.* These turn out to be the same type as those generated by the real (or pure-imaginary) trajectory structure at finite order zeros of a holomorphic quadratic differential form.

Thurston's definition of *measured foliation* goes roughly as follows: the surface X has a local product decomposition on the complement of a finite set of singularities, which determines a partition into flow boxes, each with projection to an interval parametrising the set of leaves, and singular leaves which enter or leave a (pronged) singular point. Then define equivalence classes of these generated by Whitehead moves on a pair of interlinked singular points. The result is an almost purely topological entity on the surface, but with a metric ingredient, a real transverse measure (ie. a continuous function approximable by taking intersection numbers, representing distance on the transverse leaf intervals) thrown in. For more details, see the notes from the Orsay Seminar *Travaux de Thurston sur les surfaces*, Asterisque volume (Soc. Math. de France, 1979).

Then to tie this notion in with T_g, we must attach the boudary sphere to the Teichmüller space, we need to represent both as subsets in the same projective space.

For this purpose, we regard each hyperbolic marked surface as a countable set of hyperbolic lengths, which means that we have functions $\ell(\gamma) \in \mathbf{R}_+ \mid \gamma \in \mathcal{L}$, defined on the set of loops in the surface up to free homotopy : need to use again the fact (noted earlier in $2) that each loop has a unique geodesic representative in its homotopy class with well-defined length in the hyperbolic metric. In other words we have a (hyperbolic) h-length function $\ell : T(X) \to \mathbf{R}_+^{\mathcal{L}}$.
[Useful fact: Note that this function is locally determined by suitable choice of just $6g - 6 + n$ lengths, but no single choice will do everywhere in $T(X)$.]

To compactify, need projective classes of h-length functions and then we have two more key results of Thurston, although with earlier work by Dehn (it seems) on the same ideas, that
Theorem. *(a) The infinite vector of h-length functions embeds the T_g on $\mathbf{P}(\mathbf{R}^{\mathcal{L}})$ (as a kind of open cone) and the closure is a topological $N-$sphere, where $N + 1 = dim\ T_{g,n}$.*

(b) The action of $\Gamma_{g,n}$ on loop set $\mathcal{L}$ extends to this closed disc.

Next we apply the Brouwer Fixed Point Theorem to obtain interior or boundary fixed points for infinite order elements and analyse the possibilities: interior fixed points (isolated or not), fixed foliations, eg from a Dehn twist, and two distinct boundary fixed points (foliations) which generates the picture of a pseudo-Anosov homeomorphism.
[Need to say more about this later.]

(3.4) Comments on geodesic laminations. Mentioned finally Nielsen's approach by lifing homeomorphism of a surface to the universal covering disc $\mathcal{U}$, with fixed points on the boundary circle, which generates the famous *geodesic laminations* on a hyperbolic surface X, thus leading naturally to Thurston's alternative approach to describing the boundary of T_g. Refer to Casson-Bleiler book *'Automorphisms of surfaces after Nielsen & Thurston'* (LMS Student Texts) for more on this.

Chapter 4.

Complex geometry of T_g. Teichmüller discs. The complex of loop systems in a surface and how it relates to the earlier sections, including F-N coordinates and the action of the MCGroup.

(4.1) Thurston's Axis Theorem for pseudo-Anosov mappings. This is a precise analogue of the picture from the torus case, with an infinite h-line joining the two fixpoints of a hyperbolic torus automorphism (see Exercise 1.1).

Outlined the Bers approach to existence via Teichmüller's theorem: this requires preliminary work on the complex analytic approach to Teichmüller theory, through the notion of extremality for the mappings (assumed to be quasiconformal or even C^1) which are evolving as we follow a path along the geodesic axis. Also we wish to make a point of fitting these deformations within the general theory, and so gave a statement of the big theorem of Teichmüller which began the modern theory of moduli of Riemann surfaces:

THEOREM. *Given any two marked points* $[X_j, f_j] \in T_{g,n},\ j = 1, 2)$, *there is a unique extremal qc mapping* $f : X_1 \to X_2$ *such that* $f \circ f_1 = f_2$ *up to Teichmüller-equivalence.*

Notes. 1) This result, properly interpreted, implies that the Teichmüller spaces are *straight spaces* in the sense of Buseman.

2) There is much more to be said about these geodesic rays connecting the given points. See later discussion of *T-rays* and *T-discs*.

(4.2) Beltrami's equation and complex deformations. Broadening the discussion of the deformation theory, there was a brief discussion of the complex structure on $T_{g,n} = T(X)$: it is induced by invoking the theory of Beltrami's equation for the *distortion* of a homeomorphism and dependence of solutions w_μ on the complex (measurable) function μ on X which constitutes the distortion parameter – no details given.

Also make the point that there is an approach via Earle-Eells which brings out the relation with harmonic mappings and calculus of variations. This continues today with work of Wolpert and Wolf.

See the course notes by Fred Gardiner for more on this aspect of Teichmüller theory.

(4.3) Mapping tori. The action of a pseudo-Anosov element f of MCG on its T-axis produces a circle as quotient, and there is a natural way to connect this to the mapping torus

$$M(f) = X \times [1 \leq t \leq 1]/\{(x, 0) \sim (f(x), 1) \mid x \in X\}$$

which is a closed 3-manifold fibering over S^1 with fibre X and with f as monodromy. Mention Thurston's very important hyperbolisation programme and

THEOREM.*For the 3-manifolds $M(f)$ fibering over the circle with pseudo-Anosov holonomy, there exists a hyperbolic structure, unique up to isometry.*

This raises the natural problem of exploring ways of inter-relating the hyperbolic structure of $M(f)$ (e.g. numerical invariants of Kleinian groups) with the geometry of the action of f on Teichmüller space (the axis, translation distance,...).

[Could say more: eg. the results of Wolpert and Daskalopoulos-Wentworth on existence of unique WP axis for any pseudo-anosov element.]

(4.4) The curve complex. Define the simplicial complex $\mathcal{T}(X)$ of systems

of disjoint loop classes in x. How it looks in genus 2, as a countable collection of triangles joined along edges, with infinitely many faces sharing any edge. It is a locally infinite CW complex of dimension $3g - 4 + n$ if $X = X_{g,n}$.
PROPOSITION. *The curve complex is connected and simply connected when* $3g - 3 + n \geq 2$.

In fact, Harer showed around 1983 that $\mathcal{T}(X)$ is homotopically a wedge of infinitely many $2g - 2 + n$-spheres, which then leads on to many facts about the group theoretic status of mapping class groups, such as the fact that they satisfy a generalisation of Poincaré duality and have finite virtual cohomological dimension. Also, there are generalisations of the curve complex for nonzero n and r which lead to fairly precise results on the cohomology, such as stability of the rational cohomology groups $H_k(\Gamma^r_{g,n}, \mathbf{Q})$ for large enough g, $g > 3k$.

Brief mention of the original point for introducing the curve complex: to compactify the moduli space by addition of the singular objects which are the result of pinching X along curves: these are surfaces with nodes, which occur in the Deligne-Mumford compactification of $\mathcal{M}_g$. The action of the MCG on the vertices of $\mathcal{T}$, by permuting the homotopy classes of simple loops, extends easily to an action as simplicial automorphisms on this complex, which we return to in the 6th lecture.

EXERCISE (4.1) Show that this action in the case of genus 1 preserves the Farey tesselation of the h-plane defined in an earlier Exercise.

EXERCISE (4.2) Describe in detail the structure of the complexes $\mathcal{T}_{g,n}$ for $g = 1, 2$ and $n = 0$ or 1.

EXERCISE (4.3) Show that up to topological conjugacy, there are precisely $1 + [\frac{g}{2}]$ distinct vertices, orbits for the action of Γ_g. This is the number of vertices in the quotient (virtually simplicial) complex.

EXERCISE (4.4) Choose a vertex of $\mathcal{T}_g$, a simple loop α in X. Determine the structure of the *link* of this vertex in $\mathcal{T}$, which is the sub-complex consisting of all simplices $\sigma = \{\alpha_j : j = 1, \ldots, k$ in $\mathcal{T}$ such that $\sigma \cup \alpha$ is a simplex in $\mathcal{T}$.
[*Hint*: look at the case $g = 2$ first: the answer involves the previous exercise.]

[We shall need to come back to the action on $\mathcal{T}_g$ in more detail, so as to have a good picture of the projective embedding of $\widehat{\mathcal{M}}_g$, for discussing Veech curves, for instance.]

Interlude: Grothendieck Dessins and Belyi's Theorem.

(I.1) Belyi's Theorem. A **Grothendieck dessin** or **line drawing** is a connected finite graph drawn in a compact surface with the property that each complementary piece of surface is topologically a polygon, a disc with finitely many vertices on the edge.

On somewhat mysterious grounds, but based partly on his theory of algebraic fundamental groups for varieties, G. suggested that there was an intimate link with algebraic number theory, pointing out the relationship between such dessins and two apparently unrelated notions: representations of a certain discrete (extended triangle) group (essentially the classical modular congruence subgroup $\Gamma(2)$ on the one hand and the Galois theory of the field of algebraic numbers on the other.

Around 1979, G.V. Belyi proved a result which brought out the significance of these ideas:

THEOREM. *If a projective algebraic curve X, thought of as a compact Riemann surface, is defined by a set of polynomial equations with coefficients in an algebraic extension of the rational field $\mathbf{Q}$ then there is a meromorphic function on X whose only singular values are the three points $0, 1$ and ∞.*

The converse result was already known; it follows from a theorem of A. Weil on fields of definition for algebraic varieties. A more elementary proof of this may be found in the recent article by G. Gonzalez-Diez (*Around Belyi's Theorem*, preprint, Un. Autonoma Madrid).

Grothendieck later (around 1983) wrote an extended account of his ideas and their place in an elaborate conjectural theory of 'an-abelian fundamental groups' in algebraic geometry. Thanks partly to a beautiful article by [G. Shabat and Voevodsky] in the Grothendieck Festschrift, these ideas on dessins have given birth to a rich cottage industry thriving on the fringes of three disciplines, each of which normally demands very heavy machinery (terminology and weight of background theory): algebraic geometry, number theory and complex analytic geometry of Riemann surfaces. Even more interestingly perhaps, they are related via work of V. Drinfeld, Y. Ihara and others to all sorts of contemporary work in arithmetic geometry and theoretical physics, including a class of quasi-Hopf algebras arising in the study of quantum groups.

(I.2) Dessins and Teichmüller curves in moduli space. We examine the relationship with Teichmüller discs, a special kind of deformation of complex structure on the underlying surface which carries a dessin. It turns out that a type of dessin with sufficiently large symmetry group – the *symmetry group of a dessin* (X, K) is the set of all automorphisms of the supporting curve which preserve the graph K – determines a complex analytic curve (surface with punctures), immersed in the corresponding moduli variety, and passing through the point which represents the preferred complex algebraic curve. In fact, many points of moduli varieties $\mathcal{M}_g$ which represent curves definable over the field of algebraic numbers $\overline{\mathbf{Q}}$, belong to at least one such modular family, and any such curve has a finite cover which does.

Examples of well-known surfaces falling into this category include the class of hyperelliptic curves with defining equation

$$y^2 = x^n + 1, \qquad (n \geq 5)$$

and the Fermat curves

$$u^n + v^n = 1.$$

For these examples, see discussion in section 6 below, based on articles by Veech (Inv. Math. **97**(1989) and GAFA **2** (1992).) and also by WJH (in *'Mapping Class Groups and Moduli Spaces,'* Contemp. Math. Vol.**150**).

(I.3) Examples of pseudo-Anosov homeomorphisms. To complement the earlier discussion of Thurston's classification of mapping classes, we comment on the link with Thurston's original construction of pseudo-Anosov mapping classes in Γ_g; see his article in Bull. Amer. Math. Soc. **19** (1988) for instance.

Take loops α, β which fill up the surface S, that is to say, such that the complement of the loops is a union of topological cells, each with an even number of sides, which are labelled α_j or β_k according to the loop from which they come. Choose centres for these components of $S - \{\alpha \cup \beta\}$ and a dual cell decomposition of S into quadrilaterals which follows from joining centres of adjacent components by edges labelled as v = vertical for α, h = horizontal for β.

Figure: Dual cell decomposition of S.

Exercise. Draw a local picture of a surface with two loops which fill, make the subdivision and hence supply the correct figure here.

Next, make each of these subdivided quadrilateral cells into a unit square, remembering the v/h dichotomy. Then there is a singular Euclidean structure on S obtained by using the squares to determine local charts; some of the corners contribute to a finite set of cone singular points (where more than four corners meet). Moreover, twists about α or β act as translations in the developing map and holonomy representation which are obtained in the Euclidean plane from this affine structure — in fact, the group generated by them is usually (but not always) a subgroup of finite index in $SL_2(\mathbf{Z})$. In any case, it is therefore possible to find many homeomorphisms, expressed as words in the twists α and β, which correspond under this representation to hyperbolic elements of $SL_2(\mathbf{Z})$ and which therefore define pseudo-Anosov homeomorphisms of S. For

a detailed account of this construction, see either Veech (Inv. Math. 1989) or Earle & Gardiner (Contemp. Math **201**, 1997).
NOTES: 1. It follows from Veech's earlier work (on the so-called Teichmüller geodesic flow) that the points of T_g which possess a representation like the one constricted here, in terms of branched Euclidean geometric structures with holonomy in $GL_2(BZ)$, are in some sense dense rational subset of the quotient cotangent bundle over the moduli space.
2. There is an analogy to be made here with the Hedlund-Hopf ergodic theorem for the geodesic flow on the unit tangent bundle of a Fuchsian group with finite area quotient. In a separate development on the same theme, H. Masur (in 'Riemann Surfaces and related topics',Ann. of Math. Studies vol.97, 1981) proved that the set of geodesic rays constructed by Thurston projects to a dense subset of the moduli space $\mathcal{M}_g$.

Lecture 5.

This talk broke away from the established provisional pattern of topics.
We covered more on mainly Belyi's theorem and the Thurston examples given in (I.3) and discussed the fact that this insight of Grothendieck results in a very complicated interaction between mapping class groups and the absolute Galois group of the algebraic numbers $\overline{\mathbf{Q}}$.

Chapter 6.

Recent work on mapping class groups.Compare with lattice subgroups in higher dimensional Lie groups and rigidity. Symmetry properties: Teichmuller discs and fuchsian subgroups; .
(6.1) More on uses for the complexes $\mathcal{T}_{g,n}$. There is a growing list of applications for these curious simplicial complexes in current research. A short list of items which use it includes Harer's work on the homology of the modular groups and the stable cohomology ring, Ivanov's work on rigidity of mapping class groups, Minsky's (and co-workers) proof of Thurston's ending lamination conjecture for Kleinian surface groups, the coarse geometry of augmented Teichmüller spaces (Weil-Petersson metric completion) in the sense of Gromov, and the quasi-isometry classification of the mapping class groups in the sense of Gromov. Also (at least indirectly) it is involved in Kontsevich's proof of the Witten conjecture on Chern classes of intersections of certain line bundles on the Deligne-Mumford moduli spaces $\widehat{\mathcal{M}}_{g,n}$.

Unfortunately there is no time now to provide a detailed discussion or even a full list.

(6.2) Royden's Theorem and the curve complex $\mathcal{T}_g$. The result of Masur mentioned in (I.3) has been applied by N.V. Ivanov to give a distinctive new proof of the famous Theorem of H. L. Royden on isometries of Teichmüller space.
THEOREM. *The isometry group of T_g in the Teichmüller metric is the mapping class group Γ_g.*

The proof involves two steps:

Any isometry of T_g induces a simplicial automorphism of the curve complex $\mathcal{T}_g$;

The automorphism group of $\mathcal{T}_g$ is the Teichmüller modular group Γ_g.

This second step is an important result (of Ivanov). It extends to all values of g, n except $(0,4)$, $(0,5)$ and $(1,n)$ with $n = 0,1,2$.

The identification of the MCG as the group of all simplicial automorphisms is a precursor of similar results on other actions of Γ_g on infinite complexes related to $\mathcal{T}_g$, which have indicated the way to a detailed analysis of the innate 'coarse geometry' of the group itself, along lines first established by Gromov.

In particular, there is a so-called pants complex of complete pants decompositions, also known as the Hatcher-Thurston complex, which is a 2-dimensional cell complex very similar to the dual 2-skeleton of $\mathcal{T}_g$, and Jeff Brock has proved that this complex with the standard euclidean metric on each simplex is quasi-isometrically equivalent to the Teichmüller space with the Weil-Petersson metric.

(6.3) An analogy with Out(F_n) . A recent result was outlined (by WJH with M. Korkmaz) which shows by elementary methods that there are no homomorphisms (apart from the trivial one) between Γ_g and Γ_h when $g > h \geq 1$.

Higher dimensional symmetry in $\mathcal{M}_g$ (and its absence) is also of great interest; examples show that such patterns do exist in these spaces. Note however that a counter trend also exists, rigidity theorems which assert that some geometric object has relatively few symmetries/automorphisms. In this context one has the Royden theorem which was mentioned earlier, and a more recent instance, the Masur-Kaimanovich Theorem: *all homomorphisms from a higher rank lattice to a mapping class group are virtually trivial.*

(6.4) Teichmüller discs in T_g and their projections to moduli space. In the last part of the final lecture, I included a summary of recent work on these, which are in fact examples of rank 1 hyperbolic space symmetry in moduli space – this is the only type of complete symmetry broadly possible – finishing with results by Veech, Hubert & Schmidt and McMullen on T-discs with large stabiliser in the mapping class group. In the present context, it is apposite to mention the link with Singerman's work on hyperbolic surface tesselations, on which some notes were circulated earlier. They are related to a class of triangle groups which we mention briefly below.

(6.5) Hecke groups and modular families. The Hecke groups G_n play a fundamental role in several distinct parts of mathematics; they are intrinsic to the construction of a class of Teichmüller disc families of Riemann surfaces which include the examples discovered by Veech.

Let X be a compact Riemann surface which admits a regular tesselation by hyperbolic n-gons; such a surface corresponds to a torsion-free subgroup Γ of finite index in a Hecke trinagle group G_n of type $\{2, n, \infty\}$,, by a fundamental (though elementary) theorem on maps due to Singernman. In particular, the case $n = 3$ relates X to the classical Fary tesselation of the hyperbolic plane associated with the modular group, i.e. $G_3 = SL_2(\mathbf{Z})$. To obtain X from such a subgroup Γ, one adds the cusps to the finite volume surface $\mathcal{U}/\Gamma$. The tesselation is determined by projecting a certain subset of the standard tesselation of $\mathcal{U}$ by $\{2, n, \infty\}$ triangles, arising as half of a fundamental domain for G_n in §1: the G_n-orbit of the hyperbolic line L determines a tiling of $\mathcal{U}$ by ideal n-gons, which projects to a finite tesselation of S whose vertices correspond to the cusps of Γ. Regularity of the tesselation means that a fixed number m of polygons meet at each vertex.

Final Exercises:

EXERCISE (6.1). Draw a pair of regular $n-$gons, adjacent along a vertial edge of each, with $n > 4$. Identify (using Euclidean translations) each pair of parallel edges. These mappings generate the holonomy group of a developing map for a geometric structure as discussed in the Interlude. Show that the resulting quotient space is a cone surface X_n with $\epsilon(n)$ vertices, where $\epsilon(n) = 1$ if n is odd, 2 if n is even. What is the genus of X_n?

EXERCISE (6.2). Draw the singular graph of the corresponding vertical trajectories of the quadratic differential given by dz^2 where z is the coordinate in the plane of the polygons. [Hint: the singular points are at the vertices of the quotient.] How many vertical cylinders does the decomposition have?

See Veech's paper in Inv. Math. for more details and a different analysis.

To be completed later. Appendix: TeichmüllerVeech curves in moduli space.

We indicate how to produce from Belyi covering data a corresponding Teichmüller curve in the moduli space. This is true for the Riemann surfaces obtained from the above tesselation patterns and holds in greater generality too. It connects directly with the above Interlude on Grothendieck's "dessins" and the theorem of Belyi, distinguishing the class of algebraic curves defined over a number field in terms of triangle groups.

Take a hyperbolic fundamental polygon $P \cup \overline{P}$ for any genus 0 finite volume Fuchsian group G having a reflection symmetry. Assume that G possesses a holomorphic character-automorphic form ω of weight 4 that is real only on part of $\partial P \cup \partial \overline{P}$, with finite order zeros at the cusps. Notice that in particular the Hecke group patterns all possess a form of this kind.

Then we consider the T-disc defined by a covering of the base surface $X(G) = \mathcal{U}/G$ using the character $\chi = \chi(\omega)$ with quadratic differential ω_* obtained from pulling back ω, as defined earlier. Let $K \le G$ be a finite index torsion-free normal subgroup contained in the kernel of the character χ_ω, representing a surface $X = \mathcal{U}/K$. The form ω being K-invariant, we obtain an induced T-disc in $T(K)$ which projects to a T-disc in T_g, the Teichmüller space of the compactified surface X_0. The centre point of this embedding is a representative marked surface $[X_0]$ with a quadratic differential ω_0 such that the T-disc defined by ω_0 on X_0 recovers the original T-disc, i.e. $\pi(e_\omega) = e_{\omega_0}$, up to composition perhaps with a disc automorphism.

We note that there is also an inclusion of co-compact Fuchsian groups $K_0 \hookrightarrow G_0$ which represents this branched covering surface.

Under suitable restrictions on the group G, it follows that this T-disc has a large modular stability group.

THEOREM 6.1 *For any genus 0 finite volume Fuchsian group G with cusps and torsion, having the shape and form ω postulated above, the corresponding Teichmüller-disc $\mathbf{D}$ in T_g has stabiliser in Γ_g isomorphic to a non-elementary subgroup of G.*

OUTLINE PROOF. We normalise G so as to locate the fixed point of an elliptic element x at the origin. In the representation T-disc given by the form ω_0, this implies that there is a finite order rotation symmetry for the corresponding vertical trajectory pattern of the form ω_0 on X_0. The critical trajectories of ω emanate from cusps of K and they are contained in the G orbit of the edge subset of $P \cup \overline{P}$. It follows that the critical graph of ω_0 is compact in X_0, which means that ω_0 is of Jenkins-Strebel type, with all vertical (non-critical) trajectories closed (for more on this type of form, see [?]).

Now consider the T-ray from the origin to $1 \in \partial\mathbf{D}$. After perhaps multiplying ω_0 by a non-zero constant, 1 is a cusp of G and this ray represents shrinking the lengths of the collection of closed vertical trajectories to zero. We want to show that the disc $\mathbf{D}$ is invariant under some composition of twists about a representative collection of core loops, one from each cylinder in the non-critical trajectory pattern of the (Jenkins-Strebel) form ω_0 on X_0. In the present situation, one mapping class of this type is induced by conjugation of K with the smallest power t^ν of the primitive parabolic element t of G fixing the cusp 1 that preserves each trajectory cylinder. Note that $1 \leq \nu \leq m$, where m is the local degree at the cusp 1 of the covering $X_0 \to \mathbf{P}_1$. It follows that the modular stabiliser of the T-disc contains at least the non-elementary subgroup $\langle x, t^\nu \rangle$ of G. $\diamond$

As background to this result, it may be helpful to consult Earle & Gardiner's paper in the Contemp. Math. volume 'Extremal Riemann surfaces' (AMS 1997) which provides a detailed analysis of Teichmüller discs stabilised by a mapping class.

If G is a Fuchsian triangle group with cusps and torsion, then the stability subgroup obtained in Theorem 6.1 has finite index in G and is therefore itself a finite volume Fuchsian group. As a result, it makes sense to ask whether the modular image of the T-disc $\mathcal{D}$ in $\mathcal{M}_g$ has a natural projective space completion, and this is in fact the case. Addition of a point at each cusp occurs naturally in the projective embedding of the moduli space $\widetilde{\mathcal{M}}_g$ of stable genus g Riemann surfaces, which is a compact complex V-manifold containing $\mathcal{M}_g$ as a Zariski open subset.

This eventually implies our final result,(not given in detail in the lectures).

THEOREM 6.2 *For each point $\{X\}$ of the moduli space $\mathcal{M}_g$ represented by a compact surface X which is a Galois covering of $\mathbf{CP}^1$ with ramification over three points, there is a complete algebraic curve C in the stable compactification $\widehat{\mathcal{M}}_g$ passing through $\{X\}$, isomorphic to some intermediate quotient surface of*

X.

The points of the curve C lying in the boundary of $\overline{\mathcal{M}}_g$ correspond to surfaces with nodes obtained by shrinking closed trajectories as one approaches the cusps of the stability subgroup. A suitable co-compact triangle group $\widehat{G}$ with signature $\{2, m, n\}$ and a corresponding finite permutation representation of it provide the underlying geometric structure on C.