

Teichmüller spaces, triangle groups and Grothendieck dessins

William J. Harvey

March 15, 2006

Abstract

This survey article considers moduli of algebraic curves using techniques from the complex analytic Teichmüller theory of deformations for the underlying Riemann surfaces and combinatorial topology of surfaces. The aim is to provide a readable narrative, suitable for people with a little background in complex analysis, hyperbolic plane geometry and discrete groups, who wish to understand the interplay of combinatorial, geometric and topological processes in this area.

We explore in some detail a natural relationship with Grothendieck dessins, which provides both an appropriate setting in which to describe Veech curves (a special type of Teichmüller disc) and also a framework for relating complex moduli to arithmetic data involving a field of definition for the associated algebraic curves.

Contents

1	Historical overview.	3
2	Grothendieck dessins & Thurston's examples.	7
2.1	Dessins.	7
2.2	Clean dessins and Galois covers of \mathbb{P}^1	8
2.3	Thurston's examples.	9
3	Teichmüller theory.	10
3.1	Deformations of complex structure on a surface.	10
3.2	Teichmüller spaces and modular groups.	12
3.3	Families of Riemann surfaces.	15
3.4	Hyperbolic surfaces inside moduli space.	17
3.5	Example: a T-disc parametrising tori.	19
3.6	Deformation T-discs embed.	20
3.7	Veech's examples.	21

3.8	Hecke triangle groups and polygonal surface tilings.	22
3.9	Fermat curves represented by T-discs.	25
3.10	Regular tessellations and smooth coverings	25
3.11	Return to Thurston's examples.	26
4	Triangle groups and modular families.	27
4.1	Tessellations and triangle groups.	27
4.2	Associated Teichmüller families.	28
4.3	Recent work on symmetric structures in \mathcal{M}_g	33
4.4	Completion of a T-disc with large stabiliser.	34
4.5	Consequences; complete curves in modular varieties.	34
5	Towards a space of all Q-curves: a universal modular family.	35
5.1	A moduli space of Fuchsian groups.	35
5.2	A representation of the absolute Galois group.	36
5.3	Modular forms with closed trajectories.	38
5.4	Arithmetically defined curves in moduli space.	39

Introduction.

The theory of moduli for algebraic curves began with Riemann's famous papers on Abelian functions, but it took many years before a further ingredient, Teichmüller's ideas on how to deform the underlying topological surface so as to effect a change in the complex analytic structure, produced a rigorous formulation and construction of the *variety of moduli* \mathcal{M}_g , whose points represent bijectively each isomorphism class of non-singular curve of the given genus.

We begin by describing in outline the theory of moduli for complex algebraic curves. The special case of genus 1 is discussed briefly, both as an avatar and in order to describe some special types of object to be used later, classical modular forms and Riemann surfaces, but we concentrate on the curves of genus at least 2, whose Riemann surface is uniformised by the hyperbolic plane, using the elaborate machinery of Teichmüller spaces introduced by Ahlfors and Bers. A method of construction is then given for explicit complex analytic models within the metric structure of these moduli spaces, which exhibits through the medium of hyperbolic plane geometry a large class of curves including those with defining equation over some algebraic number field. These *Teichmüller disc* models of surfaces have achieved much prominence recently through work of Veech and others on dynamical properties exhibited by the set of closed geodesics on the surface, relating to interval exchange transformations and billiard trajectories on Euclidean polygons. Here we focus on the symmetry properties of the complex curves so defined and the existence of Belyi structures on them; in a later section we show that there is a collection \mathcal{B} which comprises a union of complex affine algebraic curves contained in the moduli space $\mathcal{M} = \bigcup \mathcal{M}_g$ of all (conformal

classes of) compact Riemann surfaces (viewed as hyperbolic 2-dimensional orbifolds) which has an intrinsic arithmetical aspect. The space \mathcal{B} is a (reducible) analytic variety on which the absolute Galois group of the algebraic numbers $\text{Gal}(\overline{\mathbb{Q}})$ acts in a natural way.

To be more precise about the framework of the family $\mathcal{B}_g = \mathcal{B} \cap \mathcal{M}_g$, each member (irreducible component) is an immersed complex affine curve in some modular variety \mathcal{M}_g of nonsingular curves of genus $g > 1$, indexed by a central base point and unit tangent deformation vector, and the whole collection forms part of the tautological geometric structure inherited from the covering Teichmüller space and the holomorphic vector bundle $\Omega^2(\mathcal{V}_g)$ of holomorphic quadratic forms on the universal Teichmüller-family \mathcal{V}_g of Riemann surfaces. Veech refers to this structure, or its unit sphere subbundle, as the Teichmüller geodesic flow. The precise relationship between the space \mathcal{B}_g and the \mathbb{Q} -rational points of the moduli spaces $\mathcal{M}_g, \mathcal{C}_g$ is unknown, but we show in section 5 that for each curve in \mathcal{B}_g , after completion within the Deligne-Mumford compactification $\widehat{\mathcal{M}}_g$, there is a ramified covering (with degree bounded above by a linear function of g) which belongs to the conformal isomorphism class of compact Riemann surface defined by the central point.

This survey is based on a revised version of the preprint [26]. There is little that is original here, beyond the selection of material and point of view taken, which emphasizes wherever possible the hyperbolic geometry and group theory. However, later results presented without attribution are the author's responsibility.

1 Historical overview.

The study of algebraic curves and their deformations has a venerable history, beginning with specific investigations of simple types of curve, and progressing to very sophisticated methods of projective geometry. With Riemann's introduction of the underlying topological surface and his spectacular results on theta functions and periods of abelian integrals (Abel-Jacobi theory) ([54]), it became clear that to answer the most fundamental questions about a given projective curve, it is necessary to consider variations of the curve, and in order to maintain the link with abelian integrals one must address the problem of how to vary the curve in all possible ways which preserve the particular topological structure. Riemann showed that the topological genus, the number of handles of the Riemann surface of the curve, dictates the dimension (1 if $g = 1$, $3g - 3$ when $g \geq 2$) of the *moduli space* of local deformations of the curve within the (complex) projective space in which it lies. However, the precise structure of the space of moduli remained an enigma until a more comprehensive account of the topological nature of surfaces became available, including the introduction of homotopy as a refinement of the relation of homology for integration theory along paths. Despite much progress on the analytic side with the proof of uniformisation and the classification of Fuchsian groups, the efforts of Klein, Fricke and (independently) Poincaré left the global moduli problem unresolved, and

it was not until the introduction of quasiconformal mappings and their relation to quadratic differential forms by Teichmüller during the 1930's that a genuine theory of deformations of Riemann surfaces became possible. This was developed during the period after 1950 by Ahlfors and by Gerstenhaber and Rauch, but it was only after the landmark Princeton Conference on Analytic Functions of 1957 [2] that the complex deformation theory of Riemann surfaces took flight, together almost simultaneously with the Kodaira-Spencer theory for compact complex manifolds in general.

Grothendieck's introduction [22] of the notion of dessin d'enfant (defined below) brought into sharper focus the interaction between combinatorial surface theory, the complex geometry of algebraic curves and their algebraic number coefficient fields: it implies that the arithmetic field of definition for a curve is in some manner an intrinsic feature of the hyperbolic geometry underlying the transcendental uniformisation of the curve. A primary goal in this article is to illuminate some aspects of that perception by systematic use of the theory of Teichmüller spaces, as developed by Ahlfors and Bers; we shall highlight the relationship between a certain class of holomorphic quadratic differential forms on the Riemann surface of an algebraic curve (the Jenkins-Strebel forms) and the existence of a model of the surface (as a quotient of the hyperbolic plane) within the corresponding modular variety. The totality of these (immersed) model surfaces is a set on which the absolute Galois group of the algebraic numbers acts, a pattern which is in accord with the conjectural picture of [22]. It may equally be viewed as an organised combinatorial framework for the natural actions of the various mapping class groups and subgroups which permute markings on the various levels of ramified surface coverings that constitute the branching locus, or the orbifold singularities, in each \mathcal{M}_g .

A *Grothendieck dessin* or *line drawing in a surface* is a connected graph drawn in a compact surface with the property that each complementary piece of surface is a polygonal disc. During the late 1970s, Grothendieck proposed that there was an intimate link between this purely combinatorial object and arithmetic geometry of curves, and pointed out the relationship between dessins and two distinct, previously unrelated notions: representations of a certain discrete (extended triangle) group on the one hand and the absolute Galois group \mathcal{G} of the field of algebraic numbers on the other.

In 1979, this insight received clear confirmation when G.V. Belyi proved a fascinating theorem [6] bringing out the full significance of this idea: *If a projective algebraic curve X is defined by a set of polynomial equations with coefficients in a number field, an algebraic extension of the rational field \mathbb{Q} , then there is a meromorphic function on X whose only singular values are the three points $0, 1$ and ∞ .* The converse is also true, following indirectly from older work of A. Weil on the field of definition for an algebraic variety; for this aspect of the Belyi theorem, the reader may consult recent articles by J. Wolfart [71] and G. Gonzalez-Díez [18].

Grothendieck later wrote [22] an extended account of his ideas as part of a CNRS research proposal, placing it in an elaborate conjectural theory of 'anabelian' fundamental groups in geometry. This celebrated manuscript has

received widespread unofficial distribution; subsequently, thanks partly to an influential article by G. Shabat and V. A. Voevodsky [61] in the Grothendieck Festschrift, the notion of dessin gained a wider audience and established itself as an independent entity on the fringe of three broad estates – algebraic geometry, number theory and complex analytic geometry. Each of these areas demands heavy investment in basic language and background theory as prerequisites for understanding and progress, and so this insertion of a new elementary structural connection between them has been widely welcomed.

To display the link, we focus on the relationship between a subclass of dessins and a special kind of deformation of the Riemann surface structure. The latter arises from the initial data of a complete complex algebraic curve, carrying the dessin as part of its uniformisation in the hyperbolic plane, and an additional choice of prescribed flat geometric pattern coming via the Belyi ramified covering from the Euclidean plane. These two pieces of data determine a complex affine deformation curve (Riemann surface with punctures), immersed in the corresponding modular variety and passing through a central point which represents the base point of the deformation, the compact Riemann surface associated with the complete curve. It will follow from the basic link between automorphic forms and holomorphic deformations that any point of a modular variety which represents a curve definable over $\overline{\mathbb{Q}}$ belongs in an appropriate sense to at least one such family. As seen from the perspective of Belyi's theorem, a primary source of this type of structure comes from curves with sufficiently large automorphism group. Simple examples of familiar curves falling into this category include Wiman's class of hyperelliptic curves with affine equation $y^2 = 1 - x^n$, for $n \geq 5$, the famous Klein quartic given (in projective form) by $x^3y + y^3z + z^3x = 0$ and the Fermat curves $x^n + y^n = z^n$, for $n \geq 4$.

The theory of holomorphic families of compact Riemann surfaces is founded on the analytic properties of the Teichmüller spaces $T_{g,n}$ established by Lars Ahlfors and Lipman Bers during the decade following 1960; these spaces are a countable set of connected complex manifolds, one for each pair of non-negative integers g, n with $2g - 2 + n > 0$, whose points represent marked complex structures on an n -pointed genus g surface up to holomorphic equivalence. There is a small list of special low values of g, n which represent surfaces and deformation spaces familiar from classical work:- $T_{0,n}$ is a single point for $n = 1, 2, 3$ and $T_{0,4} = \mathcal{U} = T_{1,0}$ is the upper half plane. In this way, the totality of compact Riemann surfaces (non-singular curves) with n -point subsets is organised into a union of disjoint spaces representing distinct topological types. Each Teichmüller space carries intrinsically a structure of complex manifold of dimension $3g - 3 + n$ and a complete global metric, defined by Teichmüller as the logarithm of the least overall conformal distortion (measured in the sup norm) involved in deforming one complex structure on a surface to another. If the surface under consideration has genus 1, the deformation space is equivalent to the classical identification of (conformal classes of) marked tori, $X = \mathbb{C}/L$, with points of the upper half plane, $\mathcal{U} = T_1$, on which the modular group $\Gamma(1) = SL(2, \mathbb{Z})$ acts by fractional-linear transformations, in effect changing the basis elements of the

lattice $L \cong \pi_1(X)$, and preserving the Poincaré metric. Thus, in this special case, the deformation space carries an intrinsically richer geometric structure of hyperbolic plane geometry and we sometimes denote it by \mathcal{H}^2 when focussing on this viewpoint. In general, the *mapping class group* $\text{Mod}_{g,n} = \pi_0 \text{Diff}(X)$ of the (n -punctured) surface X acts as the *Teichmüller modular group* by changing the marking on a reference surface. When $n = 0$, the quotient of Teichmüller space T_g by this action of Mod_g is *Riemann's moduli space* \mathcal{M}_g , which parametrises biholomorphic equivalence classes of (closed, genus g) Riemann surfaces, while if $n = 1$ the resulting quotient is the *modular curve* \mathcal{C}_g , the total space of the fibred *modular family* of genus g surfaces.

Teichmüller also defined a special type of deformation within the spaces T_g , which is now called a *Teichmüller (geodesic) disc*; we sometimes refer to it as a *T-disc*. They are described in more detail in section 3: for the purposes of this outline, we mention only that inside any $T_{g,n}$ they form a natural class of complete complex submanifold of dimension 1, isometric to the Poincaré metric model of the hyperbolic plane \mathcal{H}^2 , and that they are in plentiful supply, through each point and in any given direction, forming an integrated part of the Teichmüller geodesic flow.

It is natural to consider the action of the mapping class group $\Gamma_{g,n}$ on the set of all T-discs in $T_{g,n}$. Because the modular action is biholomorphic and discontinuous, it preserves the set of T-discs and the stability subgroup preserving a given disc is then isomorphic to a Fuchsian discrete group of hyperbolic isometries. It is not hard to show (see [14] for instance) that a sufficiently general T-disc will have trivial stabiliser; but how large can the stability group of a T-disc be? One might naturally look first for examples with a cocompact Fuchsian group as stabiliser, since that type of family would be of interest for both the topology of the modular variety and the algebra of the mapping class group, and it would provide a valuable tool for constructing complete families of Riemann surfaces. The search is fruitless, however, since any T-disc contains geodesic rays whose projection is divergent in the moduli space [45], which implies at once that the quotient of a T-disc by its stabiliser cannot be compact. Thus, *there is no totally geodesic, complete, complex suborbifold in any of the higher dimensional moduli spaces $\mathcal{M}_{g,n}$, and one must use quite different methods to produce complete curves in the modular varieties of closed surfaces \mathcal{M}_g (c.f.[36], [19]).*

The situation with stabilisers of T-discs becomes more interesting if the compact quotient condition is relaxed. In fact, as we explain in section 4, in any given T_g there is a large set of marked surfaces $[S_n]$ which define centre points of T-discs stabilised by (noncompact) *finite volume* Fuchsian groups, so that in this case the quotient in moduli space \mathcal{M}_g is a finite-area immersed Riemann surface, isomorphic to a quotient of the base surface S_n by some subgroup of its automorphism group. They are examples of a more general class of affine algebraic curve in moduli space known as a *Veech curve* to be discussed briefly later; all of the examples we describe are given by algebraic equations defined over some number field, which suggests that the geometric action of the Teichmüller modular group on the unit tangent bundle over the Teichmüller space

will form a distinctive part of the relationship conjectured in [22] between the absolute Galois group $\mathcal{G}(\overline{\mathbb{Q}})$ and the Grothendieck-Teichmüller tower, once the meaning of that object has been clarified.

For higher dimensional families of algebraic curves, there are interesting but sporadic results deriving from work of many authors, among them E. Picard (and his students), G. Shimura (and his students), R.Holzapfel, P.Deligne and G.D. Mostow, Paula B. Cohen, J. Wolfart and H.Shiga; often these employ some version of the period mapping for holomorphic 1-forms and its monodromy and have some relation to the present work, which we shall not be able to explore here.

In the next two sections, we shall describe essential background and the examples which motivate our viewpoint. The fourth describes a procedure for constructing one type of Veech-Teichmüller curve in moduli space. In the last chapter, we consider the arithmetic problem of characterising the curves of given genus g that are definable over $\overline{\mathbb{Q}}$. We work within the convenient topological framework of a single space of moduli \mathcal{D}_0 , the space of all cocompact Fuchsian groups, viewed modulo conjugacy within the Lie group $G = PSL(2, \mathbb{R})$. There is a natural structure of metric space on \mathcal{D}_0 , using the Hausdorff metric on the space of closed subgroups of the Lie group G , and by the uniformisation theorem it contains copies of every modular variety $\mathcal{M}_{g,n}$ with $3g - 3 + n > 0$. A related question asks for a precise description of all $\overline{\mathbb{Q}}$ -curves in a suitable compactification of the space of stable genus g curves $\widehat{\mathcal{M}}_g$. In emphasising the totality of hyperbolic discs with crystallographic (i.e. discrete co-finite) Fuchsian stabilisers, we hope to bring out a general perspective which places dessins within the tautological geometric framework underpinning the patchwork collection of Teichmüller spaces which cover the various modular families and which in the process generate the space \mathcal{D}_0 : at the same time one needs the entire collection of lattices (finite co-volume Fuchsian groups) and Teichmüller space inclusions to reflect the wide range of possible Belyi representations of complex curves defined over $\overline{\mathbb{Q}}$: every meromorphic function which has 3 ramification values determines a covering of the complex projective line with the requisite properties and each *admissible cusp form* of weight 4 produces a Teichmüller disc and, thereby, candidate members of the set of $\overline{\mathbb{Q}}$ -points of the space of moduli. This space may provide an appropriate starting point for the (still incomplete) process of constructing the ultimate arithmetic modular family described in the Grothendieck Esquisse [22].

2 Grothendieck dessins & Thurston's examples.

2.1 Dessins.

A *dessin d'enfant* or simply *dessin* is defined to be a connected (finite) graph in a compact surface, whose complement is a union of cells and which has a bipartite structure on the vertices. It is convenient to denote the labels for vertices of the graph with \bullet and \star .

To keep the exposition simple, we restrict attention to orientable surfaces and to a special type of dessin, concentrating on the subclass of dessins which arise from pulling back the standard triangulation of $\mathbb{C}P^1$ through a *Belyi function*, a holomorphic branched covering mapping $\beta : X \rightarrow \mathbb{C}P^1$ with three critical values $0 (= \bullet)$, $1 (= \star)$ and $\infty (= \circ)$. This consists of a topological decomposition of the Riemann sphere into two triangles U and L , with interiors the upper and lower halves of the complex plane respectively, with vertices given by the above three symbols and disjoint edges joining them pairwise along the real axis. In addition, we shall often assume some regularity property for the function β , to be spelled out in the next subsection; in an appropriate sense, the restricted class of dessins to be used forms a natural cofinal subclass of the general pattern. For a broader study, the reader may consult a range of recent survey articles ([31], [56]).

2.2 Clean dessins and Galois covers of \mathbb{P}^1

A *Belyi function* $f : X \rightarrow \mathbb{C}P^1$ is called *clean* if every point in the inverse image of \star has ramification of order 2. The corresponding dessin in X , defined as

$$K_f = f^{-1}\{\bullet \text{---} \star\},$$

is then called *clean* if each vertex mapping to (and labelled) \star has valency 4 in the graph K_f .

It is not difficult to see that every dessin has a standard subdivision for which this property holds: for instance, one can take the triangular subdivision (see [61]) which arises by replacing the function f by $\tilde{f} = 4(f - f^2)$.

The transition from combinatorial surface topology to complex analysis and the modular group is made by the classical representation of the punctured plane $\mathbb{C} - \{0, 1\}$ as the congruence modular surface of level 2, $X(2) = \mathcal{U}/\Gamma(2)$. One finds with the help of the monodromy theorem that the unbranched covering mapping f^* , obtained by puncturing X at all labelled vertices, is determined by a finite index subgroup Γ of $\Gamma(2)$, for which f is just the projection of each Γ -orbit to the corresponding $\Gamma(2)$ -orbit. Consequently there is a uniformised picture of the dessin in the upper half plane which arises as a quotient under the group Γ of the cell subdivision of \mathcal{U} into ideal fundamental triangles for $\Gamma(2)$ - for more details here, consult [31] for instance.

To gain insight into the picture in the hyperbolic plane, it is helpful to begin from the standard triangular fundamental domain for the full modular group, which determines the barycentric subdivision of the $\Gamma(2)$ -invariant triangulation.

In the context of the analysis of general (orientable) dessins in [22], one considers instead permutation representations of the *cartographic group* \mathcal{C}_2^+ , which is isomorphic to the congruence subgroup $\Gamma_0(2)$ containing $\Gamma(2)$ with index 2. This group consists of all projective transformations

$$\gamma : z \mapsto \frac{az + b}{cz + d} \in \Gamma(1) = PSL(2, \mathbb{Z}), \quad \text{with } c \equiv 0(2).$$

It is not hard to see, using a standard choice of fundamental domains for the groups concerned, that passage from f to \tilde{f} is defined by extending the classifying inclusion $\Gamma < \Gamma(2)$ to $\Gamma < \Gamma_0(2)$, so that there is again a tessellation of \mathcal{U} by hyperbolic triangles which determines a (clean) dessin on the surface X .

A dessin is called *balanced* when both (a) every vertex has the same valency and (b) each cell has the same number of edges. A *Galois dessin* is a graph-and-cell decomposition obtained by lifting the standard division of $\mathbb{C}P^1$ through a Belyi mapping which is also a Galois covering.

It is a simple exercise in finite group actions to see that Galois dessins are balanced, but the converse is not necessarily true. These matters are discussed at greater length in the collection of articles [56].

2.3 Thurston's examples.

We use the term *Thurston decomposition* of a compact surface S to refer to a cell decomposition of S determined by two loops which *fill it up*, partitioning the surface into polygonal cells with an even number of edges, which are segments coming from each loop alternately.

Thurston's original 1975 construction [66] was motivated by his work on the classification of surface mappings up to free homotopy, and in particular by the search for a counterexample to an old conjecture of Jakob Nielsen that an algebraically finite mapping class, i.e. one induced by a surface mapping for which all homological eigenvalues are roots of unity, must be periodic. To construct examples of the kind of map which contradicts this, he took a certain product of twists along a pair of loops α, β that *fill up* the surface: this means that the loops, taken to be in general position with no triple intersection points, are such that the complementary regions of S are all simply connected cells with at least four edges. This type of decomposition by two loops determines a dessin on S with the property that all vertices of type \star have valency 4: this is achieved by placing a vertex \circ in each cell and an edge to it from each vertex of the graph. In fact this procedure also yields a dual partition of the surface into quadrangles, each with two vertices labelled \star , which is a subdivision of the cell partition of S dual to the original one; omitting the \bullet vertices gives the dual quadrilateral structure.

The method used by Thurston to produce pseudo-Anosov homeomorphisms from this combinatorial structure is simple but very ingenious: because the surface S has been decomposed into quadrilaterals, it has a branched piecewise Euclidean structure given by the requirement that each cell is to be a unit square. This determines a structure of compact Riemann surface too, by the standard removal of singularities at corners, but the important point is that in the Euclidean structure the Dehn twist mappings along each loop act as *translation by an integer*. Indeed, one can approach the construction of the surface in reverse from a set of square tile pieces by first joining them in line following the path α and then making the necessary identifications of edge segments on the resulting cylinder; in the local coordinate of the resulting geometric structure, a twist along the core of the cylinder is clearly an integral translation. For the β

loop, one follows a similar path of identifications and again we obtain as monodromy of the geometric structure around the loop an integer translation in the orthogonal direction. Now the group generated by the two twists is a subgroup, often of finite index, in the modular group $SL(2, \mathbb{Z})$. Furthermore, it is clear that any word in the two twist mappings that gives a modular group element with two real eigenvalues (a hyperbolic element) will define a pseudo-Anosov mapping class for the surface; with a more careful choice of word, the mapping can be arranged to act trivially on homology (see [66] for more details).

In this construction we see a first glimpse of how dessins and the mapping class group interact, although the complex geometry of moduli space is not yet in evidence.

3 Teichmüller theory.

This section presents the background theory of complex deformations of curves, mostly without proofs but with some effort to give at least an intuitive grasp of the results which are important. To this end, we also include some specific examples of the kind of extremal deformation which is to be highlighted.

3.1 Deformations of complex structure on a surface.

A convenient framework for studying analytic moduli of surfaces with all shapes and sizes in an integrated manner is provided by the universal Teichmüller space developed by Ahlfors and Bers; we provide a brief sketch only, sufficient to make it possible to give a unified approach to the kind of deformation which we need in the last section. Detailed accounts may be found in several textbooks [16], [52], [23].

The first fundamental problem encountered in deformation theory is the question of specifying precisely what kind of structure can reasonably be used. We require our object (which is to be deformed) to have no local symmetry, in the sense that there is no object nearby that is isomorphic to it: this is *local rigidity*. In the case of Riemann surfaces, it turns out that one needs, in addition to the categorical object concerned, the additional data of a topological *marking*, which is defined to be a homeomorphism $f : X_0 \rightarrow X$, from a fixed reference surface X_0 , viewed up to free homotopy. This is the appropriate condition to deploy, because *rigidity* of a Riemann surface X marked in this sense follows from the fact that holomorphic self-mappings are represented faithfully in the action induced on the homotopy (or homology) of the surface: a conformal self-mapping (distinct from the identity) cannot be trivial on homotopy. One prefers to employ a homotopy marking rather than a homologically defined one because of the more direct link with covering spaces: also, with hindsight, it turns out that this provides a clearer view of the often intricate interactions of the groups involved.

It is a consequence of the theory of Beltrami equations, to be outlined next, coupled to the uniformisation theorem of Riemann and Koebe, that for every

(finite volume) Riemann surface X with universal covering the upper half plane \mathcal{U} , provided with a *marking* $f : X_0 \rightarrow X$, there is a representation of (X, f) as a normalised homeomorphism $h : \mathbb{R} \rightarrow \mathbb{R}$ with $h(0) = 0$, $h(1) = 1$. Here $\mathbb{R} \cup \{\infty\} \sim S^1$ is the boundary of the upper half plane \mathcal{U} and these maps h , normalised to fix ∞ , are in fact *quasisymmetric* homeomorphisms of the boundary real line, arising as boundary values of quasiconformal self mappings $f : \mathcal{U} \rightarrow \mathcal{U}$ which cover the marking of X . All the essential ingredients of real and complex analysis on the Riemann surface are transferred to analogues defined on the universal cover $p : \mathcal{U} \rightarrow X$ which transform suitably under the discrete (Fuchsian) group of deck transformations of the covering, which is a subgroup of the direct isometry group of the hyperbolic plane. This process delivers within a single universal space all deformations of all Riemann surfaces.

To explain this in more detail, we make the following definitions.

DEFINITION. A *Beltrami coefficient* μ on a hyperbolic Riemann surface X , is a measurable complex function on the universal covering \mathcal{U} transforming under the fundamental (covering) group Γ as a tensor of type $(-1, 1)$, with sup norm less than 1.

Thus μ is an element of $B_1(X)$, the open unit ball in

$$L_{-1,1}^\infty(X) \cong \{\mu \in L^\infty(\mathcal{U}, \mathbb{C}) \mid \mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z) \text{ for all } \gamma \in \Gamma\}.$$

The corresponding *Beltrami equation* is a partial differential equation for a locally homeomorphic complex-valued coordinate function $w : X \rightarrow \mathbb{C}$:

$$w_{\bar{z}}(z) = \mu(z)w_z. \tag{1}$$

The Ahlfors-Bers theory (see [3]) provides a unique normalised solution homeomorphism w_μ to this equation in \mathcal{U} , called *quasiconformal* or simply ‘qc’, which extends to a homeomorphism h of the boundary $\partial\mathcal{U}$, compatible with the given Fuchsian group Γ . Here, the process of normalisation amounts to factoring out the natural action on the set of solutions by post-composition with the group $G = PSL_2(\mathbb{R})$ of all real Möbius transformations, the direct hyperbolic isometries of \mathcal{U} : if w is a solution to the equation, then so is the composition $\gamma \circ w$ with $\gamma \in G$. We shall usually employ the standard normalisation, which requires w to fix 0, 1 and ∞ . Using the above invariance property of the Beltrami form, a computation with partial derivatives shows that each composition of the form

$$\gamma_\mu = w_\mu \circ \gamma \circ w_\mu^{-1}, \quad \text{with } \gamma \in \Gamma,$$

has zero conformal distortion, and it follows that there is a deformed covering group $\Gamma_\mu = w_\mu \Gamma w_\mu^{-1}$ of real Möbius transformations, quasiconformally conjugate to Γ . This determines a geometric isomorphism between the two Fuchsian groups, induced by this qc-homeomorphism between the surfaces, which gives a genuine distortion of the original complex structure if the map is not conformal,

The crucial notion of Teichmüller equivalence for two Beltrami coefficients is introduced next; this reduces the infinite dimensional spaces of Beltrami coefficients to the (finite dimensional) deformation spaces appropriate for these marked structures.

DEFINITION. Two Beltrami coefficients μ and μ' are called *Teichmüller equivalent* if the corresponding marked groups Γ_μ and $\Gamma_{\mu'}$ are conformally equivalent, that is, conjugate by some real Möbius transformation in G .

This condition, sometimes abbreviated to T-equivalence, implies that the corresponding *normalised homeomorphisms* $h = h_\mu$ of \mathbb{R} , with $h(0) = 0$ and $h(1) = 1$, depend only on the Teichmüller class of μ . The set of all the normalised maps h_μ for arbitrary measurable $\mu \in B_1(L^\infty(\mathcal{U}))$, without any group compatibility condition, is the *universal Teichmüller space*, denoted $T(1)$. It transpires that there is a purely intrinsic analytic criterion which characterises the homeomorphisms of the boundary circle $\mathbb{R} \cong S^1$ that arise in this way: they are called *quasisymmetric*. See for instance [16]. The detailed analysis of boundary behaviour need not concern us here, but becomes highly important for infinite dimensional Teichmüller spaces, arising from surfaces with more complicated boundary, including curves or infinitely many punctures, where attention switches to asymptotic behaviour. See [17] for details of this fresh aspect of the theory.

DEFINITION. The *Teichmüller space* of the Fuchsian group Γ (and by abuse of notation, of the Riemann surface $X = \mathcal{U}/\Gamma$) is the set of T-equivalence classes of Beltrami coefficients for the group Γ .

An inclusion of Fuchsian groups $\Gamma_1 \leq \Gamma_2$ induces a natural relationship between spaces of Beltrami forms $B_1(X_1) \supseteq B_1(X_2)$, for the surfaces $X_1 = \mathcal{U}/\Gamma_1$ and $X_2 = \mathcal{U}/\Gamma_2$, where $B_1(X_1) \cong B_1(\mathcal{U}, \Gamma_1) \supseteq B_1(\mathcal{U}, \Gamma_2) \cong B_1(X_2)$. This projects to an inclusion $T(X_1) \supseteq T(X_2)$. Thus, in particular, $T(1)$ has the significant *universal property* that it contains copies of every Teichmüller space $T(\Gamma)$. For many purposes, however, this space is too vast and has too great a diversity of subspaces, and one has to focus on the individual spaces to gain real insight into the geometry of the modular families.

3.2 Teichmüller spaces and modular groups.

One strongpoint of the Ahlfors-Bers formalism lies in the coordinated description it gives for the action of the *modular groups* of the various Fuchsian groups and surfaces.

Recall first the standard (topological) definition of the *mapping class group* of a surface (or possibly orbifold) $X = \mathcal{U}/\Gamma$ as the group of isotopy classes of homeomorphisms of X to itself preserving any punctures or cone points; equivalently one may use homotopy classes instead, since these notions coincide in two dimensions. For compact surfaces, it is easily shown that each homotopy

class of homeomorphism contains quasiconformal representatives and this does extend to orbifolds and surfaces with finitely many punctures. We may apply the method of lifting in the theory of branched coverings to represent each homeomorphism of X as the projection from a Γ -compatible homeomorphism f of \mathcal{U} which induces a *geometric automorphism* α of the covering group Γ by conjugation:

$$\gamma \mapsto f \circ \gamma \circ f^{-1} = \alpha(\gamma).$$

Of course, one must verify that distinct choices in lifting, or a homotopic variation, lead to the replacement of f by $\gamma_0 \circ f$ where $\gamma_0 \in \Gamma$, and that the effect on the automorphism α is to compose it with an inner automorphism of Γ .

If $\text{mod}(\Gamma)$ denotes the group of (geometric) automorphisms of Γ and $QC(X)$ denotes the group of quasiconformal (qc) self-homeomorphisms of the compact (possibly orbifold) Riemann surface X , with $QC_0(X)$ the normal subgroup of null-homotopic ones, then from the above procedure we obtain a homomorphism from $QC(X)$ to $\text{Aut}(\pi_1(X))$, which projects to a homomorphism Φ from the *geometric mapping class group* to the algebraic one: the subgroup Inn of inner automorphisms of $\Gamma = \pi_1(X)$ arise from distinct liftings of a homeomorphism homotopic to the identity and writing $\text{Mod}(X)$ for the quotient $\text{Aut}(\pi_1(X))/\text{Inn}$, we have

$$\Phi : QC(X)/QC_0(X) \cong \pi_0 \text{Diff}^+(X) \longrightarrow \text{Mod}(X) \leq \text{Out}(\pi_1(X)).$$

The presence of torsion or punctures in the Fuchsian group Γ renders the proofs more difficult but does not change their validity. Furthermore, *the morphism Φ is surjective: that is, we can realise each individual automorphism $\alpha : \Gamma \rightarrow \Gamma$ by a quasiconformal homeomorphism $f = f_\alpha$ of \mathcal{U}/Γ* – this is a classical theorem due to Nielsen and Fenchel. For more details of that result, see for instance [23], chapters 1 & 9. A more detailed discussion of the algebraic and geometric versions of modular group for orbifold surfaces which includes the general isomorphism result is given in [27].

Employing the geometric definition of mapping class group, we write the fundamental *action on Teichmüller space* as

$$\rho_\alpha([f]) = [f \circ f_\alpha^{-1}],$$

with $[f]$ denoting Teichmüller-class, which defines a group action on $T(\Gamma)$ with kernel the subgroup $\text{Inn}(\Gamma)$ of inner automorphisms. In this way, we obtain the important representation of the quotient $\text{Mod}(X) = \text{mod}(\Gamma)/\text{Inn}(\Gamma)$, as the *Teichmüller modular group* $\text{Mod}(\Gamma)$ of (biholomorphic) automorphisms of $T(\Gamma)$; if Γ represents a compact surface with genus g and n punctures, this group is often denoted $\text{Mod}_{g,n}$ or Mod_g when $n = 0$.

A fundamental paper of Royden [55] analyses the metric structure on $T(\Gamma)$ which is induced by using the supremum of the distortion function $\mu \in B_1$ to measure the distance between marked compact surfaces, $[X_j, f_j]$ with $f_1 = w_\mu \circ f_0$: write $k = \text{ess.sup}\{|\mu(z)| \mid z \in X_0\}$ and set

$$d([X_0, f_0], [X_1, f_1]) = \frac{1}{2} \ln \left(\frac{1+k}{1-k} \right). \quad (2)$$

This is *Teichmüller's metric* on $T(\Gamma)$ – see for instance [16] for a detailed account – and it is not hard to see that it is invariant under the action of the modular group. Indeed, one needs only to redraft the definition of the metric in terms of the qc mappings which interconnect the two marked surfaces:

$$d([X_0, f_0], [X_1, f_1]) = \frac{1}{2} \ln K((f_1 \circ f_0^{-1})). \quad (3)$$

Putting in the action of $\alpha \in \text{Mod}(X)$ leaves this expression unchanged.

Royden proved the following important result.

Theorem 1 *The Teichmüller modular group Mod_g is the full group of Teichmüller-metric isometries of T_g .*

The result was extended to all finite dimensional Teichmüller spaces by Earle and Kra: the notation g, n refers to the topological surface data of genus g with $n \geq 0$ marked special points and is explained later on.

$$\text{Isom}(T_{g,n}) \cong \mathcal{M}_{g,n}. \quad (4)$$

This group action is of pivotal importance in the Teichmüller theory, because the action is properly discontinuous on $T(\Gamma) = T_{g,n}$ with discrete orbits (see for instance [16], [52]), producing as quotient the desired *moduli space* $\mathcal{M}_{g,n}$ of conformal classes of marked genus g surfaces.

It can be proved (with some work) that the natural complex structures on the Banach spaces of Beltrami forms project to make the Teichmüller spaces into complex manifolds; one shows that the tangent space at a marked point $[X_0, f] \in T(X)$ is dual to the complex vector space of holomorphic quadratic differential forms on X_0 in the case of a closed surface, with extra vanishing conditions at any cone points or punctures. From the Riemann-Roch theorem, it is known that this space has finite dimension $3g - 3 + n$, which tallies with the dimensions of the spaces $T(\Gamma)$. One then shows that this almost complex tangential structure is integrable. Furthermore, there is a Hermitian metric on the tangent bundle which derives from the Petersson inner product on the quadratic differential forms and it was shown by Weil, and independently by Ahlfors, that this metric is Kähler. In fact, Wolpert was able to prove that every moduli space has a projective embedding, using Kodaira's work on this type of structure.

NOTE. A full account of the complex analytic theory of these spaces is best given by considering the Bers embedding, which requires more time and space than we can afford: the reader should consult the standard texts cited earlier for this important work.

A compact genus g Riemann surface with a non-trivial group G of biholomorphic automorphisms will give rise to a marked point in $T(X) = T_g$ which

has G as the subgroup of $\text{Mod}(X)$ stabilising the marked point; similarly, a finite group of homeomorphisms of a compact surface always produces a marked Riemannian metric on the surface, hence a Teichmüller point with this as its automorphism group. The famous *Realisation Problem of Hurwitz-Nielsen* asked whether, given a finite subgroup H of $\text{Mod}(X)$, there is a choice of marked point $[\mu] \in T(X)$ for which the stability group is H ; this may be rephrased as follows.

Is there a finite group of homeomorphisms H_1 which realises a given finite group of mapping classes H , in the sense that $H_1 \cong H$ with $H_1 \leq QC(X)$ and $\Phi(H_1) = H$?

This problem was studied, and partially solved, by Nielsen and Fenchel and various special cases were proved by later authors over more than 30 years until it was finally resolved in general by Kerckhoff [33] using Thurston's powerful theory of earthquakes. Different proofs have been given by Wolpert (using convexity properties of the hyperbolic length functions on Teichmüller space) and more recently in an interesting paper by Gabai [15], which uses methods reminiscent of Nielsen's and also explains how to relate it to fundamental problems in the geometry and topology of 3-manifolds.

Notice, however that there is no natural overall choice of a realisation, in the following sense.

Theorem 2 *No homomorphism exists from $\text{mod}(\Gamma)$ into the group $QC(\mathcal{U})$ which would split the epimorphism Φ .*

For inverse homomorphisms into the subset of C^∞ diffeomorphisms of X , the impossibility of such a splitting was proved by Morita [51], using his theory of characteristic classes for surface bundles and moduli spaces. This restriction has been removed recently by work of Epstein and Markovic.

3.3 Families of Riemann surfaces.

At this point, it is appropriate to indicate how to construct the corresponding *universal holomorphic family of Riemann surfaces* modelled topologically on $X = \mathcal{U}/\Gamma$, with the space $T(\Gamma)$ as base. This is a tautological *holomorphic fibre space* $V(\Gamma)$ over the Teichmüller space, obtained as follows.

1. Place over each point $[\mu]$ of $T(\Gamma)$ the corresponding h_μ -deformed hyperbolic disc \mathcal{U}^μ , on which the deformed group Γ_μ operates.
2. Take the quotient of this space $F(\Gamma) \cong T(\Gamma) \times \mathcal{U}$ by the discrete action of Γ on the fibres as Γ_μ on \mathcal{U}^μ .

Note that although we use the notation h_μ , both this normalised map and the deformed disc depend only on the T-class $[\mu]$. The resulting space $V(\Gamma)$ is a complex analytic manifold which is locally a topological product of the fibre

orbifold and an open disc in the base $T(\Gamma)$. However the product structure is holomorphically nontrivial because of the nature of these families of quasi-conformal mappings h_{mu} inter-relating the nearby fibre surfaces. This type of deformation is trivial only when there is a conformal map of \mathcal{U} inducing it.

The same fibre space structure is also realisable real-analytically by a standard bundle-theoretic procedure known as homogeneous reduction, from the *representation space* $R_0(\Gamma, G)$ of all discrete injective homomorphisms ρ of the Fuchsian group Γ into the direct-isometry group $G \cong PSL_2(\mathbb{R})$ of the hyperbolic plane \mathcal{U} . We assume for simplicity that the quotient surface $X = \mathcal{U}/\Gamma$ is compact or possibly finite area, with finitely many cone points or punctures; the *topological type* (g, n) records the genus g and the total of $n \geq 0$ special points. Since the spaces $T(\Gamma)$ depend as complex manifolds only on the topological data and not on the cone/cusp nature of the marked points, we usually refer to the moduli spaces as $T_{g,n}$ and $\mathcal{M}_{g,n}$; modular groups are a little less flexible, and $\text{Mod}_{g,n}$ will refer to the case of n punctures.

The target Lie group G operates on the space R_0 by conjugation: every element $\rho(\gamma)$ in the image of the given representation is sent by $t \in G$ to $t\rho(\gamma)t^{-1}$, conjugation by the fixed element t in G . In this way, each representation $\rho \in R_0(\Gamma, G)$ determines a conjugacy class of G -similar ones.

Now this action on R_0 is *proper*, and each G -orbit, a collection of marked groups conformally equivalent to the group $\rho(\Gamma)$, may be viewed as the fibre of a map to the quotient space of conjugacy classes, which is none other than our Teichmüller space $T(\Gamma)$ of normalised marked Fuchsian groups – a comparison of these two perspectives may be found in papers of Kerckhoff [32],[33]; see also recent work of many authors (Fock, Goncharov, Kashaev, Penner and others) on a quantisation of the theory of moduli which exploits these two approaches to geometric structure both on surfaces and on the Teichmüller spaces.

To convert the fibration $R_0(\Gamma, G) \rightarrow T(\Gamma)$ into a family of Riemann surfaces, we note that the group $G = PSL_2(\mathbb{R})$ may be viewed as the unit tangent bundle of the hyperbolic plane, a principal PSO_2 -bundle; this may be converted into a holomorphic family of hyperbolic planes by taking the quotient modulo the rotation subgroup $PSO(2)$ of G . This space is denoted $F(\Gamma)$. Finally one obtains a fibration with Riemann surfaces as fibres by taking the quotient by the action of Γ via the representations $\rho \in R_0(\Gamma)$ on each fibre disc, as a subgroup of G : to see this in more detail, the reader may consult [43] or [52].

There is another description of the holomorphic fibre space $F(\Gamma)$ as the space of all pairs

$$(t, z), \quad \text{with } t = [w_\mu] \in T(\Gamma) \text{ and } z \in w_\mu(\mathcal{U});$$

for details of this aspect of the theory, which involves the analytic properties of the Schwarzian derivative and the Bers embedding of each Teichmüller space in the Banach space of holomorphic functions on a half plane (with the hyperbolic L^∞ norm), the reader may consult [8] or [16].

Following on from the fibre space property, the action of the Teichmüller-modular group on the base space – as the group $\text{Mod}(\Gamma)$ (of mapping classes for the surface or orbifold $X = \mathcal{U}/\Gamma$) on $T(\Gamma)$ – lifts to an action on $F(\Gamma)$ which

is effective (except when $g = 1$ or 2), and the quotient space is a family of hyperbolic discs \mathcal{U}_μ over $\mathcal{M}_{g,n}$. The discrete group Γ acts on this family via quasiconformal deformations Γ_μ on \mathcal{U}_μ as fibre-preserving maps to produce the *modular family* $\mathcal{C}_{g,n}$ of Riemann surfaces of (generic) genus g with $n \geq 0$ special points. The quotient fibration

$$\pi : \mathcal{C}_{g,n} \mapsto \mathcal{M}_{g,n}$$

has *orbifold singularities* which stem from symmetry properties of certain (non-generic) Riemann surfaces. These arise in the following way.

Proposition 1 *At a point x of $\mathcal{M}_{g,n}$ which represents a Riemann surface S_x with direct symmetry group H , the fibre $\pi^{-1}(x) \subset \mathcal{C}_{g,n}$ is the quotient surface S_x/H .*

This fact is easily seen from the covering space properties of the hyperbolic uniformisation: Riemann surface symmetries lift to hyperbolic disc isometries, and these contribute to the quotient action producing the fibre surface in the construction using $R_0(\Gamma, G)$ outlined above; see [43] for more details.

NOTE. It is an important ingredient of the Fuchsian group approach to moduli that the marking procedure for the group representing a surface with distinguished points provides a canonical way to incorporate the *forgetful map* which forgets one chosen distinguished point (or a designated set of points).

To explain this process, we recall the detailed geometric information present in the datum of a marked Fuchsian group Γ with quotient surface $X = \mathcal{U}/\Gamma$ of finite type: there is a finite presentation for Γ involving the standard commutator relator arising from the canonical surface symbol and further algebraic relations, one for each conjugacy class of torsion or boundary parabolic element. We have by implication an ordered finite set of special points on X , with interior cone points of angle $\frac{2\pi}{n}$ for order n torsion generators and a puncture for each parabolic generator, manifested as a *cusp* of Γ , a pointlike boundary component of X which may be filled in by a standard procedure to yield a hyperbolic Riemann surface X^* with one less boundary point. This process makes it possible to formulate precisely (and prove) the following *Bers fiber-space theorem*, which identifies the fiber space $F(\Gamma)$ as a Teichmüller space in its own right:

Theorem 3 *Let $X = \mathcal{U}/\Gamma$ be a compact Riemann surface uniformised by a (finite co-area) Fuchsian group Γ . Then*

$$F(\Gamma) \cong T(\Gamma'),$$

where Γ' is a Fuchsian group uniformising the punctured surface $X' = X - \{x_0\}$, where $X = \mathcal{U}/\Gamma$

It leads also to an identification between Teichmüller modular groups. Under the same hypotheses, we have

Theorem 4 *The modular group $\text{mod}(\Gamma)$ is a finite index subgroup of $\text{Mod}(\Gamma')$.*

In particular, $\text{mod}(\Gamma) \cong \text{Mod}(\Gamma')$ when Γ is a closed surface group, i.e. torsion free and co-compact.

The latter result is the counterpart of the final terms in the topologist's long exact sequence for the homotopy groups of a fibration for the holomorphic fiber spaces $F_{g,n} \rightarrow \mathcal{M}_{g,n}$. In general, these two modular groups are only commensurable, that is, they possess isomorphic 'pure' subgroups which correspond to mapping classes preserving all of the marked points.

3.4 Hyperbolic surfaces inside moduli space.

One of our aims is to exhibit and study finite area Riemann surfaces which are totally geodesic suborbifolds of moduli space. We saw earlier the examples discovered by Thurston; a generalisation of this construction was obtained later by Veech and others. In order to describe some of these results, we need a preliminary account of complex Teichmüller geometry.

DEFINITION. A *Teichmüller geodesic disc* is a set of marked complex structures on a Riemann surface X_0 arising from the following construction:- fix a quadratic differential form ϕ on X_0 . This determines a complex 1-parameter family of deformations of X_0 with several alternative descriptions:

(i) A characteristic type of parametrisation of X_0 by *complex affine charts* is determined by ϕ : locally away from the zeros of ϕ , write $\phi = dw^2$ to get local parameters w up to transition functions of the form $w \mapsto \pm w + c$. Equivalently, set

$$w = \int_{z_0}^z \sqrt{\phi(t)}.$$

Following Veech [67], one may call such an atlas an \mathcal{F} -structure on X_0 .

Now for each ε with $|\varepsilon| < 1$, define a new \mathcal{F} -structure on the underlying topological surface S_0 by rotating each chart through $\arg \varepsilon$ and expanding the real foliation of $\mathbb{R}^2 = \mathbb{C}$, while contracting the imaginary (vertical) one, by the mapping

$$z = x + iy \mapsto K_\varepsilon^{\frac{1}{2}} x + iK_\varepsilon^{-\frac{1}{2}} y \quad \text{where} \quad K_\varepsilon = \frac{1 + |\varepsilon|}{1 - |\varepsilon|}.$$

If $\arg \varepsilon = 0$, so that $0 < \varepsilon < 1$, the family of \mathcal{F} -structures so defined is called the *Teichmüller ray* at X_0 in the direction ϕ .

(ii) Write $\nu_\varepsilon(z) = \varepsilon \bar{\phi}(z)/|\phi(z)|$ and solve the Beltrami equation (1)

$$\frac{\partial w}{\partial \bar{z}} = \nu_\varepsilon \frac{\partial w}{\partial z}$$

for this family of complex dilatations; conjugating the marking of X_0 by the family ω_ε of solution homeomorphisms (suitably normalised to remove the G

ambiguity) yields a one-dimensional holomorphic family of Riemann surfaces over the unit disc \mathbb{D} .

It is an exercise in the basic workings of quasiconformal mappings to see that these two descriptions determine the same complex 1-parameter deformation of the central surface X_0 .

There are at least two further valuable ways to view this deformation. One stems from the affine Euclidean geometry of the system of local charts determined by a quadratic differential; this is the point of view employed by Veech and also by Kerckhoff, Masur and Smillie [34]; it is described very neatly in the survey article [14] and we describe one of Veech's examples in these terms later in this section. The second, perhaps more topological, view is older and was developed originally by Jenkins and Strebel, and later by Gardiner (for details, see for instance [30], [64], [16]); it makes use of the concept of *critical trajectories* for a quadratic differential. These are the ϕ -horizontal curves on X_0 through the zeros of ϕ in the initial system \mathcal{F}_0 of Euclidean charts: the set of all (horizontal) critical trajectories determines a decomposition of the surface into a union of annuli and spiral regions, each foliated by simple horizontal trajectories. One may then pose an *extremal stretching* problem on the Riemann surface S_0 , determined by the topological structure of this singular foliation arising from the quadratic differential ϕ : a *weight* is assigned to each component of the noncritical set and one seeks a metric on the surface which minimises the distortion overall in a sense determined by the set of weights.

This last point of view is valuable because it links the classical complex analytic methods with the ideas of Thurston on deformations of hyperbolic structures, which bring into play the geometry of foliations of a surface with 'finitely pronged' singularities to set up the space of measured foliations which determine a natural real spherical boundary of Teichmüller space. This *Thurston boundary* does of course include points representing the endpoints of the T-rays emanating from the central point $[X_0]$ as described above, but differs in an essential way from the *Teichmüller boundary*, also a sphere, which is obtained by considering the set of all the boundary points of T-rays from a fixed central base point, for reasons arising from the subtle dependence of the latter sphere on the choice of base point.

The interested reader should consult Kerckhoff's paper [32] for details of this fundamental distinction: he establishes the crucial fact that whereas the Teichmüller modular group action extends naturally to the topologically-based Thurston boundary with highly significant consequences (see [66]), it cannot extend continuously to the more rigidly constructed Teichmüller boundary.

3.5 Example: a T-disc parametrising tori.

The motivating example of this basic construction comes from the deformations of a torus: recall that marked tori X_τ correspond bijectively to $\tau \in \mathcal{U}$. We need to relate this to the T-disc construction, in which we employed the con-

formally equivalent unit disc as parameter space. To define the deformation homeomorphisms from a reference point, say $i \in \mathcal{U}$, to $\tau \in \mathcal{U}$, we use the real affine mapping $\tilde{f}_\tau : x + yi \mapsto x + y\tau$

$$\begin{array}{ccc} \tilde{f}_\tau : & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & \downarrow & & \downarrow \\ \text{which induces } f_\tau : & \mathbb{C}/\Lambda_i & \longrightarrow & \mathbb{C}/\Lambda_\tau \end{array}$$

and this homeomorphism is *extremal* within its homotopy class in Teichmüller's sense, that is, it has the least overall distortion measured by making the supremum of the local stretching function on X as small as possible. To obtain the local distortion, one calculates

$$\frac{\bar{\partial} f_\tau}{\partial f_\tau} = \frac{(1+i\tau) d\bar{z}}{(1-i\tau) dz}, \quad \text{where } \partial = \frac{\partial}{\partial z}, \bar{\partial} = \frac{\partial}{\partial \bar{z}}$$

and z is the local parameter ($z = x + iy$) induced on the complex torus by the quadratic differential form dz^2 on \mathbb{C} . This can be rewritten as $\varepsilon \bar{\phi}/|\phi|$ where $\varepsilon = \frac{1+i\tau}{1-i\tau} \in \mathbb{D}$, the unit disc, and ϕ is the projectsd holomorphic form dz^2 on \mathbb{C}/Λ_i .

In the present case, the Teichmüller distance between i and τ as defined earlier turns out to be (see for instance [1])

$$d(i, \tau) = \frac{1}{2} \log \left(\frac{1+|\varepsilon|}{1-|\varepsilon|} \right).$$

This is of course Poincaré's metric on \mathcal{U} , giving it the structure of real hyperbolic plane.

3.6 Deformation T-discs embed.

In general, let X be a Riemann surface, and let $\phi \in \Omega^2(X)$, $\phi \neq 0$. The Teichmüller deformation specified by ϕ defines a holomorphic mapping e_ϕ of \mathcal{U} (or \mathbb{D}) into $T(X)$ via the following map into $B_1(X) \subseteq L_{-1,1}^\infty(X)$

$$e_\phi(\varepsilon) = \varepsilon \frac{\bar{\phi}}{|\phi|} \quad \text{for } |\varepsilon| < 1).$$

Proposition 2 *The mapping e_ϕ is a proper holomorphic injection of the disc \mathbb{D} into $T(X)$.*

PROOF. The map e_ϕ is proper because as $|\varepsilon| \rightarrow 1$, the norm $\|e_\phi(\varepsilon)\| \rightarrow 1$, which implies that the distortion is becoming arbitrarily large and so $e_\phi(\varepsilon)$ tends to the boundary $\partial T(X)$. The map is injective, either by Teichmüller's uniqueness theorem [7] or by the uniqueness of solutions to Beltrami's equation, and it is holomorphic because the complex structure on $T(X)$ is inherited from that of the Banach space of distortion measures $B_1(X)$. \square

The global metric between points in Teichmüller space, as defined earlier, is in fact realised by the Poincaré metric for points in a T-disc. This is part of the original approach introduced by Teichmüller who proved that any two points are joined by a Teichmüller ray, and used the formula 2 to define it. As a consequence, every embedded Teichmüller-disc is totally geodesic and this leads to another result of Royden (op. cit.) which identifies Teichmüller's metric with the canonical *Kobayashi metric* on this complex manifold. In the present context, we can re-state this classic result, generally known as 'Teichmüller's Theorem', as follows.

Theorem 5 *For any two distinct marked Riemann surfaces in T_g , there is an embedded T-disc which contains them, unique up to an isometry of the disc, and the distance between them is given by Poincaré's hyperbolic metric.*

For a proof of this theorem, which inaugurated the global theory of moduli, see [4], [7].

3.7 Veech's examples.

A recipe using Euclidean geometry for explicit construction of a geometric structure on a surface using holomorphic differential forms is given next; this is part of a general procedure for endowing a surface with a covering by Euclidean polygonal charts, which W. Veech has called *\mathcal{F} -structures*. They are also called *translation surfaces* by some authors, since the surfaces concerned are determined by identifications of edges of polygons using Euclidean translations.

The examples we describe involve surfaces which may be represented explicitly as ramified coverings over $\mathbb{C}P^1$: they are taken from a large class discovered by Veech ([67],[69]).

Let X_n , with $n \geq 5$ and odd, be the compact hyperelliptic surface of genus $g = (n - 1)/2$ with affine equation $y^2 = 1 - x^n$ and consider the holomorphic 1-form $\omega = dx/y$. It has a zero of order $2g - 2$ at the single point where $x = \infty$. The quadratic form $q = \omega^2$ defines a Teichmüller disc in T_g , which turns out to have particular significance.

Veech considered these structures in connection with the dynamical study of billiard trajectories in a Euclidean isosceles triangle T with angles $(\frac{\pi}{n}, \frac{\pi}{n}, \frac{(n-2)\pi}{n})$. The relationship with X_n is based on the following construction, involving an associated triangle T_ζ with vertices at 0, 1 and $\zeta = e^{2\pi i/n}$.

Let $P = \bigcup_{\ell=0}^{n-1} \zeta^\ell T_\zeta$; multiplication by ζ determines a cyclic Euclidean rotation symmetry of P such that the quotient mapping is an n -fold covering of the plane near 0. Let Q be the equivalent polygon formed with $\tilde{T}_\zeta = -T_\zeta$, and then identify pointwise in $P \cup Q$ the pairs of outside edges in corresponding triangles $\zeta^\ell T_\zeta, \zeta^\ell \tilde{T}_\zeta$ using a translation (the corresponding edges are parallel). The result is a closed surface with a local complex analytic structure except at the projection of the corners, where the union of all the corner angles adds up to

more than 2π . After extending the complex structure by removing these isolated singularities at the corners – using Riemann’s classical approach to resolving isolated singularities – this process determines a closed Riemann surface X_ζ (with nontrivial conformal symmetry group) having a local structure over \mathbb{P}^1 given by the following composite map $p_2(z) := f \circ p_1$:-

$$\begin{array}{ccc}
 X_\zeta & & \\
 p_1 \downarrow & & p_2(z) := f \circ p_1 \\
 \mathbb{C}P^1 = X_\zeta / \langle z \mapsto \pm z \rangle & \xrightarrow{f} & X_\zeta / \langle z \mapsto \pm \zeta z \rangle \quad \text{in } P \cup Q
 \end{array}$$

The arrow labelled f is the standard mapping $z \mapsto z^n$, after moving the origin of z -coordinates to the centre of the polygon P . It is not difficult to infer from this geometric data that X_ζ is the compact Riemann surface X_n defined above: for instance, observe that there is an order n symmetry of X induced by lifting $z \mapsto \zeta z$ which fixes three points, the two centre points and the single orbit of corner points. Also it follows that the 1-form dz in $P \cup Q$ lifts to ω on X_ζ . Veech proves the following result, a striking link between the hyperbolic geometry of the disc and the complex geometry of Teichmüller space and the Teichmüller modular group.

Theorem 6 *The stabiliser in the modular group Mod_g of the Teichmüller disc determined by the differential $q = \omega^2$ is a Fuchsian triangle group*

$$H_n = \left\langle \left(\begin{array}{cc} 1 & 2 \cot \frac{\pi}{n} \\ 0 & 1 \end{array} \right), \left(\begin{array}{cc} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{array} \right) \right\rangle = \langle \sigma, \beta \rangle$$

which is isomorphic to $\langle x_1^n = x_2^2 = 1 \rangle$ via $x_1^{g+1} x_2 = \sigma, x_1 = \beta$.

3.8 Hecke triangle groups and polygonal surface tilings.

In this section, we explore some of the hyperbolic geometry underlying Theorem 6. The triangle group H_n obtained in the theorem is known as a *Hecke group* because of Hecke’s classic work [28] on this kind of Fuchsian group in connection with Dirichlet series satisfying a functional equation.

A fundamental region for the action of H_n on the hyperbolic plane \mathcal{U} may be obtained by a standard geometric construction: take a quadrilateral F with a reflection symmetry which fixes the diagonal joining one ideal vertex at ∞ and a second vertex having interior angle $2\pi/n$. It is simplest to view F as the union of two abutting hyperbolic triangles T, \bar{T} , each with angles $\pi/n, 0$ and $\pi/2$. Place the vertices of T with one of angle π/n at i , the zero angle at ∞ and the third at the foot $\pm\xi$ of the hyperbolic perpendicular from i to the line $\text{Re} z = \cot(\pi/n)$ and let the triangle \bar{T} be the reflection of T in the imaginary axis. Just as the action of an order n rotation divides a Euclidean regular n -gon

into n congruent isosceles triangles or, equivalently, into n quadrangles with angles $\{2\pi/n, \pi/2, (n-2)\pi/n, \pi/2\}$, there is a hyperbolic regular ideal n -gon P' with all vertices at infinity, which is the result of applying to F the order n hyperbolic rotation x_1 fixing i .

Consider now the union of P' and the (hyperbolic) translate $\sigma(P')$, which is a congruent polygon sharing a vertical geodesic side. It is not hard to show that the two operations x_1 and σ generate a discrete group of isometries of \mathcal{H} which turns out to be G_n : the union of all translates of the quadrilateral F tessellate \mathcal{H} and these two elements are side-pairing elements of F which fit into the framework of the classical theorem of Poincaré on fundamental polygons for discrete hyperbolic isometry groups ; for more details of this result, see [41], [43], or [50].

We can apply Proposition 2 and the homogeneity of the hyperbolic isometry group G acting on \mathcal{U} to arrange the present pattern neatly into the (local and global) complex geometric structure of the Teichmüller space $T(X_n)$, by placing the reference point i at the marked surface X_n and, furthermore, fixing the T-ray from i in the direction of q to be the upper part of the imaginary axis.

It is important to maintain contact here with the symmetric properties of the Euclidean tessellation picture imposed on X_n by the form q . Note that the element x_2 of order 2 (which may be chosen to fix the $\pi/2$ -corner point of T) corresponds to interchanging the Euclidean polygons P and Q via the halfturn $z \mapsto -z$, whereas it is represented on the quadrangle domain F by two conjugate involutions, each fixing one of the $\pi/2$ -corners $\pm\xi$. Also, the isometry x_1 of order n fixing i corresponds to the lifting of $z \mapsto \zeta z$ on each polygon. It is not hard to show using the geometry of the hyperbolic plane that within the tessellation of \mathcal{U} by ideal quadrangles $\gamma \cdot F$, $\gamma \in H_n$, we recover the original decomposition of X_n into a pair of (hyperbolic) regular ideal polygons $P', Q' = x_2(P')$ with n vertices.

To produce an algebraic description of the hyperbolic surface X_n , it is convenient to consider the quotient space X of \mathcal{U} by the commutator subgroup $C_n = [H_n, H_n]$. To see that this is indeed the same Riemann surface, one needs to note that the commutator quotient group $\mathbb{Z}_2 \oplus \mathbb{Z}_n$ acts as a group of bi-holomorphic automorphisms of X , with precisely the same symmetry pattern (fixed points etc) as that of $X_n = \{P \cup Q\} / \sim$ taken modulo the symmetries of the dihedral n -gon. Of course, the ideal (zero-angle) vertices, which determine either one or two points at ∞ on the quotient surface depending on the parity of n , must be added to create a compact surface. By adjoining cusps to X , we obtain a compact surface. A fundamental domain for C_n may be formed by first taking the union of n abutting (horizontal) translations of the domain F for G_n to produce a lift of the punctured polygon $P - 0$ and then adjoining a translate of this by the involution x_2 . The result is now recognisable as X_n , at least when n is odd: observe that it has genus $g = (n-1)/2$ (via the Riemann-Hurwitz formula), an order n symmetry with three fixed points (at the C_n orbits $[i]$, $[0]$ and ∞) and an involution symmetry induced by conjugation with x_2 on \mathcal{U}/C_n , which must be hyperelliptic because it fixes the $n+1 = 2g+2$ points on the quotient surface arising from the n right-angle corner points of P' and the (single)

orbit of cusp vertices.

In the piecewise-Euclidean model X_ζ of the surface, the critical graph of the holomorphic 1-form q consists of a union of vertical line segments through the vertices of the polygons P and Q , all of which are identified to a single point of X_ζ ; the complement is a set of $k = (n - 1)/2$ disjoint cylinders. When viewed in the hyperbolic model as the surface X_n , the cylinders lift to strips which comprise the connected components of the complement of the critical graph \mathcal{K} . In this way, it follows that a set of disjoint geodesic curves tending to the single cusp at ∞ may be taken as representative liftings of the critical trajectories of the holomorphic form φ of weight 2 on \mathcal{U} whose square defines $q = \omega^2$ on X_n . It also follows that the k^{th} power of the parabolic element σ of H_n determines a covering map which preserves each cylinder, and so defines a product of Dehn twists about the core curves of the cylinders.

These facts can also be verified directly by matrix calculations within the \mathcal{F} -structure, which are similar to the reasoning given earlier in Thurston's original examples; see for instance [14] for a careful analysis along these lines.

Because this T-disc $\mathbb{D} = \mathbb{D}_q$ has a lattice group stabiliser in the modular group Mod_g , any sequence of points in \mathbb{D} is modular-equivalent (by elements stabilising the T-disc) to either a convergent sequence or a sequence tending to the cusp of H_n and so divergent in \mathcal{M}_g : it follows that the image in the quotient moduli space is a closed (but non-compact) sub-orbifold of the complex V-manifold \mathcal{M}_g . This proves (in outline) the following theorem, also due essentially to Veech [67].

Theorem 7 *The image in \mathcal{M}_g of the Teichmüller disc through \overline{X}_ζ corresponding to the form q is a complex analytic subspace isomorphic to the affine complex line, $\mathcal{U}/H_n = X_\zeta/\text{Aut}(X_\zeta)$.*

This result motivates the following definition.

DEFINITION. A *Teichmüller - Veech disc* is a Teichmüller disc in a Teichmüller space T_g for which the stabiliser in the Teichmüller modular group is a lattice subgroup of the hyperbolic isometry group G .

The image in \mathcal{M}_g is then a finite area hyperbolic surface, called a *Veech curve* or sometimes a Teichmüller curve.

NOTES. 1. Concealed here within the orbifold structure of the moduli space is a paradoxical appearance by the *hyperelliptic involution* J of the central surface \overline{X}_ζ : as a modular automorphism of Teichmüller space, J fixes the point i of \mathcal{U} corresponding to \overline{X}_ζ and sends the 1-form φ to its negative. However this implies that it must fix *every point* of the Teichmüller-Veech disc we have constructed. Thus the stabiliser of the point i in the hyperbolic plane (representing the T-disc) \mathcal{U} is the cyclic group $\langle x_1 \rangle$ and not the full automorphism group of X_ζ , the surface which underlies this point of T_g . In fact the element x_2 is a representation of the hyperelliptic involution of X_ζ (on the universal covering space \mathcal{U}) and lies in the kernel of the holonomy homomorphism $f \mapsto Df$ from

the group of affine selfmaps of the \mathcal{F} -structure to $GL(2, \mathbb{R})$. Therefore x_2 is killed by the restriction homomorphism from the modular group stabiliser of the T-disc \mathcal{U} to the isometry group of the T-disc. On the other hand, the hyperelliptic involution j is induced by this order 2 generating element x_2 , when viewed as acting on the universal covering of the central fibre of the T-disc, but this is illusory. The point of our T-disc actually fixed by x_2 represents a different Riemann surface from X_ζ , having an additional involution automorphism; Veech [67] calls this involution a hidden symmetry.

2. The case of regular n -gons with n even follows a related but slightly different pattern which is detailed in [68]. In fact, using the discussion of Veech's examples given by [14] it follows that for every $n \geq 5$, the hyperelliptic curve X_n , of genus $g = [n - 1/2]$, enjoy the property that there are two independent Teichmüller-discs centred on the corresponding point in \mathcal{M}_g , determined by the 1-forms $\omega_1 = dx/y$ and $\omega_2 = xdx/y$, the second corresponding to a construction of the same surface as a translation surface quotient from a single regular polygon with $2n$ sides; in the latter case the T-disc has as stabiliser a $\{n, \infty, \infty\}$ triangle group.

3.9 Fermat curves represented by T-discs.

The *Fermat curves* Y_n are determined by an affine equation

$$x^n + y^n = 1$$

and have genus $g = (n - 1)(n - 2)/2$, and so when $n \geq 4$ they have hyperbolic uniformisation and fit into a similar picture with respect to the same Hecke triangle groups H_n as above. Note that the corresponding compact Riemann surface is obtained by adding just one or two points at infinity depending on parity.

The tessellation of Y_n by regular n -gons may be seen in several ways, for instance by using a similar description of a tailoring pattern for Y_n using $2n$ copies of the polygon P , as in [68]. A second method, less reliant on diagrams and explicit pastings, comes from consideration of the collection of finite index subgroups of a given Hecke group H_n and the corresponding quotient surfaces covering the affine plane \mathcal{U}/H_n . [25]. One sees by either method that the automorphic form of weight 4 for $C_n = [H_n, H_n]$ on \mathcal{U} which determines the quadratic differential form $q = \omega^2$ on X_n also induces the form $q_1 = dx^2/y^{2n-2}$ on the covering surface Y_n ; this may be viewed as arising by pullback on forms via the holomorphic covering. In the case of the Fermat curve, however, the involution x_2 in H_n does not preserve the polygonal tessellation of the surface, because the hyperelliptic involution does not lift to Y_n . The (lattice group) stabiliser in the modular group of the surface \bar{Y}_n for this Teichmüller disc is in fact precisely the index-2 subgroup H'_n that omits this element. This subgroup has fundamental set a union of two right triangles interchanged by x_2 , and hence is easily seen to be a triangle group of type (n, n, ∞) .

Theorem 8 *The image in \mathcal{M}_g of the Teichmüller disc through the complex Fermat curve \bar{Y}_n corresponding to q_1 is a complex curve isomorphic to the affine line, \mathcal{U}/H'_n .*

See [69] for detailed proofs of this and other precise stability results.

3.10 Regular tessellations and smooth coverings

In this section, we show how to produce many other examples of hyperbolic discs in T_g with large stabiliser, by an immediate inference from the argument used for the Fermat surfaces.

Let \tilde{X} be any smooth genus h covering surface of the (compactified) commutator surface X_n for the Hecke group H_n with $n \geq 3$. Then the same reasoning as above with the pullback of the form ω shows that there is a corresponding T-disc in the Teichmüller space T_h centred at $[\tilde{X}]$ with stabiliser of finite index in H_n . In some sense, it is the same data that is involved here: when lifted to a system of covering surfaces, the T-disc given by ω lives in some inverse limit of Teichmüller modular objects which identifies commensurable structures occurring among all finite coverings of X_n . Dennis Sullivan has called this type of limit object a *solenoidal surface* or *Riemann surface lamination*; our examples are individual levels in the profinite completion of the (complex) algebraic curve X_n .

Of course, one can proceed in analogy with this process for any example of *Veech curve*, by which we mean a Teichmüller disc with stabiliser a lattice Fuchsian group, and achieve further examples with the same property.

Note. A point which merits further exploration is the need to correlate the various modular groups arising in this situation. The result, in the restricted case where the coverings of moduli spaces are required to be finite cyclic and smooth, is a new type of generalised modular group which is called the *commensurability modular group* by Nag and Sullivan [53].

3.11 Return to Thurston's examples.

In this section we explain how to generalise the examples discussed earlier.

Take two distinct finite systems of loops which together *fill up a compact surface*, that is, a pair $\alpha = \{\alpha_1, \dots, \alpha_m\}$, $\beta = \{\beta_1, \dots, \beta_n\}$, such that loops in the same set are pairwise disjoint and homotopically distinct, and with α -loops intersecting β -loops transversely in such a way that the complement is a union of topological discs, each with boundary containing an even number of segments of alternately α and β curves and at least four in number, so as to ensure that the loops have no inessential intersections. Choose centres for these disc components of $S - \{\alpha \cup \beta\}$; there is a dual cell decomposition of S into quadrilaterals which arises from joining centres of adjacent components by edges labelled as $v =$ vertical for α , $h =$ horizontal for β .

Now an \mathcal{F} -structure is defined on the surface in the same way as before, by viewing the surface as covered by unit square in the Euclidean plane, with

the induced orientation and $\alpha =$ vertical, $\beta =$ horizontal labelling. This again defines an \mathcal{F} -structure on S . Furthermore, a twist about any α or β curve acts as an integer translation when viewed (in the holonomy representation) in this affine structure on the real plane below; the group generated by them is often a finite index subgroup of $SL_2(\mathbb{Z})$.

We note, however, that the subgroup of the stability group of a T-disc generated by twists may have infinite index, as in the example of a surface with genus 2 described by Earle and Gardiner [14]. It is constructed from a 6-by-1 rectangle – six squares in a line – with edge identifications given by a translation by 6 from one end to the other and four involutions, operating in pairs on top and bottom sets of edges, with two edges labelled a_j of length 1 and two labelled b_j of length 2, to produce a handle from each, to give a surface symbol $a_1 b_1 a_1^{-1} b_1^{-1} c a_2 b_2 a_2^{-1} b_2^{-1} c^{-1}$. It is not difficult to see that this structure has twist subgroup generated by the matrices

$$\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix},$$

which has infinite index in the modular group $\Gamma(1)$, while the full stabiliser is commensurable with $\Gamma(1)$.

It follows from Veech's work on the Teichmüller geodesic flow on the bundle of quadratic differentials Q_g over T_g (see [68]) that the points of T_g which possess such a representation are dense; we shall make use of this fact in the next chapter. A rather different density property for these structures was proved in a paper of H. Masur [44]: the set of Teichmüller-geodesics given by the original Thurston prescription (using single α and β curves) projects to a dense subset of the unit tangent bundle to moduli space. As a consequence, there exist Teichmüller geodesic rays whose projection is dense in \mathcal{M}_g .

4 Triangle groups and modular families.

The Hecke groups H_n and the more general triangle (p, q, ∞) -groups appear as a crucial ingredient in several areas of mathematics. We shall see in this section that they are naturally related to the construction of many Teichmüller disc families of Riemann surfaces and in particular to the examples discovered by Thurston and Veech.

4.1 Tessellations and triangle groups.

Let X be a compact Riemann surface which admits a regular tessellation by hyperbolic n -gons. We describe how such a surface admits reconstruction from a torsion-free subgroup Γ of finite index in a Hecke group H_n ; this follows from a fundamental theorem about maps on surfaces (see for instance [63]). In particular, the case $n = 3$ relates surfaces X with a decomposition into equilateral triangles to the classical Farey tessellation of the hyperbolic plane associated with the classical modular group, $H_3 = \Gamma(1) = PSL_2(\mathbb{Z})$. The

compact surface $X = X_\Gamma$ obtained from the (finite co-volume) Fuchsian group Γ has finitely many *cusps* added to the (finite area) surface \mathcal{U}/Γ in the usual way, one for each Γ -orbit of boundary points at which the stabiliser is non-trivial. The holomorphic projection $\pi : \mathcal{U}/\Gamma \rightarrow \mathcal{U}/H_n \cong \mathbb{C}$, induced from the identity map on the universal covering \mathcal{U} , extends to a Belyi map $f : X \rightarrow \mathbb{P}^1$ from the compactification, with each cusp of X mapped by f to the single cusp $\{\infty\}$ of H_n which compactifies the plane \mathcal{U}/H_n to $\mathbb{C} \cup \infty = \mathbb{P}^1$. The tessellation of X is then determined by projecting part of the standard H_n -invariant subdivision of \mathcal{U} into $\{2, n, \infty\}$ triangles. Such a triangle arose in the preceding section as half of a fundamental domain F : in that picture, the G_n -orbit of the hyperbolic line $L = \{\Re z = \cot(\pi/n)\}$ determines a tiling of \mathcal{U} by ideal n -gons, projecting to a finite tessellation of X whose vertices correspond to the cusps of Γ . To say that the tessellation of X by n -gons is *regular* means that a fixed number m of polygons surround each vertex, which is equivalent to the statement that the total width of each cusp is the same fixed number.

Notes. 1. The data of a regular m -valent tiling by n -gons of a compact surface can be specified combinatorially by prescribing a torsion-free subgroup of a $\{2, n, m\}$ -triangle group, but it is the cusped version of the picture that provides the link with Teichmüller surface families.

2. In [63] a convenient method is given to generate *all* hyperbolic surface tessellations in similar fashion, by a universal construction in \mathcal{U} governed by representative subgroups of the modular (cartographic) subgroup $\Gamma_0(2)$. This will be employed to give a succinct description of a wide class of modular curve families in the next subsection.

3. The equivalent combinatorial concept for more general triangle $\{p, q, \infty\}$ -groups is a *hypermap* (see [31]): here the corresponding universal construction involves the principal congruence subgroup $\Gamma(2)$ (which has index 2 in $\Gamma_0(2)$) and subgroups Γ of finite index in that group.

4.2 Associated Teichmüller families.

We wish to produce, from the covering data $\{X, f\}$ of the preceding section, a corresponding curve in the moduli space of X . This will apply not only to the Riemann surfaces tiled by regular n -gons, which correspond to ‘balanced dessins’, but also to the broader class of complex orbifold structures given as *clean Belyi coverings* of \mathbb{P}^1 ; in this case the surface is the quotient of the disc by a finite index subgroup of a (p, q, ∞) -triangle group $\Gamma_{p,q}$. The collection of pairs (algebraic curve & map) so generated are part of the totality of dessins and of course, by the theorem of Belyi, each is defined over a number field – in fact, they must be part of the $\overline{\mathbb{Q}}$ -structure of the modular varieties themselves by Weil’s results [70] on the field of definition for a variety. Further discussion of this may be found in [71] and in [18].

By the classification of surface maps and hypermaps, we may assume that

our covering is given by a Fuchsian group Γ of finite index in one or another of the groups $\Gamma_0(2)$, $\Gamma(2)$ or $\Gamma_{p,q}$. As before, X will denote the compact surface obtained from filling in the cusps on \mathcal{U}/Γ .

Take a tessellation of the hyperbolic plane \mathcal{U} by a standard fundamental set Q for $\Gamma_0(2)$, which is a hyperbolic quadrangle with ideal vertices at -1 , 0 , 1 and ∞ bisected into triangles by the imaginary axis \mathcal{I} . Now we indicate two further ways to approach the construction of quadratic differential forms for Γ , the first one similar to an observation of [14].

Let $\omega(z) = dz^2/(z^3 - z)$ denote the form on $Y = \mathbb{C}P^1$ with simple poles at the point set $B = \{0, \pm 1, \infty\}$. We consider the pullback of ω to a compact surface X through some Belyi function $F : X \rightarrow \mathbb{P}^1$, which we assume factors through another function $f : X \rightarrow Y$, ie. $F = f^2$. The pullback to X , $f_*(\omega)$ will have a framed horizontal/vertical structure inherited from that of ω on Y , and if f is totally branched over B the form will be holomorphic on X .

For instance, this structure lifts to a geometrically similar \mathcal{F} -structure on the elliptic curve E given by the Weierstrass equation $w^2 = z^3 - z$, which is a double covering of Y branched over the set B , and which has a precise description in terms of the standard \wp -function for the lattice $\mathbb{Z} + \mathbb{Z}i$ corresponding to E .

It is not difficult to verify [14] (cf. [67]) that the affine stabiliser of the induced \mathcal{F} -structure on Y is the congruence subgroup $\Gamma(2)$. Also, one sees from this specific model that for all such pairs X, F the stabiliser of the T-disc for the form $F^*(\omega)$ has finite index in $\Gamma(1)$, by a simple topological argument. Of course, when lifted to general coverings of Y , the form ω_f will have simple poles at any unramified points of $f^{-1}(B)$.

The second method works only for surfaces X covering the modular surface $X_0(2) = \mathcal{U}/\Gamma_0(2)$ and produces forms of an arithmetic nature on X from familiar modular forms and depends on a simple fact: the group $\Gamma_0(2)$ possesses a holomorphic cusp form ω_1 of weight 8 that is real on the imaginary axis \mathcal{I} with a zero of order 3 at each of the two cusps. This form is defined to be

$$\eta(\tau)\eta(2\tau)^4$$

where $\eta(\tau)$ is Dedekind's well-known (character-automorphic) form for $\Gamma(1)$,

$$\eta(\tau) = e^{2\pi i\tau/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \text{for } \tau \in \mathcal{H}, \text{ and } q = e^{2\pi i\tau}.$$

For any subgroup $\Gamma < \Gamma(1)$ of finite index with all cusp widths even, one verifies easily that the lifted form to X_Γ has only even order zeros at the cusps, hence has a square root ω_1 which belongs to the space $S(\Gamma, 4)$ of weight 4 cusp forms. Furthermore, the form ω_1 has critical trajectories which interconnect the cusps of the surface X in some finite graph, which includes among its edges the projection of the axis \mathcal{I} and whose pattern depends only on the structure of the covering. In addition, for any conformal automorphism α of X which fixes the cusps, one can average the form over the elements α^ν to obtain a cusp form ω_2 with (at least) two independent directions in which all critical trajectories

interconnect cusps and all noncritical trajectories are closed. This property, which is shared by the form ω defined earlier on Y , turns out to be crucial in ensuring a lattice group as Teichmüllermodular stabiliser.

We consider T-discs \mathbb{D}_ω for such forms. In order to proceed as canonically as possible, and to avoid the confusion arising from the plethora of inclusion relations between the various Teichmüller spaces associated to surface coverings, in what follows we place modular group activities within the universal group action of mod (1), the group $QS(S^1)$ of all quasi-symmetric homeomorphisms of the boundary circle, on Teichmüller space $T(1)$, universal Teichmüller space, the normalised quotient by the subgroup (isomorphic to $G = PSL_2(\mathbb{R})$) of Möbius mappings of the circle.

Let $\Gamma \leq \Gamma_0(2)$ be a finite index normal subgroup contained in the kernel of the finite character $\chi_\omega : \Gamma_0(2) \rightarrow S^1$, and representing a surface $X = \mathcal{U}/\Gamma$, so that $\omega(\gamma(z)) = \chi_\omega(\gamma)\gamma'(z)^{-2}\omega(z)$ for all $\gamma \in \Gamma_0(2)$. If the form ω is Γ -invariant, we obtain an induced T-disc \mathbb{D}_ω in $T(\Gamma) \cong T_{g,n}$, where n is the total number of cusp and elliptic torsion points. This will then be mapped *injectively*, via the (holomorphic) mapping $\pi : T(\Gamma) \rightarrow T_g$ which forgets the data of all torsion and cusp points, landing in the Teichmüller space of the compactified surface X_0 .

Theorem 9 *centre point projects* The centre point of this composite embedding is a representative marked surface $[X_0]$ with a quadratic differential ω_0 such that the T-disc defined by ω_0 on X_0 realises the embedding $\pi \circ e_\omega$.

PROOF. The compact marked surface $[X_0]$ will certainly have a quadratic form ω_0 given by the obvious extension to the cusps from ω on \mathcal{U}/Γ . In fact, because we have a holomorphic image of the original T-disc, this disc is also a T-disc, since the distortion at each point is constant, with finitely many exceptional critical points. \square

We would like to characterise if possible the Veech T-discs presented in this manner, which have lattice modular stability group. From the earlier study of examples within Hecke groups H_n , it is clear that one way to do this is to restrict attention to particular covering surfaces X obtained from the commutator subgroup of the triangle group H , where there is always a good supply of (meromorphic) automorphic forms which have the following crucial property :-

DEFINITION. An automorphic form φ of weight 4 (or quadratic differential) for a finite co-volume Fuchsian group Γ is of *Jenkins-Strebel type* if the set of critical real trajectories of φ is compact in X_Γ .

For such forms, the complementary noncritical set - which has full measure in X - is a union of cylinders, each comprising a free homotopy class of closed trajectories. Incidentally, the forms occurring in Veech's examples clearly have

the J-S property in the vertical direction: one changes the direction of the trajectories from horizontal to vertical by a quarter-turn rotation of the Euclidean structure which means switching to the form $-\varphi$. It is natural to widen the class of J-S differentials to include all forms of finite L^1 norm which have *some* direction in which the trajectory pattern has this property.

We note that forms of J-S type are dense in the space $A^1_4(X)$ for *every* compact Riemann surface [13]; their construction is usually effected by either analytic or combinatorial means and no clear arithmetic connection has yet been made. For more details, including the essential part played by J-S forms in the solution of a host of extremal problems in conformal mapping and analysis, see for instance [14], [16],[64].

In all the examples described in section 3, where Γ is a finite index subgroup of a Hecke triangle group, it was possible to use the structure of the covering (or equivalently a fundamental domain for the group) triangulated in canonical fashion, to produce J-S forms from the hyperbolic geometry; our next result follows this line of approach. We shall later (in section 5) restrict attention to subgroups of the modular group $PSL_2(\mathbb{Z})(= H_3)$, where a more direct approach is possible.

Theorem 10 *Let H be a Fuchsian triangle group, with (at least) one torsion point and one cusp; assume that there is a reflection symmetry Σ fixing a cusp and a torsion point. Let ω be an invariant holomorphic quadratic differential on some smooth Belyi cover $X_1 = \mathcal{U}/\Gamma_1$ of the compact orbifold Riemann sphere X_H , which is Σ -symmetric and corresponds to some character of H . Then the corresponding Teichmüller-disc \mathbb{D}_ω in $T(X_1) \cong T_{g,k}$ has stabiliser in the corresponding modular group $Mod_{g,k}$ isomorphic to a subgroup of H of finite index containing Γ_1 .*

PROOF. We move to a conjugate of H in the unit disc so as to locate the fixed point of one torsion generator x at the centre 0 of the disc with the real ray from 0 to 1 fixed by the symmetry Σ and horizontal for the form ω . In the representation $\mathbb{D}_\omega = \pi \circ e_\omega(\mathbb{D})$, this implies that there is a finite order rotational symmetry for the Euclidean structure on the central reference surface of this disc, which is a compact abelian covering surface X_1 on which the form ω is defined, and with all cusps filled in. By the assumptions made on the group H and the form, the critical trajectories of ω emanate from cusps of Γ_1 and are contained in the H -orbit of the edge subset of some Σ -symmetric fundamental polygonal set $P \cup \bar{P}$, with $\bar{P} = \Sigma(P)$ and P containing the ray \mathcal{I} . It follows that the critical graph of ω is compact in X_1 , which implies that ω is of Jenkins-Strebel type, with all non-critical real trajectories closed.

Now consider the variation of complex structure on the surface as we follow the Teichmüller ray from the origin to $1 \in \partial\mathbb{D}$: 1 is a cusp of Γ_1 at which ω vanishes and this ray determines a deformation of X_1 by shrinking the lengths of all closed vertical trajectories to zero. We claim that the disc \mathbb{D} is invariant under a reducible (unipotent) mapping class which effects some composition of twists about a representative set of core loops, one from each cylinder in the non-critical

trajectory pattern of the form ω on X_1 . To verify this, let t be the primitive parabolic element of H fixing the cusp 1, and consider the effect of applying conjugation ad_t to the (marked) subgroup Γ_1 : the rule $\text{ad}_t(k) = tkt^{-1}$ defines an element of $\text{mod } \Gamma_1 \cong \text{Mod}(\Gamma_1')$, the mapping class group of the pointed surface X_1^* . This element is reducible in the Thurston classification of mapping classes [66], since it preserves the set of vertical core curves, but it acts as some cyclic permutation on the set of cusps and also on the set of trajectory cylinders of ω . A mapping class of the required kind is induced on the compact surface by applying such an action at the level of Γ_1 using the smallest power t^ν of the element t that preserves each trajectory cylinder. If we denote by d the degree at the cusp 1 of the covering $X_0 \rightarrow \mathbb{P}_1 \cong X_H$, then $1 \leq \nu \leq d$. It follows that the modular stabiliser of \mathbb{D}_ω contains at least the nonelementary subgroup $\langle x, t^\nu \rangle$ of H .

To see that the modular stabiliser contains the subgroup Γ_1 , we recall the discussion of section 3 where the structure of the Bers-Teichmüller fibre spaces was sketched out: in the current situation, we have a T-disc embedded in the universal Teichmüller space $T(1)$ on which the group H acts as a subgroup of $\text{Mod}(1)$. To descend to $T(X_1)$ and its fibre space $F(X_1)$ involves passing to the classes of qc mapping invariant under Γ_1 , isomorphic to the iterated fibre space $F^{(k)}(X_1)$ of moduli for k -pointed surfaces. The forgetful map π which fills in all cusps then defines a projection to the genus g Teichmüller space T_g .

It follows from the definition of the successive Bers fibrations, beginning with the surface X_1 as fibre in $F(\Gamma_1)$ over the point $0 = [X_1] \in T(X_1)$, and the Bers fiber space theorem (discussed in section 3.3) that the stabiliser in $\text{mod}_{g,k} \cong \text{Mod}_{g,k+1}$ of the fibre space over the disc \mathbb{D}_ω contains the group Γ_1 . In fact this stabiliser is a subgroup of the k -strand braid group of the surface X_1 , and one has an extension of groups within the modular group Mod_g of the smooth compact surface X_1 :-

$$1 \longrightarrow \pi_1(X_1) \longrightarrow E \longrightarrow \pi_1(X_1^*) = \Gamma_1 \longrightarrow 1.$$

The action of Γ_1 on the kernel (isomorphic to the fundamental group of the compact fibre surface) is by conjugation on E as a subgroup of $\text{mod}(X_1)$. After some careful working out, following the discussion in [14] of the relationship with the real group of automorphisms of the Euclidean planar structure on X_1 , $GL(2, \mathbb{R})$, this also turns out, to identify Γ_1 with the subgroup which Veech in [67] calls $\text{Aff}(u)$, the stabiliser of the \mathcal{F} -structure of the form ω .

Now there is a complex analytic subspace of the tangent bundle to $F^{(k)}(X_1)$ that is preserved under the action of E ; it constitutes one sheet of the Teichmüller geodesic flow on $T(X_1)$ lifted to the fibre space. Alternatively, one may view it as an example of the type of construction of analytic complex surface given in [21]. It follows that the modular stabiliser of \mathbb{D}_ω has finite index in H . \square

We describe one way to prove existence of the forms required in this theorem. Work with an inclusion of co-compact Fuchsian groups $\Gamma_0 \leq H_0$ which

provides a model of the ramified surface covering which completes the Hecke group picture: one follows the method detailed in [50] to construct generators of fractional weight for the ring of all holomorphic character-automorphic forms on the co-compact genus 0 Fuchsian group H_0 ; each of the generating forms has a simple zero at just one vertex. It is then possible to produce (by transferring to the punctured surface and lifting to the universal cover) weight 4 multiplicative holomorphic forms ω of the same type on the group H with zeros only at designated cusp vertex points of the hyperbolic tessellation of X determined by the group inclusion $\Gamma_0 < H_0$.

Corollary 1 *Let H be a triangle group of type $\{2, q, \infty\}$. Then there are holomorphic forms for the commutator surface of H whose modular stability group has finite index in H .*

The cusp forms of Jenkins-Strebel type for the groups arising in the Veech examples emerge directly from the Euclidean geometry on a given polygonal model surface, as Veech observes in [67]. The commutator surface has $\varepsilon = \gcd(2, q)$ cusps, and the subgroup of mapping classes produced by the theorem at a torsion point of the disc coincides in this case with either the whole group H or a triangular subgroup of index 2, generated in each case by the two elements x^ε and t .

For further details and developments from the above result, the reader may consult [38], which discusses Teichmüller discs stabilised by a pseudo-Anosov mapping class, and [14], where a clear description is given of some further illuminating examples of polygonal type with detailed formulae for the relation between the T-disc deformation and the affine polygonal \mathcal{F} -structure from which it is defined.

For Riemann surfaces X given by finite index subgroups Γ of general triangle groups, one may construct cusp forms of J-S type using the methods indicated above, but a more natural way follows a (less direct) path through the tessellations by ideal quadrangles induced from the standard fundamental region for $\Gamma(2)$; again one sees from the topology that the stabiliser in the Teichmüller modular group of the cusped surface X contains the subgroup Γ of $\Gamma(2)$ which classifies X and so is of finite index, by examination of the structural properties of the geometric fibre space over the Teichmüller disc.

4.3 Recent work on Veech curves in \mathcal{M}_g .

It is unclear at present how to characterise the types of Fuchsian group that occur as the Veech (modular) stability group of a Teichmüller disc in given genus, despite the great activity in this area in recent years. For the case of genus 2, however, the picture is now completely understood, thanks largely to striking recent work by C. McMullen and others, which has produced a classification partly based on an explicit type of Euclidean decomposition of a compact (genus 2) surface X as a translation surface with rectangular L-shape. With specific side-lengths, these structures will have lattice Fuchsian stabiliser, and by bringing to the fore the dynamical structure of the Teichmüller flow on such an X ,

he provides a striking trichotomy for the case of Abelian T-discs, where the form $q = \omega^2$ with ω an Abelian 1-form. The papers [46] and [48] bring out the presence of hidden complex rank 2 symmetry in \mathcal{M}_2 by establishing the existence of T-discs \mathbb{D}_q with $q = \omega^2$, $\omega \in \Omega^1(X)$ with rectangular L-shape and side lengths in a quadratic number field $\mathbb{Q}(\sqrt{d})$, such that the projected image of the complex curve in the Siegel moduli space of dimension 2 Abelian varieties has closure a Hilbert modular surface; this is a homogeneous algebraic surface, the quotient of a product of upper half planes $\mathcal{U} \times \mathcal{U}$ by an (indiscrete) image in $PSL_2(\mathcal{O}_d)$, where \mathcal{O}_d denotes the ring of integers in the real quadratic field tied to the Euclidean structure on X , and it is a clear sign of the importance of Teichmüller-Veech curves that this study has uncovered such rich structures within these familiar varieties of dimension 3 – it is well known that the complete varieties $\widehat{\mathcal{M}}_3$ and $\widehat{\mathcal{A}}_3$ are birationally equal to \mathbb{CP}^3 . Hubert and Lelièvre [?] have produced a highly refined analysis of the situation concerning square-tiled surfaces in \mathcal{M}_2 , including asymptotic estimates of the growth of high genus curves of this type.

Other recent work by Hubert and Schmidt [29] and by McMullen [47] provides interesting geometrically defined examples of T-discs in low genus ($g \leq 4$) with infinitely generated Fuchsian stabilisers.

In a different but related direction concerning families of algebraic curves, work of Cohen and Wolfart [9] showed that uniformising hyperbolic discs for a large class of algebraic curves of genus $g \geq 2$ admit *modular embeddings* into Shimura varieties. This means that there is an embedding into a Siegel variety \mathcal{A}_g parametrising polarised Abelian varieties of genus g , using constructions involving the hyperbolic plane representation of the curve. One of their construction methods is related to the special examples of Veech’s T-disc structures we examined where the quadratic differential is the square of a 1-form, and by this (completely independent) process it follows that, for instance, many finite index subgroups of $\Gamma(1)$ embed in the Siegel modular group $Sp(2g, \mathbb{Z})$ of the relevant genus as the symplectic monodromy group of the closed 1-form defining the branched Euclidean structure on the surface. An essentially equivalent construction of embedded T-discs in the Torelli-Siegel space of a Riemann surface was given much earlier by [39]; also compare with the similar result mentioned above in the discussion of [46].

At the same time there have been important applications of the dynamical study of the bundle $\Omega^1(V_g)$ of holomorphic 1-forms on the Teichmüller curve V_g and the various subspaces into which it divides under the action of $SL_2(\mathbb{R})$ on the \mathcal{F} -structure coordinates which each point of the fibre space $\Omega^1(V_g)$ represents. The (restricted) Teichmüller geodesic flow is determined by this action and specifying the partition of $2g - 2$ defined by the set of orders occurring in the divisor of zeros of the 1-form produces a connected component. Eskin and Okounkov [?] have computed the volumes of some of these spaces by linking them to the numbers of distinct branched coverings of tori of given degree. See also [37] for more precise geometric analysis of the dynamics of this restricted geodesic flow generated on the moduli space of Abelian differentials.

4.4 Completion of a T-disc with large stabiliser.

We continue with the assumptions of section 4.2: Γ is a group of finite index in a group H of signature type $\{p, q, \infty\}$. Because the stability subgroup obtained in Theorem 10 is itself a lattice group, it makes sense in the light of the discussion in section 3.8 to ask whether the modular image of the T-disc \mathbb{D} in \mathcal{M}_g has a natural completion. This follows naturally in fact: the process of adjoining a point at each cusp of the immersed surface occurs naturally in the projective embedding of the *moduli space* $\widehat{\mathcal{M}}_g$ of *stable genus g algebraic curves* constructed by Deligne and Mumford [12], which is a projective (*ergo* compact Kähler) complex V-manifold containing \mathcal{M}_g as a Zariski open subset. For background on this completion of the space of non-singular curves the reader may consult [40], [23],[24] and [72].

We refer again to the discussion in section 4.2 before Theorem 10 as prelude to the next result, first stated in this form in [25], although it involves mainly a reinterpretation of parts of Veech's work.

Theorem 11 *For each point $\{X\}$ of the moduli space \mathcal{M}_g represented by a compact surface X which is a clean Galois Belyi covering of $\mathbb{C}P^1$, there is a complete algebraic curve C in the stable compactification $\widehat{\mathcal{M}}_g$ passing through $\{X\}$ isomorphic to some quotient surface of X by an abelian subgroup of $\text{Aut}(X)$.*

PROOF. The existence of the affine T-curves was proved in Theorem 10. Examples such as the original Veech-Wiman hyperelliptic curves and Fermat curves were met earlier. We consider the natural completion of such a curve in the projectively embedded variety $\widehat{\mathcal{M}}_g$ of stable curves. The points of a completed curve C lying in the boundary divisor of $\widehat{\mathcal{M}}_g$ then correspond, not necessarily bijectively, to surfaces with nodes obtained by passing to the cusps of the stability subgroup, which produces a noded surface by collapsing the vertical trajectories of the form. The curve C is an abelian quotient because of the construction we used.

There is a co-compact triangle group with signature $\{2, m, n\}$, where n is the l.c.m. of the ramification degrees at the points lying over ∞ , and a corresponding finite index subgroup representing the same curve with cyclic symmetry which has identical ramification structure with this Belyi covering and determines a smooth surface with the same underlying conformal structure as X . Since this is a smooth surface, the periods will in fact satisfy the condition that m divides the l.c.m. of 2 and n while n divides $\text{lcm}\{2, m\}$. \square

4.5 Consequences; complete curves in modular varieties.

We add some brief comments on the last theorem.

Firstly, it is worth noting that the immersed curves C in moduli space are likely in general to have singularities such as self intersections, although these can always be removed by passage to a finite sheeted covering of $\widehat{\mathcal{M}}_g$. In fact it follows immediately from a topological property of the orbifold structure that

there is a finite Galois cover of the modular variety which is a compact complex (projective) manifold. (This is proved in [40].) However, the genus of the resulting lifted Teichmüller Veech curve is then much larger and may not be easy to estimate.

A second point concerns the general occurrence of surface groups as subgroups in Teichmüller modular groups.

The hyperbolic plane $\mathcal{U} = \tilde{X} \cong \mathbb{D}$ forms the fibre over the point $x = [X]$ of \mathcal{M}_g , with $X = \mathcal{U}/K$, say, for some Fuchsian group $K \cong \pi_1(X)$. This determines a different holomorphic embedding of a disc into the space $F_{g,k} \cong F(K) = T(K') \cong T_{g,k+1}$, and in the situation of section 4.2 the Fuchsian group K which is acting on \mathbb{D} , and which is naturally contained in the corresponding extended modular group $\text{mod}(K)$, survives the projection homomorphism to the group $\text{Mod}(K) \cong \text{Mod}_{g,k}$ for some $k > 0$. However, it was proved by Kra [38] (and independently by Nag) that such fibre discs in Teichmüller spaces cannot be Teichmüller-geodesic. Consequently these two constructions determine completely distinct ways to realise Fuchsian groups acting on the upper half plane as geometric subobjects of the Teichmüller modular transformation group actions $\text{Mod}_{g,k}$.

As a final point, we observe that the degree of any covering which may occur in Theorem 11 for a T-curve in a given moduli space is bounded above by a linear function of g , at most $4(g+1)$ since this is the maximum order of any abelian automorphism group for a complete curve of genus $g \geq 2$. This bound may not be sharp. It is also unclear whether other complete (not necessarily Teichmüller geodesic) curves exist in \mathcal{M}_g with larger automorphism group.

5 Towards a space of all \mathbf{Q} -curves: a universal modular family.

Among the candidate theories which point towards the desired goal of a universal space of $\mathbf{Q} = \overline{\mathbb{Q}}$ -rational curves, so far achieved only in $g = 2$, there are links with Teichmüller theory stemming from both the holomorphic and combinatorial sides of the picture. The relationship we pursue here connects a class of finite index subgroups of the modular group $PSL(2, \mathbb{Z})$ and a subset of the space of conjugacy classes of co-compact Fuchsian groups and there is evidence which suggests a deeper resonance with algebraic number theory. In particular there is a clear relationship with a modular action of $\text{Gal}(\overline{\mathbb{Q}})$ given in work of J.G. Thompson [65], which we summarise below.

5.1 A moduli space of Fuchsian groups.

The space \mathcal{D} of all lattice subgroups (defined to be discrete subgroups with finite volume quotient) of the Lie group $G = PSL(2, \mathbb{R})$ has a natural structure of metric space as defined by Chabauty [11]. Within this there is a subset \mathcal{D}_0 consisting of all lattice subgroups with co-compact quotient. According to a fundamental theorem of A. M. Macbeath [41] this set is open in \mathcal{D} . Furthermore

the geometric isomorphism type of the discrete groups in a connected component is fixed, and each component $\mathcal{D}(\Gamma)$ is, by standard results on Teichmüller-theory described earlier, a fibre space over the space of moduli for X_Γ the corresponding type of hyperbolic closed surface or 2-orbifold, with the fibre over the modular point (conformal class of surface) $x = [\Gamma]$ isomorphic to the unit tangent bundle over the quotient surface/orbifold \mathcal{U}/Γ ; the base is the moduli space $\mathcal{M}(\Gamma)$. This follows from results by Weil, by Macbeath [41], [43] and by Harvey (see [23], chapter 9).

A structure of stratified space in the sense of Whitney can also be put on \mathcal{D}_0 , which takes note of the various topological types of orbifold, and there is a natural way to form the relative closure of \mathcal{D}_0 in the Chabauty space of G ; see again for instance [23] chapter 9, where it is shown that the topological boundary structure of the Deligne-Mumford boundary can be replicated in this setting.

Later in this chapter we shall formulate a theorem that makes the subset of conjugacy classes of finite index subgroups of (p, q, ∞) -triangle groups into both a set of $\overline{\mathbb{Q}}$ -rational base points for the moduli spaces of compact surfaces representable as some finite sheeted covering of the sphere - this is just a paraphrase of Belyi's Theorem - and also a data-bank for T-disc models of all the corresponding algebraic curves defined by these coverings within the space of G -lattices.

Thus for the case of closed surface moduli spaces, each \mathcal{M}_g will contain (after passing to a suitable finite sheeted covering $\widetilde{\mathcal{M}}_g$) complex affine models of all curves of genus g which carry a critically framed J-S form; these occur as subvarieties defined within a specific model of the moduli space, presumably with the standard \mathbb{Q} -structure that it has.

5.2 A representation of the absolute Galois group.

Let $\Gamma(1) = PSL(2, \mathbb{Z})$, the classical modular group, act as usual on the upper half plane by Möbius transformations and denote by \mathcal{U}^* the completion by the *cusps* of $\Gamma(1)$, $\mathcal{U}^* = \mathcal{U} \cup \mathbb{Q} \cup \infty$ and $X^*(1) = \mathcal{U}^*/\Gamma(1)$ the corresponding compact quotient surface. This is the unique compact Riemann surface which contains the open surface $X(1) \cong \mathbb{C}$, so it is of course the Riemann sphere, but provided with a wealth of supplementary information of an arithmetic nature, stemming from the isomorphism with the moduli space of marked complex tori, elliptic curves with their special representations coming from the classic theory of elliptic modular functions. There are many sources which the reader may consult on this fascinating topic, for instance [35] and [62].

Following Thompson ([65]), we construct an exact sequence of Galois groups associated with a certain space of meromorphic modular functions on this rational bordification, which represents from one aspect an action of the *absolute Galois group* $\text{Gal}(\overline{\mathbb{Q}})$ on Belyi coverings of the sphere; this is related to work of Ihara and others on braid and modular group representations of the absolute Galois group. The field of functions to be deployed forms a part of the regular representation of the modular group $\Gamma(1)$ on the space $\mathcal{H}(\mathcal{U})$ of functions holo-

morphic on the upper half plane with convergent Laurent-Puiseux expansions at each cusp which have the additional property that they are algebraic in the standard local parameter. For instance, near ∞ we require an expansion in the variable $q^{1/N} = \exp(2\pi i\tau/N)$ for some integer $N > 0$.

Let \mathbf{F}_0 denote the field of all functions meromorphic in $\widehat{\mathcal{U}}$ which determine functions on some finite sheeted cover of the modular surface $X(1)^*$. Clearly \mathbf{F}_0 contains the subfield $\mathbb{C}(j)$, where $j(\tau)$ denotes the classical modular invariant, and it is well known (see for instance [60],[62]) that when the group $\Gamma(1)$ acts on \mathbf{F}_0 by $\sigma(\gamma, f) = f \circ \gamma$, the fixed field is precisely $\mathbb{C}(j)$.

In fact, it is known by general results on modular functions that for any subgroup Γ of finite index in $\Gamma(1)$ there is a (meromorphic) function invariant under precisely Γ ; standard methods of Galois theory then imply that $\text{Gal}(\mathbf{F}_0/\mathbb{C}(j))$ is $\widehat{\Gamma(1)}$, the profinite completion of the modular group.

Now consider the following extension field of $\mathbb{C}(j)$ introduced in [65].

DEFINITION. The *field of algebraic modular functions* \mathbf{K} is the relative algebraic closure of $\mathbb{C}(j)$ in \mathbf{F}_0 .

We write \mathbf{G} for the Galois group of this field over $\mathbb{C}(j)$. Then there is a short exact sequence connecting Galois groups for the field extension.

Theorem 12 *The subgroup fixing the intermediate field $\overline{\mathbb{C}(j)}$ is isomorphic to $\widehat{\Gamma(1)}$ and there is a split exact sequence*

$$1 \rightarrow \widehat{\Gamma(1)} \longrightarrow \mathbf{G} \longrightarrow \text{Gal}(\overline{\mathbb{C}(j)}/\mathbb{C}(j)) \rightarrow 1. \quad (5)$$

Here a constant field Galois automorphism $\sigma \in \text{Aut } \overline{\mathbb{C}(j)}$ acts on the field of algebraic modular functions via the action on coefficients of the polynomial ring $\overline{\mathbb{C}(j)}[U]$: any $f \in \mathbf{K}$ is a root of some polynomial $P(j, U)$ and, by an observation of Thompson (see [65]), the image P^σ also has a root in \mathbf{K} .

The extension is split by the section $s : \text{Gal}(\overline{\mathbb{C}(j)}/\mathbb{C}(j)) \rightarrow \mathbf{G}$ defined in the following way:-

let $f \in \mathbf{K}$ with $f = \sum_{n \geq M} a_n q^{n/N}$ near ∞ and let $\sigma \in \text{Gal}(\overline{\mathbb{C}(j)}/\mathbb{C}(j))$. Then

$$f \mapsto s^\sigma(f) = \sum_n a_n^\sigma q^{n/N}. \quad (6)$$

This establishes a base for a common action of the absolute Galois group and mapping class groups of all topological types on certain subspaces of the graded ring of modular forms. For the case of congruence subgroups, it is then possible to establish an arithmetic model for the function fields of all levels (this is done for instance in [62]) and generators for the ring of modular forms with (algebraic) integral q -coefficients at ∞ . In this sense, at least, the action of $\overline{\mathbb{C}(j)}$ on (Galois) dessins can be subsumed into this setting of a field of functions on \mathcal{U} .

For noncongruence subgroups, none of this arithmetic approach applies directly due to the lack of general methods such as a satisfactory theory of Hecke operators, but over a complex coefficient field one can define as usual the correct (integer weight k) action of $\gamma \in SL_2(\mathbb{R})$ on $\mathcal{H}(\mathcal{U})$ as:

$$f \mapsto f[\gamma]_k(z) := f(\gamma(z)) \cdot (cz + d)^{-k}, \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \quad (7)$$

The space $\mathbf{A}_4 = \bigcup_{\Gamma \leq \Gamma(1)} A_4(\Gamma)$ of all modular forms of weight 4 with poles of order at most 1 at cusps of some finite index subgroup Γ and having algebraic Fourier coefficients may then represent an appropriate arithmetic core within the cotangent bundles to moduli spaces of curves: the linear action of $\Gamma(1)$ on spaces $A_{2k}(\Gamma)$ of forms of weight $2k$, $k \geq 1$ for a subgroup Γ intertwines with that of the absolute Galois group on q -expansion coefficients. This territory has not been much explored for general (i.e. non-congruence) subgroups of $\Gamma(1)$, but the reader may consult several articles [5],[58] where this approach is applied to determining fields of definition and other arithmetic problems for non-congruence modular subgroups.

5.3 Modular forms with closed trajectories.

We return to the bridge between the class of Jenkins-Strebel modular forms and the curves in moduli space discussed in chapter 4.

Beginning with a finite index subgroup Γ of the modular group $\Gamma(1)$, let $f \in A_4(\Gamma)$ be a holomorphic automorphic form of weight 4, non-vanishing in \mathcal{U} , with zeros at some (non empty) finite subset of the cusps of Γ which includes ∞ .

In all the cases discussed in section 3, where Γ was a finite index subgroup of some Hecke triangle group, it was possible to use the structure of the covering, or equivalently a fundamental domain for the group, triangulated in a canonical way, to produce J-S forms. We restrict attention now to the classical modular group where a more direct approach is possible.

We make two observations which will be preparation for our final result. First, given any compact surface X furnished with a Belyi covering f of the sphere, there is a subgroup Γ of finite index in the modular group $\Gamma(1)$ such that the covering f is represented away from $f^{-1}(\infty)$ by the induced projection map of X_Γ onto $X(1)$, and with dessin the inverse image of the standard modular triangle decomposition of the sphere (with barycentric triangulation). From the many alternative combinatorial descriptions of the same ramified covering pattern which come from the various refinements of this polygonal decomposition of X , we choose a representation which involves subgroups of the cartographic subgroup $\Gamma_0(2)$ so as to coordinate with the joint action $\rho : \mathbf{G} \rightarrow \text{Aut}(\mathbf{F})$ on modular forms outlined in the exact sequence (5).

If the pattern of cells is a geometrical subdivision of a regular polygonal tessellation with an even number $2n$ of sides for each polygon, then there is a dessin with \bullet vertices at the corners, \star vertices at midpoints of edges and

with a \circ vertex at the centre of each cell and a unique form on the surface with zeros only at the points marked \circ and with compact critical graph. This dessin is obtained from a multivalued holomorphic form on the complex line $X_0^* = \mathcal{U}/\Gamma_0(2) \cup \infty \cong \mathbb{P}^1$ which vanishes at ∞ ; it arises by use of Singerman's theorem on universal tessellations ([63]) which gives an isomorphism between this covering and the canonical covering of \mathbb{P}^1 that realises the tessellation by regular $2n$ -gons by a specific finite index subgroup of the $\{2, 2n, \infty\}$ -Hecke group. If the cell pattern chosen is irregular at the vertices, then we need to make a preliminary extra choice of covering surface which does have such a regular structure; for instance we could take the intersection $\Gamma \cap \Gamma_0(2n)$, a normal subgroup of that congruence group.

The kind of \mathcal{F} -structure that we saw in sections 2 and 3 arose from the prescription of a quadratic differential with *two* interacting sets of closed trajectories and in addition a finite order mapping class, represented as a rotation of the coordinates for the \mathcal{F} -structure. In the Thurston-Veech square-tiled examples, there is a $\pi/2$ rotational switch from vertical to horizontal which comes in when viewed in the modular group framework, for the reason that we need to use the paving by Euclidean squares, whereas in the case of the patterns of regular $2n$ -gons one obtains a quadratic differential for a Hecke group with closed trajectories in directions making angle π/n . In that case, the rotation acts as a translation at the cusp ∞ by a rational fraction, which induces a multiplication by some $2n$ -th root of unity in the algebraic action ρ on q -expansions.

In this way we see that the appropriate class of modular form for our purposes should have *closed trajectories in at least two independent directions related by a root of unity*.

DEFINITION. An automorphic form φ of weight 4 for Γ is called *critically framed* if every non-critical trajectory is closed for the \mathcal{F} -structures obtained in two independent rationally related directions for φ on $X = \mathcal{U}/\Gamma$.

It seems quite likely that the study of these purely geometric objects should have intrinsic arithmetic interest, as they form natural ways to parametrise curves defined over number fields analogous to the intensively studied modular elliptic curves; the model curves in this setting are, however, usually noncongruence quotients.

5.4 Arithmetically defined curves in moduli space.

We can now formulate a final result, valid for any complete Belyi curve X of genus $g \geq 2$.

Theorem 13 *Given any pair X, β with X a genus g curve and $\beta : X \rightarrow X(1)$ a Belyi map, there is a corresponding Veech curve in \mathcal{M}_g , an affine algebraic curve over $\overline{\mathbb{Q}}$ arising from a critically framed modular form φ in $A_4^1(X)$ which determines a T -disc modular parametrisation e_φ centred at X .*

The proof follows directly from the definitions and preliminary discussion of the preceding section.

The set of all such triples X, β, ω forms a subset of a fibre space over the modular tower $\widehat{\mathcal{M}} = \bigcup_{g \geq 2} \mathcal{M}_g$ on which the absolute Galois group acts in the manner of (7) on Fourier q -coefficients of weight 4 modular forms. One expects that some curves will have many such immersed models in their moduli space, just as the hyperelliptic curves $y^2 + x^n = 1$ have at least two, possibly more, distinct T-disc parametrisations.

Having seen how the complex geometry of moduli spaces is linked to dessins d'enfant by way of the collection of modular Veech curves, we close by indicating another approach to the relationship between moduli, combinatorial patterns on closed surfaces and hyperbolic geometry. In a paper of Macbeath ([42]) originating from the same period as the ideas of Grothendieck and Thurston, it was shown that the set of points of \mathcal{M}_g which are represented by hyperbolic structures with Dirichlet fundamental region having an inscribed circle that is tangent to all sides, corresponds precisely to the multitude of genus g permutation representations of the modular group $\Gamma(1)$; further evidence, perhaps, of how important mathematical ideas have a life of their own.

References

- [1] L.V. Ahlfors, *Lectures on Quasiconformal Mappings*, Wadsworth and Brooks/Cole, Monterey, California, 1987.
- [2] L.V. Ahlfors, H.Behnke *et al.* *Analytic Functions*. (Proceedings of the 1957 Conference.) Princeton University Press, 1960.
- [3] L.V. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, *Annals of Math.*, **72**: 345–404, 1961.
- [4] W. Abikoff, *The Real Analytic Theory of Teichmüller Space*. Lecture Notes in Math., vol. **820**, Springer-Verlag, Berlin and New York, 1980.
- [5] A.O.L. Atkin & H.P.F.Swynnerton-Dyer, *Modular forms on non-congruence subgroups* Proc. Symp. Pure Math. XIX, Amer. Math. Soc. (1971), 1-25.
- [6] G.V. Belyi, *Galois extensions of a maximal cyclotomic field*, *Izv. Ak. N. USSR*. **43** (1979), 269-276.
- [7] L. Bers, *Quasiconformal mappings and Teichmüller's theorem*, in *Analytic Functions* (R. Nevanlinna et al., eds) Princeton U. Press (1960) 89-119.
- [8] ———, *Fiber spaces over Teichmüller spaces*, *Acta Math.* **130** (1973), 89–126.
- [9] P. B. Cohen and J. Wolfart, *Dessins de Grothendieck et variétés de Shimura*, *C. R. Acad. Sci. Paris* **315** (I) (1992), 1025-1028.

- [10] ———, *Modular embeddings for some nonarithmetic Fuchsian groups*, Acta Arithm. **56** (1991), 93-110.
- [11] C. Chabauty, *Limites d'ensembles et géométrie des nombres*, Bull. Soc. Math. France **78** (1950), 143-150.
- [12] P. Deligne and D. Mumford, *Irreducibility of the space of curves of given genus*, Publ.Math. I.H.E.S. **36** (1969), 75 - 110.
- [13] A.Douady and J.H. Hubbard, *Density of Strebel differentials*, Invent. Math. **30** (1975), 175-179.
- [14] C.J. Earle and F.P. Gardiner, *Teichmüller disks and Veech's \mathcal{F} -structures*, Contemp. Math. **201** (1997), 165-189.
- [15] A. Eskin and A. Okounkov, *Asymptotics of numbers of branched coverings of tori and volumes of moduli spaces...* Invent. Math. **145** (2001), 59 - 103.
- [16] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math. **136** (1992), 447-510.
- [17] F.P. Gardiner, *Teichmüller Theory and Quadratic Differentials* John Wiley & Sons, New York. 1987.
- [18] F.P. Gardiner& N. Lakic, *Quasiconformal Teichmüller Theory*. AMS (Providence, RI, 2001).
- [19] G. González-Diez, *Around Belyi's Theorem*, Ann. Scuo. Norm. Pisa (to appear, 2006).
- [20] G. González-Diez and W. J. Harvey, *On complete curves in moduli space I* Math. Proc. Camb. Phil. Soc. **110** (1991), 461-466.
- [21] ———, *On families of algebraic curves with automorphisms*, Contemp. Math. **311** (Amer. Math. Soc. , Providence RI 2002), 261-275.
- [22] P. Griffiths. *Complex analytic properties of certain Zariski open sets in algebraic varieties*, Ann. of Math.**94** (1971),21-51.
- [23] A. Grothendieck, *Esquisse d'un Programme*. Unpublished manuscript, circa 1982. Reprinted in LMS Lecture Notes vol. **242** (eds. Schneps-Lochak).
- [24] W. J. Harvey (ed.), *Discrete Groups and Automorphic Functions*, Academic Press, London (1977).
- [25] ———, *Modular groups and representation spaces*, Contemporary Math. **74** (1988), 205-214.
- [26] ———, *On certain families of compact Riemann surfaces*, in "Mapping Class Groups and Moduli Spaces" Contemp. Math.**150** Amer. Math. Soc., Providence, RI (1993), 137 - 148.

- [27] ———, *Drawings, Triangle groups and Algebraic Curves*, KCL Preprint MTH-97-12, pp. 1 - 24 (Feb. 1997).
- [28] W.J. Harvey & C. Maclachlan, *On mapping class groups and Teichmüller spaces*, Proc. London Math. Soc. **30** (1975), 495-512.
- [29] E. Hecke, *Über die Bestimmung Dirichlet'sche Reihen durch Functionalgleichungen*, no. 33 in Mathematische Werke, Vandenhoeck & Ruprecht, Göttingen, 1959.
- [30] P. Hubert & Th. A. Schmidt, *Veech groups and polygonal coverings*, J. Geom. Physics **35**:1 (2000), 75-91.
- [31] P. Hubert & S. Lelièvre, *Prime arithmetic Teichmüller discs in $\mathcal{H}(2)$* Israel J. Math (to appear, 2006). (Preprint: arXiv:math.GT/0401056 v2 October 2004)
- [32] J.A. Jenkins, *Univalent Functions and Conformal Mapping*. Springer-Verlag, Berlin and New York, 1965.
- [33] G. Jones and D. Singerman, *Belyi functions, hypermaps and Galois groups* Bull. Lond. Math.Soc. **28** (1996) 561-590.
- [34] S.P. Kerckhoff, *Asymptotic geometry of Teichmüller space*, Topology **19** (1980), 23–41.
- [35] ———, *The Nielsen realisation problem*, Ann. of Math. **117** (1983), 235–265.
- [36] S.P. Kerckhoff, H. Masur & J. Smillie, *Ergodicity of billiard flows and quadratic differentials*, Ann. of Math. **124** (1986), 293–311.
- [37] A.W. Knap, *Elliptic Curves*. Princeton Math. Notes **40** Princeton U. Press, 1992.
- [38] K. Kodaira, *A certain type of irregular algebraic surfaces* J. Analyse Math. **19** (1967), 207 - 215.
- [39] M. Kontsevich & A. Zorich, *Connected components of the moduli space of holomorphic differentials ...*, Invent. Math. **153** (2003), 631-678.
- [40] I. Kra, *On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces* Acta Math **146** (1981) 231–270.
- [41] ———, *The Caratheodory metric on Abelian Teichmüller disks*, J. d'Analyse Math. **40** (1981), 129-143.
- [42] E. Looijenga, *Smooth Deligne-Mumford compactifications by means of Prym level structures*, J. Alg. Geometry **3** (1994), 283-293.
- [43] A.M. Macbeath, *Groups of homeomorphisms of a simply connected space*, Ann. of Math. **79** (1964), 473-488.

- [44] ———, *Generic Dirichlet polygons and the modular group*, Glasgow Math. J. **27** (1985), 129-141.
- [45] A.M.Macbeath & D.Singerman, *Spaces of subgroups and Teichmüller theory*, Proc. London Math. Soc. **31** (1975), 211 - 256.
- [46] H. Masur, *Dense geodesics in moduli space*, in ‘Riemann Surfaces and Related Topics’, Ann. of Math. Studies, vol. **97**, (Princeton Univ. Press, 1981), 417–438.
- [47] ———, *Closed trajectories for quadratic differentials with an application to billiards* Duke Math. J., **53** (No. 2) (1986), 307–314.
- [48] C. T. McMullen, *Billiards and Teichmüller curves on Hilbert modular surfaces*, J. Amer. Math. Soc. **16** (2003), 857-885.
- [49] ———, *Teichmüller curves of infinite complexity*, Acta Math. **191**(2003), 191-223.
- [50] ——— *Dynamics of $SL_2(\mathbb{R})$ over moduli space in genus 2*, Preprint (2004).
- [51] ——— *Prym varieties and Teichmüller curves*, Preprint (2005).
- [52] J.W. Milnor, *On the 3-dimensional Brieskorn manifolds $M(p,q,r)$* , in ‘Knots, Groups & 3-Manifolds’, Ann. of Math. Studies, No. **84** (Princeton Univ. Press 1975), 175-225.
- [53] S. Morita, *Geometry of Characteristic Classes*, Transl. of Math. Monographs vol. 199, (Amer. Math. Soc., Providence, RI, 1999).
- [54] S. Nag, *Complex Analytic Theory of Teichmüller Spaces*, John Wiley and Sons, New York, 1988.
- [55] S. Nag & D. Sullivan, *Teichmüller theory and the universal period mapping ...*, Osaka Math J. **32** (1995), 1- 34.
- [56] Bernhard Riemann, *Gesammelte Werke*, (ed. H. Weber), Dover reprint, 1953.
- [57] H.L. Royden, *Automorphisms and isometries of Teichmüller space*, Ann. of Math. Studies vol.**66** (1971), 369–384.
- [58] L. Schneps (ed), *The Grothendieck Theory of Dessins d’enfant*, LMS Lecture Notes, vol. **200**, Cambridge Univ. Press (1994).
- [59] L. Schneps & P. Lochak (eds), *Geometric Galois Actions*, LMS Lecture Notes, vol. **242** & **243**, Cambridge Univ. Press (1997).
- [60] A.J. Scholl *Modular forms on noncongruence subgroups* Sémin. de Th. des Nombres (Paris 1985-86) Progress in Math. vol. 71 Birkhäuser (1987), 199 -206.

- [61] Th. Schmidt *Klein's cubic surface and a non-arithmetic curve* Math. Annalen **309** (1997) 533 - 539.
- [62] J-P. Serre, *Cours d'Arithmétique*, Presses Univ. de France, (2-ème éd.) 1977. [English translation: *A Course in Arithmetic*, Springer Universitext.]
- [63] G. B. Shabat and V. A. Voevodsky *Drawing curves over number fields*, in 'The Grothendieck Festschrift', Progress in Math., vol.**87** (Birkhäuser, 1991), 199–227.
- [64] G. Shimura. *Introduction to Arithmetic Theory of Automorphic Functions*, Princeton Univ. Press & Iwanami-Shoten (1971).
- [65] D. Singerman, *Universal tessellations*, Revista Matematica (Univ. Complutense de Madrid 1988) **1** (1988) 111–126.
- [66] K. Strebel. *Quadratic Differentials*, *Ergebnisse der Math.*, Springer Verlag (Berlin-Heidelberg), 1984.
- [67] J.G. Thompson, *Galois groups*, in 'Groups at St. Andrews 1989', London Math. Soc. Lecture Notes vol. **160** (1991) Cambridge Univ. Press (Cambridge, UK), 455-462.
- [68] W.P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. AMS, vol. 19 (1988), 417-431.
- [69] W.A. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards* Invent. Math. **97** (1989), 553–583. (Corrig. Ibid. (1991))
- [70] ———, *Moduli spaces of quadratic differentials* J. d'Analyse Math. **55** (1991), 117-171.
- [71] ———, *The billiard in a regular polygon*, Geom. & Funct. Anal. **2** (1992), 341-379.
- [72] A. Weil, *The field of definition of a variety*, Amer. J. Math. **79** (1956), 509-524.
- [73] J. Wolfart, *The 'obvious' part of Belyi's theorem*, in LMS Lecture Notes **242**, Schneps-Lochak, eds.
- [74] S. A. Wolpert, *Homology of the moduli space of stable curves*, Ann. of Math. **118** (1983), pp. 491 -523.