ETA FORMS AND THE CHERN CHARACTER

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ABSTRACT: The semi-topological nature of the eta-invariant of a self-adjoint elliptic differential operator derives from a relative identification with a Chern character. This remarkable semi-locality property of the eta-invariant can be seen in spectral flow formulae and many other applications [APS2, APS3, BC1, DZ1, L1]. In this paper we prove two geometric index theorems for a family of first-order elliptic operators over a manifold with boundary by computing eta form representatives for the Chern character classes of the index bundle. The eta forms occur as relative and regularized traces on infinite-dimensional vector bundles realized as the limiting values of superconnection character forms. The formulas are non-local and general, they do not require spin structures, compatibility with Clifford actions, or dimensional restrictions.

1. Introduction

Let $M \longrightarrow B$ be a smooth Riemannian fibration with fibre diffeomorphic to a closed even-dimensional spin manifold X and let $\mathbb{E} = \mathbb{E}^+ \oplus \mathbb{E}^-$ be a graded vector bundle of Clifford modules over M. Let $\mathsf{D} = \{D_z \mid z \in B\}$ be a family of compatible Dirac operators acting on the space of C^∞ sections $\Gamma(M, \mathbb{E})$. The kernels $\mathrm{Ker}(D_z) = \mathrm{Ker}(D_z^+) \oplus \mathrm{Ker}(D_z^-)$ are $\mathbb{Z}_2$-graded by the kernels of the chiral Dirac operators D_z^{+}, and if assumed to vary smoothly with z form a finite rank superbundle $\mathrm{Ker}(\mathsf{D}) = \mathrm{Ker}(\mathsf{D}^+) \oplus \mathrm{Ker}(\mathsf{D}^-)$ on B. The eta-form $\eta \in \mathcal{A}^{\mathrm{odd}}(B) = \sum_i \Gamma(B, \wedge^{2i+1}T^*B)$ of the Bismut superconnection $\mathbb{A}_t$ is the odd-degree differential form introduced by Bismut [B], Bismut-Cheeger [BC1], Berline-Verne [BV]

$$\eta = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Str}(\dot{\mathbb{A}}_t e^{-\mathbb{A}_t^2}) \; dt$$

as a canonical transgression form for the local families index formula [B, BGV]

$$\mathrm{ch}(\mathbb{A}^{\mathrm{Ker}(\mathsf{D})}) = \int_{M/B} \widehat{A}(M/B)\mathrm{ch}(\mathbb{E}) - d\eta \; . \tag{1.1}$$

Here $\mathrm{ch}(\mathbb{A}^{\mathrm{Ker}(\mathsf{D})}) = \mathrm{Str}(e^{-(\mathbb{A}^{\mathrm{Ker}(\mathsf{D})})^2}) \in \mathcal{A}(B)$ is a Chern character form for the index bundle

$$\mathrm{Ind}\,(\mathsf{D}) = [\mathrm{Ker}(\mathsf{D}^+)] - [\mathrm{Ker}(\mathsf{D}^-)] \in K(B) \; ,$$

and hence (1.1) implies the cohomological Atiyah-Singer families index theorem in $\mathrm{H}^\bullet(B)$

$$\mathrm{ch}(\mathrm{Ind}\,(\mathsf{D})) = \int_{M/B} \widehat{A}(M/B)\mathrm{ch}(\mathbb{E}) \; . \tag{1.2}$$

Thus, though not local, η defines via (1.1) a secondary topological invariant in the form of a generalized Chern-Simons form for superconnections.

Extensions to compatible Dirac operators associated to a fibration of manifolds with boundary $\pi : M \xrightarrow{X} B$ were achieved by Bismut-Cheeger [BC1] and Melrose-Piazza [MP]. For D to have a well-defined index non-local boundary conditions must be imposed. The induced boundary fibration $\pi^N : N = \partial M \longrightarrow B$ of closed Riemannian spin manifolds with fibre $Y_z = \partial X_z$ defines a family of self-adjoint compatible Dirac operators $\partial = \{\partial^z \mid z \in B\}$ acting on the infinite-dimensional bundle $\pi^N_*(\mathbb{E}^0)$ over B whose fibre at $z \in B$ is the space of sections over Y_z. ∂ occurs in the degree zero component of the induced Bismut superconnection $\mathbb{B}_t = (\mathbb{A}_t)_{|N}$ on the boundary. For simplicity we assume the operators ∂^z are invertible. Let $\Pi_> = \{\Pi^z_> \mid z \in B\}$ be the smooth family of boundary projections onto the direct sum of the positive eigenspaces of ∂^z. Then by restricting the domain of D to those sections whose boundary values lie in the kernel of $\Pi_>$ we obtain a smooth family of Fredholm operators $\mathsf{D}_{\Pi_>}$ for which one can aim to construct analogues of (1.1), (1.2). The index form $\int_{M/B} \widehat{A}(M/B)\mathrm{ch}(\mathbb{E})$ is no longer closed, rather a correcting non-local boundary Eta form term is required

$$\widehat{\eta}_{\Pi_\geq} = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}_{\,\mathrm{even}}(\dot{\mathbb{B}}_t e^{-\mathbb{B}_t^2})\; dt \;, \tag{1.3}$$

where $\mathrm{Tr}_{\,\mathrm{even}}$ is the trace over forms of even degree. The cohomological APS families index theorem states that in $H^\bullet(B)$

$$\mathrm{ch}(\mathrm{Ind}\,(\mathsf{D}_{\Pi_>})) = \int_{M/B} \widehat{A}(M/B)\mathrm{ch}(\mathbb{E}) - \frac{\widehat{\eta}_{\Pi_>}}{2} \;. \tag{1.4}$$

This was proved by Bismut-Cheeger [BC2] by an adiabatic limit argument identifying the index bundle with that for a family of Dirac operators over a closed manifold with singular metric formed by attaching a cone to X. A conceptually simpler proof and generalization to spectral sections was subsequently given by Melrose-Piazza [MP] using the b-calculus — a spectral section is a smooth family of pseudodifferential operator (ψdo) boundary projections $\mathcal{P} = \{P_z \mid z \in B\}$ differing from $\Pi_>$ by a family of finite-rank operators. Melrose-Piazza based their proof on the observation that $\mathcal{P}$ occurs as the spectral projection $\Pi^{\partial_1}_>$ of a family ∂_1 of boundary ψdos differing from ∂ by a family of finite-rank ψdos of order 0, and thus generalized (1.4) to the case where $\mathsf{D}_{\mathcal{P}}$ replaces $\mathsf{D}_{\Pi_>}$ and $\widehat{\eta}_{\Pi_>}$ is replaced by an analogously defined eta form $\widehat{\eta}_{\mathcal{P}}$.

The corresponding transgression formula for (1.4) for any spectral section $\mathcal{P}$ implies that as elements of $\mathcal{A}(B)$

$$d\widehat{\eta}_{\mathcal{P}} = 2\int_{\partial M/B} \widehat{A}(\partial M/B)\mathrm{ch}(\mathbb{E}) \;.$$

The relative eta form $\widehat{\eta}_{\mathcal{P}_1} - \widehat{\eta}_{\mathcal{P}_2}$ is thus closed, and defines a cohomology class in $H^\bullet(B)$ equal to the relative Chern character

$$\mathrm{ch}(\mathrm{Ind}\,(\mathsf{D}_{\mathcal{P}_1})) - \mathrm{ch}(\mathrm{Ind}\,(\mathsf{D}_{\mathcal{P}_2})) = \mathrm{ch}(\mathrm{Ind}\,(\mathcal{P}_2, \mathcal{P}_1)) \;, \tag{1.5}$$

where $(\mathcal{P}_2, \mathcal{P}_1)$ is a canonical family of boundary Fredholm operators defined by the spectral sections. The equality in (1.5) follows from K-theory arguments [DZ1, MP, S4].

1.1. **Theorem (I): Relative Boundary Eta forms.** To explain our constructions we begin with the case where B is just a single point. Then (1.4) reduces to the classical APS formula for the index of the single operator $D_{\Pi^{\partial}_{>}}$

$$\text{ind}\,(D_{\Pi^{\partial}_{>}}) = \int_X \widehat{A}(X)\text{ch}(E) + \frac{\widehat{\eta}(\partial)}{2}\ . \tag{1.6}$$

Now $\mathbb{B} = t^{1/2}\partial$ and (1.3) becomes the usual single operator eta-invariant

$$\begin{aligned} \widehat{\eta}(\partial) &= \frac{1}{2\sqrt{\pi}}\int_0^\infty t^{-1/2}\text{Tr}\,(\partial e^{-t\partial^2})\ dt \\ &= \text{Tr}\,(\partial|\partial|^{-s-1})|^{\text{mer}}_{s=0}\ , \end{aligned} \tag{1.7}$$

the superscript indicating the meromorphically continued trace evaluated at $s = 0$. We may rewrite (1.7) as the regularized trace

$$\widehat{\eta}(\partial) = \text{Tr}\,((\Pi^{\partial}_{>} - \Pi^{\partial}_{<})|\partial|^{-s})|^{\text{mer}}_{s=0} \tag{1.8}$$

of the involution $\Pi^{\partial}_{>} - \Pi^{\partial}_{<}$ defined by the order 0 ψdo projections

$$\Pi^{\partial}_{>} = \frac{1}{2}(I + \partial|\partial|^{-1})\ , \qquad \Pi^{\partial}_{<} = \frac{1}{2}(I - \partial|\partial|^{-1}) = (\Pi^{\partial}_{>})^{\perp}$$

onto the positive and negative spectral subspaces of ∂.

Consider the infinite Grassmannian $\text{Gr}_\infty(\partial)$ parameterizing ψdo projections P such that $P - \Pi_{>}$ is an element of the algebra $\Psi^{-\infty}(E^0)$ of smoothing operators on $\Gamma(Y, E^0)$. If $P_1, P_2 \in \text{Gr}_\infty(\partial)$, then $P_1 - P_2 \in \Psi^{-\infty}(E^0)$ is represented by a smooth kernel $k(y_1, y_2)$ with trace $\text{Tr}\,(P_1 - P_2) = \int_Y \text{Tr}\,(k(y,y))\ dy$, and so the relative variant of (1.8) exists without regularization

$$\widehat{\eta}(P_1, P_2) = \text{Tr}\,\Big((P_1 - P_1^{\perp}) - (P_2 - P_2^{\perp})\Big)\ , \tag{1.9}$$

where $P_i^{\perp} = I - P_i$. For spectral sections $\Pi^{\partial_1}_{>}, \Pi^{\partial_2}_{>}$ the additivity formula holds

$$\widehat{\eta}(\Pi^{\partial_1}_{>}, \Pi^{\partial_2}_{>}) = \widehat{\eta}(\partial_1) - \widehat{\eta}(\partial_2)\ . \tag{1.10}$$

Rewriting (1.9) as

$$\frac{\widehat{\eta}(P_1, P_2)}{2} = \text{Tr}\,(P_1 - P_2) = \text{ind}\,(P_1, P_2)\ , \tag{1.11}$$

where $(P_1, P_2) := P_2 \circ P_1 : W_1 \to W_2$ and $W_i = \text{ran}\,(P_i)$, this is equivalent to the relative index formula implied by (1.6)

$$\frac{\widehat{\eta}(P_2, P_1)}{2} = \text{ind}\,(D_{P_1}) - \text{ind}\,(D_{P_2})\ ,$$

which coincides with (1.5) for the case $B = pt$. Notice that, unlike $\widehat{\eta}(\partial_i)$, the relative eta-invariant is a homotopy invariant.

To generalize this to families of boundary problems the ψdo projection P_i is replaced by a spectral section — or, more generally, by a Grassmann section $\mathcal{P}_i = \{P_{i,z} \mid z \in B\}$, defined to be a smooth section of the fibration with fibre $\text{Gr}_\infty(\partial^z)$ at $z \in B$. $\mathcal{P}_i$ defines a smooth infinite-dimensional subbundle $\mathcal{W}_i$ of $\pi_*^N(\mathbb{E}^0)$ with a canonical connection with curvature $(\mathcal{P}_i \cdot \nabla^{\pi_*^N(\mathbb{E}^0)} \cdot \mathcal{P}_i)^2 \in \mathcal{A}^2(B, \text{End}(\mathcal{W}_i))$, which by extending by zero we can consider as a 2-form endomorphism of $\pi_*^N(\mathbb{E}^0)$, where $\nabla^{\pi_*^N(\mathbb{E}^0)}$ is the Bismut connection on $\pi_*^N(\mathbb{E}^0)$. Generalizing (1.11) we define the relative Eta-form by

$$\eta(\mathcal{P}_1,\mathcal{P}_2) = \mathrm{Tr}\left[e^{-(\mathcal{P}_1\cdot\nabla^{\pi_*^N(\mathbb{E}^0)}\cdot\mathcal{P}_1)^2} - e^{-(\mathcal{P}_2\cdot\nabla^{\pi_*^N(\mathbb{E}^0)}\cdot\mathcal{P}_2)^2}\right] \in \mathcal{A}^{2\bullet}(B)\ .$$

Here the individual exponential operators $e^{-(\mathcal{P}_i\cdot\nabla^{\pi_*^N(\mathbb{E}^0)}\cdot\mathcal{P}_i)^2}$ are families of unbounded ψdos of positive order and so not of trace class. The relative exponential operator, however, is a smooth family of smoothing operators with coefficients in $\mathcal{A}^{2\bullet}(B)$ and so has a well-defined fibrewise trace.

Theorem (I) *Let D be a smooth family of first-order elliptic operators. In $H^{2\bullet}(B)$*

$$\mathrm{ch}(\mathrm{Ind}\,(\mathsf{D}_{\mathcal{P}})) = \eta(\mathcal{P}(\mathsf{D}),\mathcal{P})\ . \tag{1.12}$$

Equivalently,

$$\mathrm{ch}(\mathrm{Ind}\,(\mathsf{D}_{\mathcal{P}_1})) - \mathrm{ch}(\mathrm{Ind}\,(\mathsf{D}_{\mathcal{P}_2})) = \eta(\mathcal{P}_2,\mathcal{P}_1)\ . \tag{1.13}$$

One has in $\mathcal{A}(B)$

$$\eta(\mathcal{P}_2,\mathcal{P}_1)_{[2k]} = \frac{(-1)^k}{k!}\,\mathrm{Tr}\left((\mathcal{P}_1\cdot\nabla^{\pi_*^N(\mathbb{E}^0)}\cdot\mathcal{P}_1)^{2k} - (\mathcal{P}_2\cdot\nabla^{\pi_*^N(\mathbb{E}^0)}\cdot\mathcal{P}_2)^{2k}\right)\ ,$$

where $\omega_{[j]}\in\mathcal{A}^j(B)$ is the j-form component of $\omega\in\mathcal{A}(B)$.

Here, the Calderon section $\mathcal{P}(\mathsf{D})$ is a Grassmann section canonically associated to D.

Note that for $k=0$, $(\mathcal{P}_i\cdot\nabla^{\mathbb{H}_N}\cdot\mathcal{P}_i)^0 = \mathcal{P}_i$, and hence that the degree zero component of (1.13) coincides with the classical pointwise index identity (1.11).

For related relative eta form formulae in the context of gerbes we refer to recent work of Lott [L2].

Theorem (I) and the Melrose-Piazza Theorem [MP], stated in (1.4), imply:

Corollary 1.1. *If D is a family of compatible Dirac operators and $\mathcal{P}$ is a spectral section, then in $H^{2\bullet}(B)$*

$$\eta(\mathcal{P},\mathcal{P}(\mathsf{D})) = \int_{M/B}\widehat{A}(M/B)\mathrm{ch}(\mathbb{E}) - \frac{\widehat{\eta}_{\mathcal{P}}}{2}\ . \tag{1.14}$$

Generalizing (1.10)*, one has*

$$\eta(\mathcal{P}_2,\mathcal{P}_1) = \frac{\widehat{\eta}_{\mathcal{P}_1}}{2} - \frac{\widehat{\eta}_{\mathcal{P}_2}}{2}\ , \tag{1.15}$$

where $\widehat{\eta}_{\mathcal{P}_i} = \widehat{\eta}(\partial_i)$ are the eta-forms defined similarly to (1.3)*.*

The index formulas (1.4), (1.14) refer specifically to a family of compatible Dirac operators, though Dai-Zhang [DZ2] generalize to families of generalized Dirac operators with spectral section boundary conditions. It is worth pointing out that Theorem (I), and Theorem (II), below, hold for any smooth family of first-order elliptic differential operators and any Grassmann section, and, in fact, may be extended to significantly more general classes of boundary condition and to families of higher order elliptic operators. [Such generality is needed, for example, in local anomaly computations for supersymmetric branes.]

1.2. **Theorem (II): Relative Interior Eta forms.** Theorem (I) may, in the general sense explained in Section (1.3), be seen as a Quillen-Chern-Weil model for Theorem (II), which is the principal result of this paper.

Each of [BC2, MP] use a generalized ψdo boundary calculus to transform the family of boundary problems to a K-theory equivalent family of elliptic operators over a closed, or complete, manifold to which the methods of (1.2) can be adapted. Unlike [BC2, MP], and also Theorem (I) which transforms to the boundary, Theorem (II) computes the heat trace of a superconnection constructed *directly* on the infinite-dimensional bundle $\pi_*(\mathbb{E}|\mathcal{P})$ of domains of the family of boundary problems. Like [BC2, MP] this relies on a generalized ψdo boundary calculus, here the ψdo boundary calculus of singular Green operators developed by Grubb [G1].

There are a number of technical difficulties inherent in a direct assault on the families index, starting with the bundle structure of $\pi_*(\mathbb{E}|\mathcal{P})$. We show that this is inherited in a natural way from the structure of the boundary bundle $\mathcal{W}$ and that for Grassmann sections $\mathcal{P}_i$, $i = 1, 2$ there are induced superconnections on $\pi_*(\mathbb{E}|\mathcal{P}_i)$ with curvature forms $\mathsf{R}_i \in \mathcal{A}^2(B, \mathrm{End}(\pi_*(\mathbb{E}|\mathcal{P}_i)))$ with restricted differential operator coefficients. The relative curvature form $\mathsf{R}_1 - \mathsf{R}_2$ is trace-class in $\mathcal{A}(B, \pi_*(\mathbb{E}))$, and there is a generalized relative eta-form defined by the supertrace

$$\eta^{[M]}(\mathsf{P}_1, \mathsf{P}_2) = \mathrm{Str}\left(e^{-\mathsf{R}_1} - e^{-\mathsf{R}_2}\right) \in \mathcal{A}^{2\bullet}(B)\ .$$

But matters are complicated in this case by zeta-regularized trace terms. Let F be a pseudodifferential operator of order r and let Δ be a generalized Laplacian. Then for $m >> 0$ a resolvent trace expansion as $\lambda \to \infty$

$$\mathrm{Tr}\,(F(\Delta - \lambda)^{-m}) \sim \sum_{j=-n}^{\infty} a_j(-\lambda)^{\frac{r-j}{2}-m} + \sum_{j=0}^{\infty}\sum_{k=0}^{1}(a_{j,k}\log^k(-\lambda) + c_j)(-\lambda)^{-\frac{j}{2}-m} \tag{1.16}$$

means that the zeta-function trace $\mathrm{Tr}\,(F\Delta^{-s})$ defined by the standard trace for $\mathrm{Re}(s) > m + r/2$ extends meromorphically to all of $\mathbb{C}$ with the singularity structure

$$\Gamma(s)\,\mathrm{Tr}\,(F\Delta^{-s}) \sim \sum_{j=-n}^{\infty} \frac{\tilde{a}_j}{s + \frac{j-r}{2}} + \sum_{j=0}^{\infty}\sum_{k=0}^{1} \frac{\tilde{a}_{j,k}}{(s + \frac{j}{2})^{k+1}}\ . \tag{1.17}$$

The coefficient $\tilde{a}_{0,1}$ is, modulo a universal constant, the Wodzicki residue trace of F [W]. Remarkably, this is independent of the choice of regularizing operator Δ. On the other hand, the (Kontsevich-Vishik) regularized trace of F defined to be the constant term in the asymptotic expansion of $\mathrm{Tr}\,(F\Delta^{-s})$ around $s = 0$

$$\mathrm{TR}_\Delta(F) = \mathrm{Tr}\,(F\Delta^{-s})|^{\mathrm{mer}}_{s=0} \tag{1.18}$$

and equal to $\tilde{a}_r + \tilde{a}_{0,0}$, is not independent of the choice of regularizing operator Δ, and is not a trace functional on ψdos. The dependence on Δ is measured by a Wodzicki residue trace [KV]. An example of such a pseudo-trace is the eta invariant $\widehat{\eta}(\partial) = \mathrm{TR}_{|D|}(\Pi^{\partial}_{>} - \Pi^{\partial}_{<})$ in (1.8).

On closed manifolds this is essentially well understood. On manifolds with boundary the expansions (1.16), (1.17) are known to hold when F is a differential operator and the regularizing Laplacian Δ_P an APS-type boundary problem [G5].

We have:

Theorem (II) *In* $H^*(B)$
(1.19)

$$\mathrm{ch}(\mathrm{Ind}\,(D_{\mathcal{P}_1})) - \mathrm{ch}(\mathrm{Ind}\,(D_{\mathcal{P}_2})) = \sum_{k\geq 0} \frac{(-1)^k}{k!} \mathrm{Str}\left(R_1^k - R_2^k\right) + \frac{k+1}{k!}\ \mathrm{sTR}_{[\Delta_1,\Delta_2]}(R(D)^k)\ .$$

where with $\Delta_i = D^2_{\mathcal{P}_i}$

$$\mathrm{sTR}_{[\Delta_1,\Delta_2]}(R(D)^k) = \mathrm{Str}\left(R(D)^k(\Delta_1^{-s} - \Delta_2^{-s})\right)|^{\mathrm{mer}}_{s=0}\ . \tag{1.20}$$

Equivalently,

$$\mathrm{ch}(\mathrm{Ind}\,(\mathsf{D}_{\mathcal{P}})) = \sum_{k\geq 0} \frac{(-1)^k}{k!} \mathrm{Str}\left(\mathsf{R}^k - \mathsf{R}(\mathsf{D})^k\right) + \frac{k+1}{k!}\ \mathrm{sTR}_{[\Delta_{\mathcal{P}},\Delta_{\mathcal{P}(\mathsf{D})}]}(\mathsf{R}(\mathsf{D})^k)\ . \tag{1.21}$$

The regularization (1.20) means pointwise on the operator coefficient of $\mathsf{R}(\mathsf{D})^k$.

Since $\mathsf{R}_i^0 := \mathsf{P}_i$, the degree zero part of the right-side of (1.19) is the pointwise relative index $\mathrm{ind}\,(D_{P_1}) - \mathrm{ind}\,(D_{P_2})$. Applying the Melrose-Piazza Theorem (1.4) we obtain:

Corollary 1.2. *If D is a family of compatible Dirac operators and $\mathcal{P}$ is a spectral section, then in* $H^{2\bullet}(B)$
(1.22)

$$\sum_{k\geq 0} \frac{(-1)^k}{k!} \mathrm{Str}\left(R^k - R(D)^k\right) + \frac{k+1}{k!}\ \mathrm{sTR}_{[\Delta_{\mathcal{P}},\Delta_{\mathcal{P}(D)}]}(R(D)^k) = \int_{M/B} \widehat{A}(M/B)\mathrm{ch}(\mathbb{E}) - \frac{\widehat{\eta}_{\mathcal{P}}}{2}\ .$$

Taking differences of (1.22) the regularized trace terms cancel and one is left with the interior analogue of (1.15):

Corollary 1.3. *For spectral sections $\mathcal{P}_1, \mathcal{P}_2$ one has in* $H^*(B)$

$$\eta^{[M]}(P_1, P_2) = \frac{\widehat{\eta}_{\mathcal{P}_1}}{2} - \frac{\widehat{\eta}_{\mathcal{P}_2}}{2}\ .$$

1.3. **Relation to Quillen-Chern-Weil Theory.** A consequence of using the computational device of relative supertraces is that in the simplest non-trivial case the constructions in Thm(I) reduce to a superconnection formulation of Chern-Weil theory on the classifying space $\mathrm{BGl}(\infty)$. This is interesting because *a priori* there is no total Chern character form as the trace of an operator $e^{-\nabla^2}$ on the universal bundle $\mathcal{E}_1$ over $\mathrm{BGl}(\infty)$ in the semi-ring of vector bundles $\mathrm{Vect}(\infty)$ — however, there is a Chern character form at the level of K-theory, defined on virtual bundles.

By way of example and to explain some of the methodology of our approach we pause for a moment to outline this further.

1.3.1. *Construction of Heat Operators.* The heat operators we shall use are defined via holomorphic functional calculus, thus placing primacy on the resolvent operator. This works in Quillen-Chern-Weil theory as follows. Consider a complex super vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ over a finite-dimensional manifold B, and let L be an odd Hermitian endomorphism of $\mathcal{E}$. A superconnection [Q1] on $\mathcal{E}$ adapted to L is an odd-parity differential operator $\mathbb{A}$ on the graded complex $\mathcal{A}(B,\mathcal{E}) = \mathcal{A}(B) \otimes \Gamma(B,\mathcal{E})$, with $\otimes = \widehat{\otimes}$ the graded tensor product, such that

$$\mathbb{A}(\omega.s) = dw.s + (-1)^{|\omega|}\omega.\mathbb{A}s$$

for $\omega \in \mathcal{A}(B)$, $s \in \Gamma(B, \mathcal{E})$, and with $\mathbb{A}_{[0]} = L$, where $\mathbb{A}_{[i]}$ is the component of $\mathbb{A}$ which raises form degree by i. The curvature of $\mathbb{A}$ is the even-parity element $\mathbb{F} = \mathbb{A}^2$ of $\mathcal{A}(B, \mathrm{End}(\mathcal{E}))$.

It is convenient to utilize the scaling operator $\delta_t \in \mathcal{A}(B, \mathrm{End}(\mathcal{E}))$ of [BV] defined for $t > 0$ by $\delta_t(\omega_{[i]}) = t^{-i/2}\omega_{[i]}$ for $\omega_{[i]} \in \mathcal{A}^i(B)$, extended to endomorphisms of $\mathcal{A}(B, \mathcal{E})$ by

$$\delta_t(T) = \delta_t \cdot T \cdot \delta_t^{-1} \ . \tag{1.23}$$

We then have the scaled superconnection

$$\mathbb{A}_t = t^{1/2}\delta_t(\mathbb{A}) = t^{1/2}L + \mathbb{A}_{[1]} + t^{-1/2}\mathbb{A}_{[2]} + \dots$$

and curvature operator $\mathbb{F}_t = t^{1/2}\delta_t(\mathbb{F}) = tL^2 + \mathcal{F}_t$, where $\mathcal{F}_t$ raises exterior degree.

The heat operator

$$e^{-\mathbb{F}_t} \in \mathcal{A}^{2\bullet}(B, \mathrm{End}(\mathcal{E}))$$

is defined by the contour integral

$$e^{-\mathbb{F}_t} = \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\,(\mathbb{F}_t - \lambda I)^{-1}\,d\lambda\ , \tag{1.24}$$

where $I \in \mathcal{A}(B, \mathrm{End}(\mathcal{E}))$ is the identity automorphism and $\mathcal{C}$ is a contour surrounding the positive real axis $\mathbb{R}_+$ coming in on a ray with argument in $(0, \pi/2)$, encircling the origin, and leaving on a ray with argument in $(-\pi/2, 0)$. The resolvent operator $(\mathbb{F}_t - \lambda I)^{-1}$ is defined for any $\lambda \in \mathbb{C}\backslash\mathbb{R}_+$. For, $(\mathbb{F}_t)_{[0]} = tL^2$ is a positive Hermitian operator and hence $(tL^2 - \lambda I)^{-1}$ is defined for any $\lambda \in \mathbb{C}\backslash\mathbb{R}_+$, while the remaining terms are of non-zero form degree and hence nilpotent. Precisely,

$$\begin{aligned}(\mathbb{F}_t - \lambda I)^{-1} &= (\mathcal{F}_t + tL^2 - \lambda I)^{-1}\\ &= (tL^2 - \lambda I)^{-1}(\mathcal{F}_t(tL^2 - \lambda I)^{-1} + I)^{-1}\\ &= \sum_{k=0}^{\dim B}(-1)^k(tL^2 - \lambda I)^{-1}(\mathcal{F}_t(tL^2 - \lambda I)^{-1})^k\ . \end{aligned}\tag{1.25}$$

The last line of (1.25) is the finite algebraic Neumann series expansion since $\mathcal{F}_t(tL^2 - \lambda I)^{-1}$ is nilpotent. The heat trace asymptotics of the Chern character form $\mathrm{ch}(\mathbb{A}_t) = \mathrm{Str}(e^{-\mathbb{F}_t})$ can be thus be analyzed from the resolvent trace asymptotics.

If $\mathcal{E}$ is finite-rank this is elementary; one then has in $\mathcal{A}(B)$

$$\mathrm{ch}(\mathbb{A}_t) = \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\,\mathrm{Str}\left((\mathbb{F}_t - \lambda I)^{-1}\right)\,d\lambda\ , \tag{1.26}$$

and taking $\mathbb{A} = t^{1/2}L + \nabla^{\mathcal{E}}$, where $\nabla^{\mathcal{E}} = \nabla^{\mathcal{E}^+} \oplus \nabla^{\mathcal{E}^-}$ is a connection with curvature 2-form $R = R_+ \oplus R_-$, and setting t to zero $(\mathbb{F}_0 - \lambda I)^{-1} = \sum_{k=0}^{\dim B}(-1)^k(-\lambda)^{-k-1}R^k$ we have

$$\lim_{t\to 0}\mathrm{ch}(\mathbb{A}_t) = \sum_{k\geq 0}(-1)^k\left(\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\,(-\lambda)^{-k-1}\,d\lambda\right)\mathrm{Str}(R^k) = \sum_{k\geq 0}\frac{1}{k!}\left[\mathrm{tr}\,(R_+^k) - \mathrm{tr}\,(R_-^k)\right]\ ,$$

and so the transgression formula becomes the classical Chern-Weil formula.

The interest in this methodology is where L^2 is replaced by a family of Dirac Laplacians and $\mathbb{F}_t$ is a vertical differential operator valued form. Then (1.26) must be regularized by repeated integration by parts, with the resolvent replaced by $(\mathbb{F}_t - \lambda I)^{-m}$ for large enough m to ensure a trace class operator. To analyze $\mathrm{ch}(\mathbb{A}_t)$ for small t one computes the asymptotics as $\lambda \to \infty$ of generalized resolvent traces of the form $\mathrm{Str}(F(\Delta - \lambda I)^{-k})$ where F is a differential operator and Δ a generalized Laplacian. As $t \longrightarrow 0$ in $\mathrm{ch}(\mathbb{A}_t)$ this yields regularized trace terms and Wodzicki residue traces [W] — computed for closed manifolds in [S3]. In Thm(I) and Thm(II) residue traces cancel in the relative heat traces except for the regularized trace correction term (1.20). See also [PR] for a discussion of Wodzicki residues and determinant bundle curvature.

1.3.2. *Comparing Chern Forms on Subbundles.* Between these two extremes there is an intermediate case worthy of consideration with $\mathcal{E}^{\pm}$ Hilbert bundles with structure group reduced to Gl_1, the space of invertible operators of the form $I+g$ with g of trace-class. First, let $H = H^+ \oplus H^-$ be a graded Hilbert space where the orthogonal projections $\Pi_+, I-\Pi_+$ onto $H^{\pm}$ are of infinite rank. The infinite Grassmannian $\mathrm{Gr}_1(H)$ parameterizing projections P on H for which $P-\Pi_+$ is trace-class has homotopy type $\mathbb{Z}\times \mathrm{BGl}(\infty)$. By a theorem of Quillen [Q2], the de-Rham theorem holds for $\mathrm{Gr}_1(H)$ and so the Chern character cohomology classes $\mathrm{ch}_k \in H^{2k}(\mathrm{Gr}_1(H);\mathbb{R})$ of the canonical K-classes can be represented by differential forms. The universal bundle $\mathcal{E}_1$, with fibre $\mathrm{ran}\,(P)$ at $P\in \mathrm{Gr}_1(H)$, considered as a sub-Hilbert bundle of the trivial bundle $H\times \mathrm{Gr}_1(H)$ inherits the canonical connection $P\cdot d\cdot P$ with curvature 2-form $R = PdPdP \in \mathcal{A}^2(\mathrm{Gr}_1(H), \mathrm{End}(\mathcal{E}_1))$. Since dP is trace class the form $\omega_k = \mathrm{Tr}\,(PdP^{2k}) \in \mathcal{A}^{2k}(\mathrm{Gr}_1(H))$ is well-defined, and up to a constant it represents the primitive generator ch_k for $k\geq 1$. Although e^{-R} is a well-defined bounded operator, it is not trace-class, and so there is no total Chern character form. On the other hand, $e^{-R}-\Pi_+$ is of trace class with

$$\mathrm{Tr}\,(e^{-R}-\Pi_+) = \mathrm{index}\,(\Pi_+\circ P) + \sum_{k\geq 1}\frac{(-1)^k}{k!}\omega_k \in \mathcal{A}(\mathrm{Gr}_1(H))\ , \tag{1.27}$$

which is the relative Chern character $\mathrm{Tr}\,(e^{-R}-e^{-0_+})$ of the virtual bundle $\mathcal{E}_1-\mathcal{E}^{\mathrm{triv}}$, with $\mathcal{E}^{\mathrm{triv}}$ the trivial subbundle $H^+\times \mathrm{Gr}_1(H)$ and 0_+ the zero operator on H^+.

Theorem 1.4. *Let* $\mathcal{E} = \mathcal{E}^+\oplus\mathcal{E}^-$ *be a Hilbert bundle with connection* $\nabla^{\mathcal{E}}$ *with structure group* Gl_1*. Let* $\mathrm{Gr}_1(\mathcal{E}) \longrightarrow B$ *be the fibration with fibre the infinite-Grassmannian* $Gr_1(\mathcal{E}_z)$*. Let* $\mathbb{P}_1,\mathbb{P}_2$ *be smooth sections of* $\mathrm{Gr}_1(\mathcal{E})$*, defining smooth subbundles* $\mathcal{W}_1,\mathcal{W}_2$ *of* $\mathcal{E}$ *endowed with connections with curvature* $R_i = (\mathbb{P}_i\cdot\nabla^{\mathcal{E}}\cdot\mathbb{P}_i)^2$*. Then there are canonical superconnections* $\mathbb{A}_{i,t}$ *associated to* $\mathcal{W}_i$ *and a transgression formula for* $t>0$ *on* $\mathcal{A}(B)$

$$\mathrm{Str}(e^{-\mathbb{A}_{1,t}^2} - e^{-\mathbb{A}_{2,t}^2}) = \mathrm{index}\,(P_1,P_2) + \sum_{k=1}^{\dim B}\frac{1}{k!}\mathrm{Tr}\,(R_1^k - R_2^k) + d\beta_t\ .$$

The individual heat operators $e^{-\mathbb{A}_{1,t}^2}$ *are not trace class. Here* $\mathrm{index}\,(P_1,P_2)$ *is the pointwise index of a canonical family* $(\mathbb{P}_1,\mathbb{P}_2) : \mathcal{W}_1 \longrightarrow \mathcal{W}_2$ *of generalized Toeplitz operators with index bundle* $\mathrm{Ind}\,(P_1,P_2)\in K(B)$*, and one has in* $H^*(B)$

$$\mathrm{ch}(\mathrm{Ind}\,(\mathbb{P}_1,\mathbb{P}_2)) = \mathrm{index}\,(P_1,P_2) + \sum_{k=1}^{\dim B}\frac{1}{k!}\mathrm{Tr}\,(R_1^k - R_2^k)\ .$$

The proof of this theorem, of which (1.27) is obviously a special case, can be extracted from the proof of Theorem (I) by applying the identifications of Section 2 to the family of boundary problems $\mathsf{D}_{\mathcal{P}} = \{D_P \mid P\in \mathrm{Gr}_\infty(\partial)\}$ associated to a single Dirac operator D.

2. Preliminaries

Let X be a compact manifold with boundary $Y = \partial X$ and let $\pi : M \longrightarrow B$ be a smooth Riemannian fibration of manifolds with fibre X_z diffeomorphic to X, restricting on the boundary $N = \partial M$ to a smooth fibration $\pi^N : N \longrightarrow B$ of closed manifolds with fibre Y_z diffeomorphic to Y. Let $\mathbb{E} = \mathbb{E}^+ \oplus \mathbb{E}^-$ be a vertical super bundle over M with metric and let

$$\mathbb{E}' = \mathbb{E}_{|\mathbf{N}}$$

be the restricted vertical bundle over N graded as

$$\mathbb{E}' = \mathbb{E}^0 \oplus \mathbb{E}^1$$

where

$$\mathbb{E}^0 = \mathbb{E}^+_{|\mathbf{N}} \quad \text{and} \quad \mathbb{E}^1 = \mathbb{E}^-_{|\mathbf{N}} \ .$$

We assume a collar neighborhood

$$\mathcal{U} = [0,1) \times N$$

of the boundary such that each bundle splits isometrically. Thus if $g^{\mathbb{E}^+}, g^{\mathbb{E}^0}$ denote the metrics on $\mathbb{E}^+, \mathbb{E}^0$, then

$$\mathbb{E}^+_{|\mathcal{U}} = p^*(\mathbb{E}^0) \cong [0,1) \times \mathbb{E}^0 \ , \qquad g^{\mathbb{E}^+}_{|\mathcal{U}} = p^*(g^{\mathbb{E}^0}) \ , \tag{2.1}$$

where

$$p : \mathcal{U} \longrightarrow N$$

is the canonical projection map, and so forth. In particular, the tangent bundle $T(M/B)$ along the fibres of M is assumed to be oriented and endowed with a metric g^M such that

$$g^M_{|\mathcal{U}} = du^2 + g^N \ , \tag{2.2}$$

where g^N is the induced metric on $T(N/B)$.

Associated to $\mathbb{E}$ is the graded infinite-dimensional C^∞ Frechet bundle over B

$$\pi_*(\mathbb{E}) = \pi_*(\mathbb{E}^+) \oplus \pi_*(\mathbb{E}^-)$$

with fibre $\Gamma(X_z, E_z)$ at $z \in B$, where $E_z = \mathbb{E}_{|X_z}$ defined by

$$\Gamma(B, \pi_*(\mathbb{E})) = \Gamma(M, \mathbb{E}) \ .$$

More generally, we have the complex of C^∞ forms on B with values in $\pi_*(\mathbb{E})$

$$\mathcal{A}(B, \pi_*(\mathbb{E})) = \Gamma(M, \pi^*(\wedge T^*B) \otimes \mathbb{E}) \ ,$$

where $\otimes = \widehat{\otimes}$ is the graded tensor product. Likewise over $N = \partial M$ there is the bundle

$$\pi^N_*(\mathbb{E}') = \pi^N_*(\mathbb{E}^0) \oplus \pi^N_*(\mathbb{E}^1)$$

with fibre $\Gamma(Y_z, N_z)$ and the complex

$$\mathcal{A}(B, \pi_*(\mathbb{E}')) = \mathcal{A}(N, (\pi^N)^*(\wedge T^*B) \otimes \mathbb{E}') \ .$$

An element of $\mathcal{A}(B, \mathrm{End}(\pi_*(\mathbb{E})))$ is an endomorphism of $\mathcal{A}(B, \pi_*(\mathbb{E}))$ which supercommutes with the (pull-back) action of $\mathcal{A}(B)$ by exterior multiplication. The inclusion

$$\mathcal{A}(M, \mathrm{End}(\mathbb{E})) \subset \mathcal{A}(B, \mathrm{End}(\pi_*(\mathbb{E})))$$

is strict. Specifically, a vertical differential operator $\mathbb{D} \in \mathcal{A}(B, \mathrm{End}(\pi_*(\mathbb{E})))$ on $\mathbb{E}$ is a differential operator on $\mathcal{A}(B, \pi_*(\mathbb{E}))$ which supercommutes with $\mathcal{A}(B)$. Then $\mathbb{D}$ is a smooth section of the bundle $\mathcal{D}(\mathbb{E})$ over B with fibre the space of elliptic differential operators on $\Gamma(X_z, E_z)$ and, by definition, a smooth family of elliptic differential operators on $\mathbb{E}$. As such $\mathbb{D}$ decomposes as

$$\mathbb{D} = \sum_{i=0}^{\dim B} \mathbb{D}_{[i]} \ ,$$

where $\mathbb{D}_{[i]} \in \mathcal{A}^i(B, \mathcal{D}(\mathbb{E}))$ raises form degree by i.

Over the closed boundary N we have the graded bundle $\Psi(\mathbb{E}^{'})$ of boundary ψdos with fibre the space $\Psi(Y_z, E_z^{'})$ of ψdos on $\Gamma(Y_z, E_z^{'})$. Let $\Psi^r(\mathbb{E}_N)$ be the subbundle of ψdos of order r. A smooth family of ψdos on $\mathbb{E}^{'}$ is a smooth section $\mathbb{T} = \{T_z \in \Psi^r(Y_z, E_z) \mid z \in B\}$ of $\Psi(\mathbb{E}^{'})$, identified with a ψdo on $\Gamma(N, \mathbb{E}^{'})$ which commutes with the pull-back action of $C^\infty(B)$.

The restriction of sections of $\mathbb{E}$ over M to boundary sections of $\mathbb{E}^{'}$ over $N = \partial M$ defines an even parity vertical bundle map

$$\gamma \in \mathcal{A}(B, \mathrm{Hom}(\pi_*(\mathbb{E}), \pi_*^N(\mathbb{E}^{'})) \ ,$$

$$\gamma(\omega \otimes \psi) = \omega \otimes \psi_{|N} \ ,$$

where $\omega \in \mathcal{A}(B), \psi \in \Gamma(B, \pi_*(\mathbb{E}))$.

We consider a smooth family of formally self-adjoint first-order elliptic differential operators $\mathsf{D} = \{D_z \mid z \in B\} \in \mathcal{A}^0(B, \mathcal{D}(\mathbb{E}))$ of odd parity with respect to the grading on $\mathcal{A}(B, \mathrm{End}(\pi_*(\mathbb{E})))$, so

$$\mathsf{D} = \begin{bmatrix} 0 & \mathsf{D}^- \\ \mathsf{D}^+ & 0 \end{bmatrix} \ ,$$

where $\mathsf{D}^\pm = \{D_z^\pm \mid z \in B\} \in \mathcal{A}(B, \mathrm{Hom}(\mathbb{E}^\pm, \mathbb{E}^\mp))$ are formal adjoints – that is, symmetric for the induced metrics on $\pi_*(\mathbb{E}^\pm)$ for sections with support disjoint from from boundary. D extends to all of $\mathcal{A}(B, \mathcal{D}(\mathbb{E}))$ by

$$\mathsf{D}(\omega.s) = (-1)^{|\omega|}\omega.\mathsf{D}s \ ,$$

where $\omega \in \Gamma(M, \pi^*(\wedge T^*B)), s \in \Gamma(M, \mathbb{E})$.

We assume $\mathsf{D}^\pm$ have product structure in the collar $\mathcal{U}$

$$\mathsf{D}^\pm_{|\mathcal{U}} = \Upsilon_\pm \left(\frac{\partial}{\partial u} + \partial_\pm \right) \ , \tag{2.3}$$

where

$$\partial_+ = \{\partial_{+,z} \mid z \in B\} \in \Gamma(B, \mathcal{D}(\mathbb{E}^0)) \ , \qquad \partial_- = \{\partial_{-,z} \mid z \in B\} \in \Gamma(B, \mathcal{D}(\mathbb{E}^1))$$

are smooth boundary families of self-adjoint first-order elliptic differential operators on $\mathbb{E}^{'}$ and $\Upsilon^+ = -(\Upsilon^-)^* : \mathbb{E}_{|\mathcal{U}} \longrightarrow \mathbb{E}_{|\mathcal{U}}$ is a unitary bundle isomorphism. Then D has product structure in $\mathcal{U}$

$$\mathsf{D}_{|\mathcal{U}} = \begin{bmatrix} 0 & \Upsilon_- \\ \Upsilon_+ & 0 \end{bmatrix} \left(\frac{\partial}{\partial u} + \begin{bmatrix} \partial_+ & 0 \\ 0 & \partial_- \end{bmatrix} \right) \ . \tag{2.4}$$

For example, the isometric splitting of the metric (2.2) ensures this in the case of a family of Dirac operators associated to a fibration of spin manifolds, as in [BC2, MP].

Vertical APS-boundary problems for D arise as follows. For the moment we restrict our attention to D acting on $\Gamma(B,\pi_*(\mathbb{E})) = \mathcal{A}^0(B,\pi_*(\mathbb{E}))$. Associated to D is the subspace of interior solutions

$$\mathrm{Ker}(\mathsf{D}) = \{\psi \in \Gamma(B,\pi_*(\mathbb{E})) \mid \mathsf{D}\psi = 0 \ \text{ in } \ M\backslash N\} \ .$$

Since D is a bundle endomorphism of $\pi_*(\mathbb{E})$, a vertical differential operator on $\mathbb{E}$, then $\mathrm{Ker}(\mathsf{D})$ is fibered by the C^∞ kernels $\mathrm{Ker}(D_z) = \{\psi_z \in \Gamma(X_z, E_x) \mid D_z\psi_z = 0 \text{ in } X_z\backslash Y_z\}$. The space of vertical Cauchy data is defined by restriction of interior solutions to the boundary

$$\mathcal{W}(\mathsf{D}) := \gamma\mathrm{Ker}(\mathsf{D}) = \{\xi \in \pi_*(\mathbb{E}^{'}) \mid \xi = \gamma\phi, \ \phi \in \mathrm{Ker}(\mathsf{D})\} \ ,$$

and fibered $\mathcal{W}(\mathsf{D}) = \cup_z W(D_z)$ by the pointwise Cauchy data spaces $W(D_z) = \gamma\mathrm{Ker}(D_z)$ of classical boundary ψdo theory. These spaces are naturally graded as

$$\mathrm{Ker}(\mathsf{D}) = \mathrm{Ker}(\mathsf{D}^+) \oplus \mathrm{Ker}(\mathsf{D}^-) \ , \qquad \mathcal{W}(\mathsf{D}) = \mathcal{W}(\mathsf{D}^+) \oplus \mathcal{W}(\mathsf{D}^-)$$

with respect to the gradings on $\pi_*(\mathbb{E}), \pi_*(\mathbb{E}^{'})$. For manifolds with boundary, $\mathrm{Ker}(D_z)$ is infinite-dimensional, contrasting with elliptic regularity on closed manifolds. On the other hand, and again unlike closed manifolds, the kernels $\mathrm{Ker}(D_z)$ vary smoothly with z:

Proposition 2.1. $\mathrm{Ker}(\mathsf{D})$ *and* $\mathcal{W}(\mathsf{D})$ *are, respectively, smooth subbundles of* $\pi_*(\mathbb{E})$ *and* $\pi_*^N(\mathbb{E}^{'})$.[1] *The restriction map defines a vector bundle isomorphism over* B

$$\mathrm{Ker}(\mathsf{D}) \cong \mathcal{W}(\mathsf{D}) \ . \tag{2.5}$$

Equivalently, $\mathrm{Ker}(\mathsf{D}^\pm)$ *are smooth subbundles of* $\pi_*(\mathbb{E}^\pm)$*,* $\mathcal{W}(\mathsf{D}^+)$ *a subbundle of* $\pi_*^N(\mathbb{E}^0)$*, and* $\mathcal{W}(\mathsf{D}^-)$ *a subbundle of* $\pi_*^N(\mathbb{E}^1)$*, with isomorphisms* $\mathrm{Ker}(\mathsf{D}^\pm) \cong \mathcal{W}(\mathsf{D}^\pm)$.

Proof. The assertion is that there is an exact sequence

$$0 \longrightarrow \mathrm{Ker}(\mathsf{D}) \longrightarrow \pi_*(\mathbb{E}) \xrightarrow{\mathsf{D}} \pi_*(\mathbb{E}) \longrightarrow 0 \tag{2.6}$$

and that it is canonically split. (This holds also on all Sobolev completions [S4].) The exactness follows from the existence of the splitting map, which in turn is defined by a vertical right inverse to D. Let $\widehat{M} = M \cup_N (-M) \to B$ be the fibration of closed double manifolds constructed from M, with fibre $\widehat{X}_z = X_z \cup_{Y_z} (-X_z)$. With the product structure (2.4), D extends by a standard argument to an invertible vertical first-order differential operator $\widehat{\mathsf{D}}$ on $\Gamma(\widehat{M},\widehat{\mathbb{E}})$, where $\widehat{\mathbb{E}}_{|M} = \mathbb{E}$ and $\widehat{\mathsf{D}}_{|M} = \mathsf{D}$. Using the continuous extension operator $\mathbf{e} : \pi_*(\mathbb{E}) \to \pi_*^{\mathbf{L^2}}(\widehat{\mathbb{E}})$ to the bundle of L^2 sections of $\widehat{\mathbb{E}}$, where $\mathbf{e}(\psi) = \psi$ on M and zero elsewhere, and the restriction operator $\mathbf{r} : \pi_*(\widehat{\mathbb{E}}) \longrightarrow \pi_*(\mathbb{E})$, we define

$$\mathsf{G} := \mathbf{r}\widehat{\mathsf{D}}^{-1}\mathbf{e} \in \Gamma(B, \mathrm{End}(\pi_*(\mathbb{E}))) \ .$$

Since $\widehat{\mathsf{D}}\widehat{\mathsf{D}}^{-1} = \mathbb{I}$ on $\Gamma(\widehat{M},\widehat{\mathbb{E}})$, with $\mathbb{I}$ the vertical identity operator, by locality we have $\mathsf{D}\cdot\mathsf{G} = \mathbb{I}$ on $\Gamma(M,\mathbb{E})$, and hence G is the required splitting of (2.6).

On the other hand, we can use the inverse $\widehat{\mathsf{D}}^{-1}$ to define the vertical Poisson operator

$$\mathcal{K} = \mathbf{r}\widehat{\mathsf{D}}^{-1}\widehat{\gamma}^*\Upsilon \in \Gamma(B, \mathrm{Hom}(\pi_*^N(\mathbb{E}^{'}), \pi_*(\mathbb{E}))) \ .$$

[1]By a C^k vector bundle $\mathcal{E} \to B$ of infinite-rank we mean in the weak sense: for $z \in B$ there is an open set U_z and a fibre-preserving C^k-diffeomorphism $\mathcal{E}_{|U_z} \cong U_z \times \mathcal{E}_z$, where $\mathcal{E}_z$ is the fibre of $\mathcal{E}$ at z.

where $\widehat{\gamma}$ is the restriction map from $\widehat{M}$ to N. Then $\mathcal{K} = \{K_z \mid z \in B\}$ is a smooth family of operators taking boundary fields into interior fields coinciding fibrewise with the classical Poisson operator K_z of D_z [G1, G2]. Hence $\mathcal{K}$ is a bundle map with range in $\mathrm{Ker}(\mathsf{D})$, and which restricts to a fibrewise isomorphism

$$\mathcal{K} : \mathcal{W}(\mathsf{D}) \xrightarrow{\cong} \mathrm{Ker}(\mathsf{D}) \tag{2.7}$$

that by construction depends smoothly on $z \in B$. It is inverse to γ on $\mathrm{Ker}(\mathsf{D})$. The vertical operator

$$P_{\mathrm{Ker}} = \mathcal{K} \circ \gamma : \pi_*(\mathbb{E}) \longrightarrow \pi_*(\mathbb{E}) \ ,$$

is therefore a canonical smooth family of projections on $\pi_*(\mathbb{E})$ with range $\mathrm{Ker}(\mathsf{D})$, while

$$\mathsf{G} \cdot \mathsf{D} = \mathbb{I} - \mathcal{K} \circ \gamma = \mathbb{I} - P_{\mathrm{Ker}} = P_{\mathrm{Ker}}^{\perp} \in \Gamma(B, \mathrm{End}(\pi_*(\mathbb{E}))) \tag{2.8}$$

is the complementary vertical projection with range $\mathrm{Ker}(\mathsf{D})^{\perp}$. The first equality in (2.8) follows from the distributional Green's formula (the single operator case proved in [G2] applies directly to D). On the other hand, the restriction of $\mathcal{K}$ to the bundle of boundary sections

$$P(\mathsf{D}) := \gamma \circ \mathcal{K} \in \Gamma(B, \Psi^0(\mathbb{E}^{'})) \tag{2.9}$$

is the vertical Calderon projector with range $\mathcal{W}(\mathsf{D})$. $P(\mathsf{D}) = \{P(D_z) \mid z \in B\}$ is a smooth family of order 0 ψdo projections restricting on each fibre to the usual Calderon projector with range $\gamma\mathrm{Ker}(D_z) = W(D_z)$.

To see that $\mathcal{W}(\mathsf{D})$ is a vector subbundle of $\pi_*^N(\mathbb{E}^{'})$, because $\mathcal{P}(\mathsf{D})$ (and $P_{\mathbb{K}\mathrm{er}}$) is a smooth family and invertibility is an open condition in the space of Fredholm operators, it is sufficient to show that $P(D_z) \circ P(D_{b'}) : W(D_z) \to W(D_{b'})$ is Fredholm for z near $z^{'}$ in B, since $P(D_z) \circ P(D_{b'})$ is the identity when $z = z^{'}$. But if $\Pi_{>,z}, \Pi_{>,z'}$ are the APS spectral projections at z and $z^{'}$, it is a well known spectral flow property that $\Pi_{>,z} \circ \Pi_{>,z'} : \mathrm{ran}\,(\Pi_{>,z'}) \to \mathrm{ran}\,(\Pi_{>,z})$ is invertible for z close to $z^{'}$; for, since $\partial_z, \partial_{z'}$ are self-adjoint elliptic, perturbation theory asserts the spectrum is locally smooth, so $\dim \mathrm{Ker}(\partial_z)$ is locally constant. On the other hand, it is well known that $P(D_z) - \Pi_{>,z}$ and $P(D_{z'}) - \Pi_{>,z'}$ are smoothing operators [S1]. Hence $P(D_z) \circ P(D_{z'})$ is also locally invertible and we thus reach the conclusion.

Finally, the bundle structure on $\mathrm{Ker}(\mathsf{D})$ is immediate from that on $\mathcal{W}(\mathsf{D})$ and the fibre preserving isomorphism (2.7). □

Evidently, the (orthogonalized) Calderon section $P(\mathsf{D})$ splits with the grading as

$$P(\mathsf{D}) = \begin{bmatrix} P(\mathsf{D}^+) & 0 \\ 0 & P(\mathsf{D}^-) \end{bmatrix} \ , \tag{2.10}$$

where with obvious notation

$$P(\mathsf{D}^+) := \gamma \circ \mathcal{K}^+ \in \Gamma(B, \Psi^0(\mathbb{E}^0)), \qquad \mathcal{K}^+ = \mathbf{r}\widehat{\mathsf{D}^+}^{-1} \widehat{\gamma}^* \Upsilon \in \Gamma(B, \mathrm{Hom}(\pi_*^N(\mathbb{E}^0), \pi_*(\mathbb{E}^+))) \ ,$$

and similarly $P(\mathsf{D}^-)$, are the vertical Calderon projections for $\mathsf{D}^{\pm}$. In this way we obtain the bundle structure of $\mathcal{W}(\mathsf{D}^{\pm})$. One has

$$P(\mathsf{D}^-) = \Upsilon_+ \cdot P(\mathsf{D}^+)^{\perp} \cdot \Upsilon_+^* = -\Upsilon_+ \cdot P(\mathsf{D}^+)^{\perp} \cdot \Upsilon_- \ . \tag{2.11}$$

The (orthogonalized) Calderon section $P(\mathsf{D})$ defines a canonical section in $\Gamma(B,\Psi^0(\mathbb{E}^{'})$, which by (2.11) is determined by the section $P(\mathsf{D}^+)$ in $\Gamma(B,\Psi^0(\mathbb{E}^0)$. Relative to $P(\mathsf{D}^+)$ we define the subspace of $\Gamma(B,\Psi^0(\mathbb{E}^0))$ of *Grassmann sections*

$$\mathrm{Gr}(\pi_*^N(\mathbb{E}^0)) = \Gamma(B,\mathrm{Gr}_\infty(\partial_+)) \subset \Gamma(B,\mathrm{End}(\pi_*^N(\mathbb{E}^0))) \ ,$$

where $\mathrm{Gr}_\infty(\partial_+)\to B$ is the fibration with fibre at $z\in B$ the smooth Grassmannian

$$\mathrm{Gr}_\infty(\partial_{+,z}) = \{P_{+,z}\in\Psi^0(Y_z,E_z^0) \mid P_{+,z}^* = P_{+,z},\ P_{+,z}^2 = P_{+,z},\ P_{+,z}-P(D_z^+)\in\Psi^{-\infty}(Y_z,E_z^0)\} \ .$$

Thus a Grassmann section is a vertical projection $\mathcal{P}^+ = \{P_{+,z} \mid z\in B\}$ on $\pi_*^N(\mathbb{E}^0)$, defining a smooth family of of ψdo projections, such that

$$\mathcal{P}^+ = P(\mathsf{D}^+) + \mathsf{S} \ ,$$

where $\mathsf{S}\in\Gamma(B,\Psi^{-\infty}(\mathbb{E}^0))$ is a smooth family of smoothing operators. This means S has kernel

$$k\in\Gamma(N\times_{\pi^N}N,\mathbb{E}^0\boxtimes\mathbb{E}^0) \ ,$$

where $N\times_{\pi^N}N$ is the fibre product with respect to the projection maps $pr_i : N\times N\to N$ onto the i^{th} component, and $\mathbb{E}\boxtimes\mathbb{E} = pr_1^*(\mathbb{E}^0)\otimes pr_2^*(\mathbb{E}^0)$.

Notice that due to boundary spectral flow the family $\Pi_> = \{\Pi_>^z \mid z\in B\}$ is only locally continuous. However, when $\dim\mathrm{Ker}(D_{Y_z})$ is constant, as [BC2], then $\Pi_>\in\mathrm{Gr}(\pi_*^N(\mathbb{E}^0))$.

The primary purpose of $\mathrm{Gr}(\pi_*^N(\mathbb{E}^0))$ is a parameter space of self-adjoint global boundary conditions for the the family D. To this end, we associate to each $\mathcal{P}^+\in\mathrm{Gr}(\pi_*^N(\mathbb{E}^0))$ the Grassmann section for D

$$\mathcal{P} = \begin{bmatrix}\mathcal{P}^+ & 0\\ 0 & \mathcal{P}^-\end{bmatrix} \ ,$$

where

$$\mathcal{P}^- = -\Upsilon_+\cdot(\mathcal{P}^+)^\perp\cdot\Upsilon_- \ , \tag{2.12}$$

defining the smooth family of odd-parity APS-type global boundary problems

$$\mathsf{D}_\mathcal{P} = \begin{bmatrix}0 & \mathsf{D}^-_{\mathcal{P}^-}\\ \mathsf{D}^+_{\mathcal{P}^+} & 0\end{bmatrix} : \pi_*(\mathbb{E}|\mathcal{P}) \longrightarrow \pi_*(\mathbb{E}) \ .$$

This means

$$\mathsf{D}_\mathcal{P} = \mathsf{D} \ , \qquad \mathsf{D}^+_{\mathcal{P}^+} = \mathsf{D}^+ \ , \qquad \mathsf{D}^-_{\mathcal{P}^-} = \mathsf{D}^-$$

as operators, but with graded dense domain

$$\pi_*(\mathbb{E}|\mathcal{P}) = \pi_*(\mathbb{E}^+|\mathcal{P}^+)\oplus\pi_*(\mathbb{E}^-|\mathcal{P}^-) \ ,$$

where

$$\pi_*(\mathbb{E}|\mathcal{P}) = \mathrm{Ker}(\mathcal{P}\circ\gamma : \pi_*(\mathbb{E}) \longrightarrow \pi_*^N(\mathbb{E}^{'}))$$

and

$$\pi_*(\mathbb{E}^+|\mathcal{P}^+) = \mathrm{Ker}(\mathcal{P}^+\circ\gamma : \pi_*(\mathbb{E}^+) \longrightarrow \pi_*^N(\mathbb{E}^0)) \ , \quad \pi_*(\mathbb{E}^-|\mathcal{P}^-) = \mathrm{Ker}(\mathcal{P}^-\circ\gamma : \pi_*(\mathbb{E}^-) \longrightarrow \pi_*^N(\mathbb{E}^1)) \ .$$

Over X_z, $\mathsf{D}_\mathcal{P}$ restricts to the self-adjoint APS-type operator

$$D_{P_z} := (D_z)_{P_z} : \mathrm{dom}\,(D_{P_z}) \longrightarrow \Gamma(X_z,E_z) \ ,$$

in the usual single operator sense.

The definition (2.12) ensures $\mathsf{D}^-_{\mathcal{P}^-}$ is the adjoint family to $\mathsf{D}^+_{\mathcal{P}^+}$, in the obvious sense. Note that

$$\mathcal{P} = -\Upsilon \cdot \mathcal{P}^{\perp} \cdot \Upsilon \ .$$

$\mathcal{P}$ extends naturally to the even parity boundary operator

$$\mathcal{P} = 1 \otimes \mathcal{P} \in \mathcal{A}(B, \mathrm{End}(\mathbb{E}^{'})) \ , \qquad \mathcal{P}(\omega \otimes s) = \omega \otimes \mathcal{P}s \ ,$$

and which hence extends the smooth family of APS-type boundary problems to

$$\mathsf{D}_{\mathcal{P}} : \mathcal{A}(B, \pi_*(\mathbb{E}|\mathcal{P})) \longrightarrow \mathcal{A}(B, \pi_*(\mathbb{E})) \ ,$$

where $\mathcal{A}(B, \pi_*(\mathbb{E}|\mathcal{P})) = \mathcal{A}(M, \pi^*(\wedge T^*B)) \otimes \Gamma(M, \mathbb{E}|\mathcal{P}))$ is the dense subspace of sections annihilated by $\mathcal{P} \circ \gamma$, with γ the extended restriction map

(2.13) $$\gamma : \mathcal{A}(B, \pi_*(\mathbb{E})) \longrightarrow \mathcal{A}(B, \pi^N_*(\mathbb{E}^{'})) \qquad \gamma(\omega \otimes s) = \omega \otimes \gamma s \ .$$

A Grassmann section $\mathcal{P} = \{P_z \mid z \in B\}$ distinguishes the graded boundary subspace

$$\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^- \subset \pi^N_*(\mathbb{E}^{'})$$

with

$$\mathcal{W} = \mathrm{ran}\,(\mathcal{P}), \qquad \mathcal{W}^+ = \mathrm{ran}\,(\mathcal{P}_0), \qquad \mathcal{W}^- = \mathrm{ran}\,(\mathcal{P}^-) \ ,$$

and fibre $W_z = \mathrm{ran}\,(P_z)$ at $z \in B$. $\mathcal{W}, \mathcal{W}^{\pm}$ are canonically related to the spaces $\pi_*(\mathbb{E}|\mathcal{P}), \pi_*(\mathbb{E}^{\pm}|\mathcal{P}^{\pm})$ of interior sections by the exact sequences

(2.14) $$0 \longrightarrow \pi_*(\mathbb{E}|\mathcal{P}) \longrightarrow \pi_*(\mathbb{E}) \stackrel{\mathcal{P}\circ\gamma}{\longrightarrow} \mathcal{W} \longrightarrow 0$$

and

$$0 \longrightarrow \pi_*(\mathbb{E}^{\pm}|\mathcal{P}^{\pm}) \longrightarrow \pi_*(\mathbb{E}^{\pm}) \stackrel{\mathcal{P}^{\pm}\circ\gamma}{\longrightarrow} \mathcal{W}^{\pm} \longrightarrow 0 \ .$$

Proposition 2.2. *$\pi_*(\mathbb{E}|\mathcal{P})$, $\pi_*(\mathbb{E}^{\pm}|\mathcal{P}^{\pm})$ and $\mathcal{W}, \mathcal{W}^{\pm}$ are smooth infinite-dimensional Frechet vector bundles on B.*

Proof. We restrict our comments to $\pi_*(\mathbb{E}|\mathcal{P})$ and $\mathcal{W}$. Because

$$(P_{z_0}, P_z) = P_z \circ P_{z_0} : W_{z_0} \longrightarrow W_z$$

is Fredholm for z close to $z_0 \in B$ and is the identity operator for $z = z_0$, then, since invertibility is an open condition in Fred, the operator (P_{z_0}, P_z) is invertible for z close to z_0. Since $P_z - P(D_z) \in \Psi^{\infty}(Y_z, E_z)$, the Fredholm property for (P_{z_0}, P_z) is equivalent to that of $(P(D_{z_0}), P(D_z))$ and this is proved in Proposition 2.1. By its local invertibility, (P_{z_0}, P_z) defines around z_0 a local trivialization of $\mathcal{W}$ modelled on W_{z_0} bounded in each C^l norm. Since z_0 was arbitrary $\mathcal{W}$ is hence a smooth subbundle of the Frechet bundle $\pi^N_*(\mathbb{E}^{'})$.

The bundle structure on $\pi_*(\mathbb{E}|\mathcal{P})$ is defined by a Fredholm splitting of the exact sequence (2.14). Let χ be a C^{∞} cut-off function: $\chi(t) = 1$ when $0 \leq t < 1/4$ and $\chi(t) = 0$ when $t > 3/4$. Then we define a vertical Poisson operator extending boundary sections into interior sections with support in the collar neighborhood $\mathcal{U}$

$$\mathsf{K} \in \mathcal{A}(B, \mathrm{Hom}(\pi^N_*(\mathbb{E}^{'}), \pi_*(\mathbb{E})))$$

by defining $\mathsf{K} : \Gamma(N, \mathbb{E}') \longrightarrow \Gamma(M, \mathbb{E})$ by

$$\mathsf{K}(x) = \begin{cases} \chi(u).e^{-u\partial_y^2} & x = (u,y) \in \mathcal{U}, \\ 0 & x \in M\backslash\mathcal{U} \ . \end{cases} \tag{2.15}$$

Here $e^{-u\partial^2} \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}'))$ is the heat operator defined by the family of boundary Laplacians ∂, and as such we have

$$\gamma \circ \mathsf{K} = \lim_{u \longrightarrow 0} \chi(u).e^{-u\partial^2}(y) = \mathbb{I}_N \ , \tag{2.16}$$

where $\mathbb{I}_N$ is the vertical identity operator on $\pi_*^N(\mathbb{E}')$. We use K to define the operator

$$\mathsf{P} = \mathsf{P}(\mathcal{P}) := \mathbb{I} - \mathsf{K} \cdot \mathcal{P} \cdot \gamma \in \mathcal{A}(B, \mathrm{End}(\pi_*(\mathbb{E}))) \ , \tag{2.17}$$

with $\mathbb{I}$ the vertical identity operator on $\pi_*(\mathbb{E})$. Then

$$\mathsf{P}(\mathcal{P}) = \{\mathsf{P}(P_z) := I_z - \mathsf{K}_z \cdot P_z \cdot \gamma \mid z \in B\}$$

is a smooth family of projection operators with range $\pi_*(\mathbb{E}|\mathcal{P})$:

Lemma 2.3.

$$\mathsf{P}(\mathcal{P})^2 = \mathsf{P}(\mathcal{P}) \ .$$
$$\mathsf{P}(\mathcal{P}) = \mathbb{I} \quad \text{on} \quad \pi_*(\mathbb{E}|\mathcal{P}) \ .$$
$$\mathcal{P}\gamma \mathsf{P}(\mathcal{P}) = 0 \ .$$

In a neighborhood of $z_0 \in B$, the local map between fibres of $\pi_*(\mathbb{E}|\mathcal{P})$

$$\mathsf{P}(P_z) \cdot \mathsf{P}(P_{z_0}) : \Gamma(X_{z_0}, E_{z_0}|P_{z_0}) \longrightarrow \Gamma(X_z, E_z|P_z) \ , \tag{2.18}$$

is an isomorphism depending smoothly on z.

Proof. The first three identities are readily verified from (2.16). Note that the third identity says that $\mathrm{ran}\,(\mathsf{P}(P_z)) = \Gamma(X_z, E_z|P_z)$. To see (2.18), from the above a local trivializations of $\mathcal{W}$ is defined by the smooth family of invertible operators $P_{z_0} \circ P_z : W_z \longrightarrow W_{z_0}$ in a small neighborhood U_0 of z_0. We hence find that

$$I_z - \mathsf{K}_z P_z (P_{z_0} \circ P_z)^{-1} P_{z_0} \cdot \gamma$$

is a two-sided inverse to $\mathsf{P}(P_z) \cdot \mathsf{P}(P_{z_0})$ in U_0, while smoothness is obvious as compositions of smooth families of operators. □

The isomorphisms (2.18) define local trivializations of $\pi_*(\mathbb{E}|\mathcal{P})$ and hence complete the proof. □

By Proposition 2.2, $\mathcal{P}_i, \mathcal{P}_j \in \mathrm{Gr}(\pi_*^N(\mathbb{E}^0))$ define the canonical smooth Hermitian family of odd-parity Fredholm operators

$$L_{i,j} = \begin{bmatrix} 0 & \mathcal{P}_i \cdot \mathcal{P}_j \\ \mathcal{P}_j \cdot \mathcal{P}_i & 0 \end{bmatrix} \in \mathcal{A}^0(B, \mathrm{End}(\mathcal{W}_i \oplus \mathcal{W}_j)) \ , \tag{2.19}$$

where $\mathcal{P}_j \cdot \mathcal{P}_i$ parameterizes the operators $P_{j,z} \circ P_{i,z} : W_{i,z} \longrightarrow W_{j,z}$. We may also write

$$L_{i,j} = (\mathcal{P}_i, \mathcal{P}_j) \ .$$

In particular, associated to $\mathsf{D}_{\mathcal{P}}$ is the vertical 'scattering' operator

$$\mathcal{S}(\mathcal{P}) = \begin{bmatrix} 0 & \mathcal{S}(\mathcal{P}^+)^* \\ \mathcal{S}(\mathcal{P}^+) & 0 \end{bmatrix} \in \mathcal{A}^0(B, \mathrm{End}(\mathcal{W}(\mathsf{D}^+) \oplus \mathcal{W}^+)) \ ,$$

where $\mathcal{S}(\mathcal{P}^+) = \mathcal{P}^+ \cdot P(\mathsf{D}^+) : \mathcal{W}(\mathsf{D}^+) \longrightarrow \mathcal{W}^+$.

The choice of a Grassmann section $\mathcal{P}$ restricts D to a family of Fredholm operators. It also has the consequence that the kernels of the restricted operators no longer define a vector bundle. They do, however, still define a virtual bundle:

Proposition 2.4. *$D_{\mathcal{P}}$ defines a smooth family of Fredholm operators with kernel and cokernel consisting of smooth sections, and hence an index bundle*

$$\mathrm{Ind}\,(D_{\mathcal{P}}) \in K(B)\ .$$

Similarly, $L_{1,2} = (\mathcal{P}_1, \mathcal{P}_2)$ is a smooth Toeplitz family of Fredholm operators with kernel and cokernel consisting of smooth sections defining an index bundle

$$\mathrm{Ind}\,(\mathcal{P}_1, \mathcal{P}_2) \in K(B)\ .$$

The principal relations between these K-theory elements are as follows.

Proposition 2.5. *As elements of $K(B)$*

$$\mathrm{Ind}\,(D_{\mathcal{P}}) = \mathrm{Ind}\,(S(\mathcal{P})) \tag{2.20}$$

and

$$\mathrm{Ind}\,(\mathcal{P}_1, \mathcal{P}_3) - \mathrm{Ind}\,(\mathcal{P}_1, \mathcal{P}_2) = \mathrm{Ind}\,(\mathcal{P}_2, \mathcal{P}_3)\ ,$$

and hence

$$\mathrm{Ind}\,(D_{\mathcal{P}_1}) - \mathrm{Ind}\,(D_{\mathcal{P}_2}) = \mathrm{Ind}\,(\mathcal{P}_2, \mathcal{P}_1)\ . \tag{2.21}$$

Proofs for spectral sections can be accessed in [MP, DZ1]. The general case is proved in [S4] by a quite different method, extending to the corresponding determinant bundle isomorphisms. We omit further comment as the details are not relevant here.

The objective is to construct canonical differential form representatives for the K-class $\mathrm{Ind}\,(\mathsf{D}_{\mathcal{P}})$ by computing Chern character forms induced from the superconnections on $\pi_*(\mathbb{E})$ and $\pi_*^N(\mathbb{E}^0)$ defined as follows. A superconnection on $\pi_*(\mathbb{E})$ adapted to D is a differential operator on $\mathcal{A}(B, \pi_*(\mathbb{E}))$ of odd-parity with respect to the induced $\mathbb{Z}_2$-grading such that

$$\mathbb{A}(\omega\psi) = d\omega\psi + (-1)^{|\omega|}\omega\mathbb{A}(\psi) \tag{2.22}$$

for $\omega \in \mathbb{A}(B), \psi \in \mathcal{A}(B, \pi_*(\mathbb{E}))$, with $\mathbb{A}_{[0]} = \mathsf{D}$. Notice (2.22) implies $\mathbb{A}_{[1]}$ is a connection on $\pi_*(\mathbb{E}) = \pi_*(\mathbb{E}^+) \oplus \pi_*(\mathbb{E}^-)$ preserving the grading.

The induced superconnection on $\mathcal{A}(B, \mathrm{End}(\pi_*(\mathbb{E})))$ is defined by $\mathbb{A}a = [\mathbb{A}, a]$, where $[\ ,\ ]$ is the supercommutator. The curvature of $\mathbb{A}$ is the vertical differential operator with differential form coefficients $\mathbb{F} = \mathbb{A}^2 \in \mathcal{A}(B, \mathcal{D}(\mathbb{E}))$ with degree zero component the Dirac-Laplacian

$$\mathbb{F}_{[0]} = \mathsf{D}^2 = \begin{bmatrix} \mathsf{D}^-\mathsf{D}^+ & 0 \\ 0 & \mathsf{D}^+\mathsf{D}^- \end{bmatrix}\ .$$

For $t > 0$ we consider the scaled superconnection

$$\mathbb{A}_t = t^{1/2}\delta_t(\mathbb{A}) = t^{1/2}\mathsf{D} + \mathbb{A}_{[1]} + t^{-1/2}\mathbb{A}_{[2]} + \dots$$

with curvature operator $\mathbb{F}_t = t^{1/2}\delta_t(\mathcal{F}) = t\mathsf{D}^2 + F_t$ where F_t raises exterior degree.

A superconnection on $\pi_*(\mathbb{E}')$ defines by restriction a superconnection $\mathbb{A}'$ on $\pi_*^N(\mathbb{E}')$ adapted to ∂. We assume throughout the compatibility condition

$$\mathbb{A}_{|\mathcal{U}} = p^*(\mathbb{A}') \ , \tag{2.23}$$

where $p : \mathcal{U} \longrightarrow N$ is the canonical projection map. By functoriality $\mathbb{F}_{|\mathcal{U}} = p^*(\mathbb{F}')$, where $\mathbb{F}' = (\mathbb{A}')^2 \in \mathcal{A}(B, \mathcal{D}(\mathbb{E}^0))$ is the boundary curvature.

Though much of what follows holds for a general superconnection, Theorems (I) and (II) are proved with the canonical superconnection induced from a choice of connection on $\pi_*(\mathbb{E})$. This is defined by a compatible connection $\nabla^{\mathbb{E}}$ on $\mathbb{E}$ over M such that

$$\nabla^{\mathbb{E}}_{|\mathcal{U}} = p^*(\nabla^{\mathbb{E}'}) \ , \tag{2.24}$$

with $\nabla^{\mathbb{E}'}$ the induced boundary connection on $\mathbb{E}'$, along with a connection on the fibration $\pi : M \longrightarrow B$, meaning a choice of splitting

$$T^*M = T^*(M/B) \oplus T^*_H M \ , \tag{2.25}$$

and hence an isomorphism $\tau : T^*_H M \longrightarrow \pi^*(T^*B)$. The connection

$$\nabla^{\pi_*(\mathbb{E})} : \mathcal{A}^0(B, \pi_*(\mathbb{E})) \longrightarrow \mathcal{A}^1(B, \pi_*(\mathbb{E}))$$

is then defined by the composition

$$\Gamma(M, \mathbb{E}) \xrightarrow{\nabla^{\mathbb{E}}} \Gamma(M, T^*M \otimes \mathbb{E}) \xrightarrow{\tau} \Gamma(M, \pi^*(T^*B) \otimes \mathbb{E}) \ . \tag{2.26}$$

Hence

$$\mathbb{A}_t = \nabla^{\pi_*(\mathbb{E})} + t^{1/2}\mathsf{D} = \begin{bmatrix} \nabla^{\pi_*(\mathbb{E}^+)} & t^{1/2}\mathsf{D}^- \\ t^{1/2}\mathsf{D}^+ & \nabla^{\pi_*(\mathbb{E}^-)} \end{bmatrix} \tag{2.27}$$

maps $\Gamma(M, \mathbb{E}^{\pm}) \longrightarrow \mathcal{A}^{\pm}(M, \mathbb{E})$, where $\pm$ refer to the $\mathbb{Z}_2$-gradings, and thus extends to a superconnection on all of $\mathcal{A}(B, \pi_*(\mathbb{E}))$ by setting $\mathbb{A}(\omega s) = d\omega s + (-1)^{|\omega|}\omega\mathbb{A}s$, for $\omega \in \pi^*(\mathcal{A}(B)), s \in \Gamma(M, \mathbb{E})$. Assumption (2.23) is assured by (2.3) and (2.24).

In this case we have

$$\mathbb{F}_t = \mathsf{R}^{\pi_*(\mathbb{E})} + t^{1/2}\nabla^{\pi_*(\mathbb{E})}\mathsf{D} + t\mathsf{D}^2 \ , \tag{2.28}$$

where the curvature $\mathsf{R}^{\pi_*(\mathbb{E})} = (\nabla^{\pi_*(\mathbb{E})})^2 \in \mathcal{A}(B, \mathcal{D}(\mathbb{E}))$ of $\nabla^{\pi_*(\mathbb{E})}$ is the vertical first-order differential operator valued 2-form on B

$$\mathsf{R}^{\pi_*(\mathbb{E})} = -\nabla^{\pi_*(\mathbb{E})}_{\sigma} + R^{(M/B)} \ . \tag{2.29}$$

Here $\sigma(\xi_1, \xi_2) = -P_H[\xi_1^H, \xi_2^H]$, with P_H the projection onto $T_H M$ arising from the dual splitting to (2.25) and ξ^H the corresponding lifting of a vector field ξ on B. $R^{(M/B)}$ is the curvature tensor of $P_{M/B} \cdot \nabla^{TM} \cdot P_{M/B}$, where ∇^{TM} is the Levi-Civita connection for $g^M + \pi^*(g^B)$ for any metric on the base, and $P_{M/B}$ the projection onto $T(M/B)$ complementary to P_H. Note that $\nabla^{\pi_*(\mathbb{E})}\mathsf{D} = [\nabla^{\pi_*(\mathbb{E})}, \mathsf{D}]$ in (2.28).

The compatibility of metric, connections all other geometrical structures in the collar neighborhood $\mathcal{U}$ of N in M means that the homolog of each of these objects is induced on the boundary fibration of closed manifolds. Thus we have a splitting

$$T^*N = T^*(N/B) \oplus T^*_H N$$

which combined with (2.23) defines the connection

$$\nabla^{\pi_*^N(\mathbb{E}')} = \nabla^{\pi_*^N(\mathbb{E}^0)} \oplus \nabla^{\pi_*^N(\mathbb{E}^1)} : \mathcal{A}^i(B, \pi_*(\mathbb{E}')) \longrightarrow \mathcal{A}^{i+1}(B, \pi_*(\mathbb{E}'))$$

on $\pi_*^N(\mathbb{E}')$, and hence the superconnection $\mathbb{A}_t' = \nabla^{\pi_*^N(\mathbb{E}')} + t^{1/2}\partial$ which the Bismut superconnection extends.

3. Chern Character Forms from the Boundary

In this section we construct a canonical Chern character form representative for ch(Ind $(\mathsf{D}_{\mathcal{P}})$ from the geometry of the boundary fibration of closed manifolds. Precisely, the K-theory identities of Proposition 2.5 mean it is enough to compute the Chern character form of $L_{i,j}$ (2.19) considered as a vertical ψdo of order 0 on the super bundle $\pi_*^N(\mathbb{E}^0) \oplus \pi_*^N(\mathbb{E}^0)$.

3.1. **Construction of a relative heat operator.** A superconnection on the superbundle $\pi_*^N(\mathbb{E}^0) \oplus \pi_*^N(\mathbb{E}^0)$ adapted to an odd family of self-adjoint pseudodifferential operators $\mathbb{T} = \begin{bmatrix} 0 & \mathbb{T}_- \\ \mathbb{T}_+ & 0 \end{bmatrix} \in \mathcal{A}^0(B, \Psi(\pi_*^N(\mathbb{E}^0) \oplus \pi_*^N(\mathbb{E}^0)))$ is a differential operator of odd-parity $\mathbb{B} = \sum_{i=0}^{\dim B} \mathbb{B}_{[i]}$ on $\mathcal{A}(B, \pi_*^N(\mathbb{E}^0) \oplus \pi_*^N(\mathbb{E}^0))$ satisfying

$$\mathbb{B}(\omega.\phi) = d_B\omega.\phi + (-1)^{|\omega|}\omega.\mathbb{B}\phi$$

for $\omega \in \mathcal{A}(B), \phi \in \mathcal{A}(B, \pi_*^N(\mathbb{E}^0) \oplus \pi_*^N(\mathbb{E}^0))$ such that $\mathbb{B}_{[0]} = \mathbb{T}$.

Given $\mathcal{P}_i, \mathcal{P}_j \in \mathrm{Gr}(\pi_*^N(\mathbb{E}^0))$ we define the even element

$$\mathcal{P}_{i,j} = \mathcal{P}_i \oplus \mathcal{P}_j \in \mathcal{A}^0(B, \mathrm{End}(\pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0)) \ .$$

We then have the graded Hermitian subbundle $\mathcal{W}_i \oplus \mathcal{W}_j$ of $\pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0)$ and canonical maps

$$\mathcal{A}(B, \pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0)) \longrightarrow \mathcal{A}(B, \mathcal{W}_i \oplus \mathcal{W}_j) \ , \qquad \omega \otimes s \longmapsto \omega \otimes \mathcal{P}_{i,j}s \ ,$$

and

$$\mathcal{A}(B, \mathrm{End}(\pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0))) \longrightarrow \mathcal{A}(B, \mathrm{End}(\mathcal{W}_i \oplus \mathcal{W}_j)) \ , \qquad \omega \otimes A \longmapsto \omega \otimes \mathcal{P}_{i,j}A\mathcal{P}_{i,j} \ .$$

The latter extends to define the induced superconnection on $\mathcal{W}_i \oplus \mathcal{W}_j$

$$\mathbb{B}_{i,j} := \mathcal{P}_{i,j} \cdot \mathbb{B} \cdot \mathcal{P}_{i,j} : \mathcal{A}(B, \mathcal{W}_i \oplus \mathcal{W}_j) \longrightarrow \mathcal{A}(B, \mathcal{W}_i \oplus \mathcal{W}_j) \ .$$

The degree 0 term is the odd-parity pseudodifferential operator of order 0

$$\mathbb{B}_{[0]}^{i,j} = \mathcal{P}_{i,j}\,\mathbb{T}\,\mathcal{P}_{i,j} = \begin{bmatrix} 0 & \mathcal{P}_i\,\mathbb{T}_-\,\mathcal{P}_j \\ \mathcal{P}_j\,\mathbb{T}_+\,\mathcal{P}_i & 0 \end{bmatrix} \ , \tag{3.1}$$

while the degree 1 term is the unitary connection of even parity $\mathbb{B}_{[1]}^{i,j} = \mathcal{P}_{i,j} \cdot \mathbb{B}_{[1]} \cdot \mathcal{P}_{i,j}$ on $\mathcal{W}_i \oplus \mathcal{W}_j$.

We shall consider the asymptotic behaviour of the scaled superconnections

$$\mathbb{B}_t = t^{1/2}\delta_t \cdot \mathbb{B} \cdot \delta_t^{-1} \ , \qquad \mathbb{B}_t^{i,j} = t^{1/2}\delta_t \cdot \mathbb{B}^{i,j} \cdot \delta_t^{-1} \ .$$

For our purposes here it will be enough to consider just the canonical superconnection defined by setting $\mathbb{T}_\pm = \mathbb{I}_N$, and $\mathbb{B}_{[1]} = \nabla^{\pi_*^N(\mathbb{E}^0)} \oplus \nabla^{\pi_*^N(\mathbb{E}^0)}$, where $\mathbb{I}_N$ is the vertical identity operator on $\pi_*^N(\mathbb{E}^0)$. Thus

$$\mathbb{B}_t = \begin{bmatrix} \nabla^{\pi_*^N(\mathbb{E}^0)} & 0 \\ 0 & \nabla^{\pi_*^N(\mathbb{E}^0)} \end{bmatrix} + t^{1/2} \begin{bmatrix} 0 & \mathbb{I}_N \\ \mathbb{I}_N & 0 \end{bmatrix}$$

and with $\nabla^i = \mathcal{P}_i \cdot \nabla^{\pi_*^N(\mathbb{E}^0)} \cdot \mathcal{P}_i$

$$\mathbb{B}_t^{i,j} = \begin{bmatrix} \nabla^i & 0 \\ 0 & \nabla^j \end{bmatrix} + t^{1/2} L_{i,j} \tag{3.2}$$

is a superconnection on $\mathcal{W}_i \oplus \mathcal{W}_j$ adapted to the family $t^{1/2}L_{i,j}$. The curvature form $\mathbb{F}_t^{i,j} = (\mathbb{B}_t^{i,j})^2 \in \mathcal{A}^2(B, \mathrm{End}(\mathcal{W}_i \oplus \mathcal{W}_j))$ is then

$$\mathbb{F}_t^{i,j} = \mathcal{R}_{i,j} + t^{1/2}\nabla L_{i,j} + tL_{i,j}^2 = \begin{bmatrix} \mathcal{R}_i + t\mathcal{P}_i\mathcal{P}_j\mathcal{P}_i & t^{1/2}\nabla^{i,j}(\mathcal{P}_i\mathcal{P}_j) \\ t^{1/2}\nabla^{j,i}(\mathcal{P}_j\mathcal{P}_i) & \mathcal{R}_j + t\mathcal{P}_j\mathcal{P}_i\mathcal{P}_j \end{bmatrix} , \tag{3.3}$$

where $\mathcal{R}_i = (\nabla^i)^2$, $\mathcal{R}_{i,j} = \mathcal{R}_i \oplus \mathcal{R}_j$ are the curvature 2-forms on $\mathcal{W}_i$, $\mathcal{W}_i \oplus \mathcal{W}_j$, and $\nabla^{i,j}\alpha = \nabla^j\alpha - \alpha\nabla^i$ the induced connection on $\mathrm{Hom}(\mathcal{W}_i, \mathcal{W}_j)$. Note with $\mathcal{P}_i = \mathcal{P}(\mathsf{D}), \mathcal{P}_j = \mathcal{P}$ (3.2) and (3.3) become

$$\begin{bmatrix} \nabla^{\mathcal{W}(\mathsf{D})} & t^{1/2}\mathcal{S}(\mathcal{P})^* \\ t^{1/2}\mathcal{S}(\mathcal{P}) & \nabla^{\mathcal{W}} \end{bmatrix} , \qquad \begin{bmatrix} \mathcal{R}_{\mathcal{W}(\mathsf{D})} + t\mathcal{S}(\mathcal{P})^*\mathcal{S}(\mathcal{P}) & t^{1/2}\nabla\mathcal{S}(\mathcal{P})^* \\ t^{1/2}\nabla\mathcal{S}(\mathcal{P}) & \mathcal{R}_{\mathcal{W}} + t\mathcal{S}(\mathcal{P})\mathcal{S}(\mathcal{P})^* \end{bmatrix} .$$

Since $(\mathbb{F}_t^{i,j} - \lambda\mathbb{I}_{i,j})_{[0]} = tL_{i,j}^2 - \lambda\mathbb{I}_{i,j}$, where $\mathbb{I}_{i,j} = \mathbb{I}_i \oplus \mathbb{I}_j$ is the identity operator on on $\mathcal{A}(B, \mathcal{W}_i \oplus \mathcal{W}_j)$, and $tL_{i,j}^2$ is positive self-adjoint, then $\mathbb{F}_t^{i,j}$ has a resolvent for $\lambda \in \mathbb{C}\backslash\mathbb{R}_+$

$$(\mathbb{F}_t^{i,j} - \lambda\mathbb{I}_{i,j})^{-1} = \sum_{k=0}^{\dim B} (-1)^k (tL_{i,j}^2 - \lambda\mathbb{I}_{i,j})^{-1} \left\{ (\mathcal{R}_{i,j} + t^{1/2}\nabla L_{i,j})(tL_{i,j}^2 - \lambda\mathbb{I}_{i,j})^{-1} \right\}^k , \tag{3.4}$$

using (1.25). Since $L_{i,j}^2$ is bounded in each C^l norm we can expand this further via the infinite Neumann series for $t/|\lambda| < \|L_{i,j}^2\|_l^{-1}$

$$(tL_{i,j}^2 - \lambda\mathbb{I}_{i,j})^{-1} = -\sum_{l\geq 0} t^l \lambda^{-l-1} L_{i,j}^{2l} . \tag{3.5}$$

It follows from (3.4), which shows $(\mathbb{F}_t^{i,j} - \lambda\mathbb{I}_{i,j})^{-1}$ is (the restriction of) an unbounded differential operator, and from (3.5), that the heat operator $e^{-\mathbb{F}_t^{i,j}}$ can be defined. This is not, however, the object of interest–in particular it is not trace class and so has no Chern character. Rather we consider the relative Chern character, defined via the choice of an intermediary Grassmann section, as follows.

Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \in \mathrm{Gr}(\pi_*^N(\mathbb{E}^0))$. Then the subbundles

$$\mathcal{W}_1 \oplus \mathcal{W}_2 \quad \text{and} \quad \mathcal{W}_2 \oplus \mathcal{W}_3$$

of $\pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0)$ are endowed with the canonical superconnections

$$\mathbb{B}_t^{1,2} = \nabla^{1,2} + t^{1/2}L_{1,2} \quad \text{and} \quad \mathbb{B}_t^{2,3} = \nabla^{2,3} + t^{1/2}L_{2,3}$$

with resolvent operators

$$(\mathbb{F}_t^{1,2} - \lambda\mathbb{I}_{1,2})^{-1} \in \mathcal{A}(B, \mathrm{End}(\mathcal{W}_1 \oplus \mathcal{W}_2)) \quad \text{and} \quad (\mathbb{F}_t^{2,3} - \lambda\mathbb{I}_{2,3})^{-1} \in \mathcal{A}(B, \mathrm{End}(\mathcal{W}_2 \oplus \mathcal{W}_3)) .$$

We construct the relative heat operator on $\pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0)$ by extending the resolvents by zero on the bundles $\mathcal{W}_i^\perp \oplus \mathcal{W}_j^\perp$. By inclusion $(\mathbb{F}_t^{i,j} - \lambda\mathbb{I}_{i,j})^{-1}$ is identified with the ψdo

$$(\mathbb{F}_t^{i,j} - \lambda\mathbb{I}_{i,j})^{-1}_{|\mathbb{E}^0} := \mathcal{P}_{i,j} \cdot (\mathbb{F}_t^{i,j} - \lambda\mathbb{I}_{i,j})^{-1} \cdot \mathcal{P}_{i,j}$$

on $\pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0)$ and we have

$$(\mathbb{F}_t^{i,j} - \lambda\mathbb{I}_{i,j})^{-1}_{|\mathbb{E}^0} = (\mathbb{F}_t^{i,j} - \lambda\mathbb{I})^{-1} - (-\lambda)^{-1}\mathcal{P}_{i,j}^\perp \quad \text{in} \quad \mathcal{A}(B, \Psi(\mathbb{E}^0 \oplus \mathbb{E}^0)) , \tag{3.6}$$

where on the right-side $\mathbb{F}_t^{i,j}$ is the operator $\mathcal{P}_{i,j}\cdot\mathbb{F}_t^{i,j}\cdot\mathcal{P}_{i,j}$ and $\mathbb{I}$ is the vertical identity operator on $\pi_*^N(\mathbb{E}^0\oplus\mathbb{E}^0)$. It follows that the relative resolvent

$$(\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}-(\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}$$

is $O(|\lambda|^{-1})$ as $\lambda\to\infty$ in all C^l norms (see Proposition 3.1 below) and hence that the relative heat operator can be defined by

$$e^{-\mathbb{F}_t^{1,2}}-e^{-\mathbb{F}_t^{2,3}}=\frac{i}{2\pi}\int_{\mathcal{C}}e^{-\lambda}\left((\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}-(\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}\right)\,d\lambda\;. \tag{3.7}$$

Let $\Psi^{-\infty}(\mathbb{E}^0\oplus\mathbb{E}^0)$ be the vertical bundle of smoothing operators, as in [BGV], whose sections are smooth families of smoothing operators.

Proposition 3.1. *[1] For each* $\lambda\in\mathbb{C}\backslash\mathbb{R}_+$

$$(\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}-(\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}\in\mathcal{A}(B,\Psi^{-\infty}(\mathbb{E}^0\oplus\mathbb{E}^0))\;.$$

[2] The integral (3.7) *converges in the* C^∞*-topology to an element*

$$e^{-\mathbb{F}_t^{1,2}}-e^{-\mathbb{F}_t^{2,3}}\in\mathcal{A}(B,\Psi^{-\infty}(\mathbb{E}^0\oplus\mathbb{E}^0))\;.$$

Proof. From (3.4) and (3.6) we have

$$(\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}-(\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}=(\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1}-(\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-1}+(-\lambda)^{-1}(\mathcal{P}_{2,3}^\perp-\mathcal{P}_{1,2}^\perp)$$

$$=(-\lambda)^{-1}(\mathcal{P}_{1,2}-\mathcal{P}_{2,3})+\sum_{k=0}^{\dim B}(-1)^k(tL_{1,2}^2-\lambda\mathbb{I})^{-1}\left\{(\mathcal{R}_{1,2}+t^{1/2}\nabla L_{1,2})(tL_{1,2}^2-\lambda\mathbb{I})^{-1}\right\}^k$$

$$-(-1)^k(tL_{2,3}^2-\lambda\mathbb{I})^{-1}\left\{(\mathcal{R}_{2,3}+t^{1/2}\nabla L_{2,3})(tL_{2,3}^2-\lambda\mathbb{I})^{-1}\right\}^k$$

$$=(-\lambda)^{-1}(\mathcal{P}_{1,2}-\mathcal{P}_{2,3})$$

$$+\sum_{k=0}^{\dim B}(-1)^k\sum_{i=0}^{k}\left\{(tL_{2,3}^2-\lambda\mathbb{I})^{-1}(\mathcal{R}_{2,3}+t^{1/2}\nabla L_{2,3})\right\}^i\left\{(tL_{1,2}^2-\lambda\mathbb{I})^{-1}-(tL_{2,3}^2-\lambda\mathbb{I})^{-1}\right\}$$

$$\times\left\{(\mathcal{R}_{1,2}+t^{1/2}\nabla L_{1,2})(tL_{1,2}^2-\lambda\mathbb{I})^{-1}\right\}^{k-i}$$

$$+\sum_{k=1}^{\dim B}(-1)^k\sum_{j=0}^{k-1}\left\{(tL_{2,3}^2-\lambda\mathbb{I})^{-1}(\mathcal{R}_{2,3}+t^{1/2}\nabla L_{2,3})\right\}^j(tL_{2,3}^2-\lambda\mathbb{I})^{-1} \tag{3.8}$$

$$\times\left\{(\mathcal{R}_{1,2}-\mathcal{R}_{2,3})+t^{1/2}(\nabla L_{1,2}-\nabla L_{2,3})\right\}\left\{(tL_{1,2}^2-\lambda\mathbb{I})^{-1}(\mathcal{R}_{1,2}+t^{1/2}\nabla L_{1,2})(tL_{1,2}^2-\lambda\mathbb{I})^{-1}\right\}^{k-j-1}\;.$$

From (3.8) and (3.5) we hence obtain for any vector fields $\xi_1,\ldots,\xi_l$ on B

$$\|\nabla_{\xi_1}^{\pi_*(\mathbb{E}^0\oplus\mathbb{E}^0)}\ldots\nabla_{\xi_k}^{\pi_*(\mathbb{E}^0\oplus\mathbb{E}^0)}\left((\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}-(\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}\right)\|\leq C_l t^l|\lambda|^{-l-1}$$

and so (3.7) is convergent in each C^l norm.

The vertical operator $D_{i,j}=\mathcal{R}_{i,j}+t^{1/2}\nabla L_{i,j}$ is a differential form with differential operator coefficients. Since these are families of operators over a closed manifold we have by [BGV] Cor(2.40) that if $K\in\mathcal{A}(B,\Psi^{-\infty}(\mathbb{E}^0\oplus\mathbb{E}^0))$, then $D_{i,j}K$ and $KD_{i,j}$ are also smooth families of smoothing operators. Hence since $(tL_{i,j}^2-\lambda\mathbb{I})^{-1}$ is a vertical ψdo of order 0, then [1] is immediate from (3.8) once we have the following sublemma.

Lemma 3.2.

$$\mathcal{P}_{1,2} - \mathcal{P}_{2,3} \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0)) \ . \tag{3.9}$$

$$(tL_{1,2}^2 - \lambda\mathbb{I})^{-1} - (tL_{2,3}^2 - \lambda\mathbb{I})^{-1} \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0)) \ . \tag{3.10}$$

$$\mathcal{R}_{1,2}^k - \mathcal{R}_{2,3}^k \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0)) \ . \tag{3.11}$$

$$\nabla L_{1,2} - \nabla L_{2,3} \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0)) \ . \tag{3.12}$$

Proof. (3.9) is obvious. For (3.10) we have

$$(tL_{1,2}^2 - \lambda\mathbb{I})^{-1} - (tL_{2,3}^2 - \lambda\mathbb{I})^{-1} = t(tL_{1,2}^2 - \lambda\mathbb{I})^{-1}(L_{2,3}^2 - L_{1,2}^2)(tL_{2,3}^2 - \lambda\mathbb{I})^{-1}$$

while $(tL_{1,2}^2 - \lambda\mathbb{I})^{-1} \in \mathcal{A}(B, \Psi^0(\mathbb{E}^0 \oplus \mathbb{E}^0))$ and

$$L_{2,3}^2 - L_{1,2}^2 = (\mathcal{P}_2\mathcal{P}_3\mathcal{P}_2 - \mathcal{P}_1\mathcal{P}_2\mathcal{P}_1) \oplus (\mathcal{P}_3\mathcal{P}_2\mathcal{P}_3 - \mathcal{P}_2\mathcal{P}_1\mathcal{P}_2) \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0)) \tag{3.13}$$

since $\mathcal{P}_i - \mathcal{P}_j \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0))$.

It is enough to prove (3.11) over a local trivialization $N_{|U} = Y_{z_0} \times U$ of the boundary fibration in an open neighborhood U of $z_0 \in B$. This defines an identification $\mathbb{E}^0_{|U} = E^0_{z_0} \times U$, where $E^0_{z_0}$ is the superbundle over $Y_{z_0} = \partial X_{z_0}$, and hence a trivialization (in the weak sense) of $\pi_*^N(\mathbb{E}^0)$ over U as $\pi_*^N(\mathbb{E}^0)_{|U} = \Gamma(X_{z_0}, E^0_{z_0}) \times U$ and

$$\mathcal{A}(U, \pi_*^N(\mathbb{E}^0)) = \mathcal{A}(U) \times \Gamma(Y_{z_0}, E^0_{z_0}) \ .$$

Any superconnection $\mathbb{B}$ on $\pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0)$ then takes the local form

$$\mathbb{B} = d_U + \sum_I D_I^z dz_I \ , \tag{3.14}$$

where $dz_I \in \mathcal{A}^{|I|}(U)$, and D_I^z are differential operators on $\Gamma(Y_{z_0}, E^0_{z_0} \oplus E^0_{z_0})$ depending smoothly on z. We have over U

$$\mathbb{B}^{i,j}_{|U} = P_{i,j}^z \cdot d_U \cdot P_{i,j}^z + \sum_I P_{i,j}^z \cdot D_I^z \cdot P_{i,j}^z dz_I \ , \tag{3.15}$$

where the $P_{i,j}^z$ are elements of the fixed Grassmannian $\mathrm{Gr}_\infty(\partial_{z_0} \oplus \partial_{z_0})$ depending smoothly on z and defining $P_i^z : U \longrightarrow \mathcal{A}(U) \otimes \Gamma(Y_{z_0}, \mathrm{End}(E^0_{z_0} \oplus E^0_{z_0}))$. Fixing $P_0 \in \mathrm{Gr}_\infty(\partial_{z_0} \oplus \partial_{z_0})$ we have $P_{i,j}^z - P_0 \in \mathcal{A}^0(\Psi^{-\infty}(E^0_{z_0} \oplus E^0_{z_0}))$ and hence that

$$dP_{i,j}^z = d(P_{i,j}^z - P_0) \in \mathcal{A}^1(U, \Psi^{-\infty}(E^0_{z_0} \oplus E^0_{z_0}) \ , \tag{3.16}$$

is a smooth family of smoothing operators.

From (3.15)

$$(\mathcal{R}_{i,j})_{|U} = P_{i,j}^z dP_{i,j}^z \wedge dP_{i,j}^z + \sum_I (P_{i,j}^z \cdot D_I^z \cdot P_{i,j}^z) dz_I \wedge dP_{i,j}^z \tag{3.17}$$

$$+ \quad P_{i,j}^z \sum_I \sum_{l=0}^{\dim B} \frac{\partial}{\partial z_l}(P_{i,j}^z \cdot D_I^z \cdot P_{i,j}^z) dz_l \wedge dz_I + \left(\sum_I (P_{i,j}^z \cdot D_I^z \cdot P_{i,j}^z) dz_I \right)^2 \ . \tag{3.18}$$

From (3.16) the first two terms on the right-side are in $\mathcal{A}(U, \Psi^{-\infty}(E^0_{z_0} \oplus E^0_{z_0}))$. The terms in (3.18) are non-smoothing, but their difference in $\mathcal{R}_{1,2} - \mathcal{R}_{2,3}$ are smooth families of smoothing operators. For the first term in (3.18) we have

$$\sum_I \sum_{l=0}^{\dim B} \left(P_1^z \frac{\partial D_I^z}{\partial z_l} P_1^z - P_2^z \frac{\partial D_I^z}{\partial z_l} P_2^z \right) dz_l \wedge dz_I + F(\mathcal{P}_{1,2}, \mathcal{P}_{2,3})$$

$$= \sum_I \sum_{l=0}^{\dim B} (P_{1,2}^z - P_{2,3}^z) \frac{\partial D_I^z}{\partial z_l} P_{1,2}^z - P_{2,3}^z \frac{\partial D_I^z}{\partial z_l} (P_{1,2}^z - P_{2,3}^z) dz_l \wedge dz_I + F(\mathcal{P}_{1,2}, \mathcal{P}_{2,3}) \ ,$$

where $F(\mathcal{P}_{1,2}, \mathcal{P}_{2,3})$ is a sum of terms involving derivatives of $\mathcal{P}_{i,j}$ and hence smoothing. Since the composition of a differential operator and a smoothing operator is smoothing, this is in $\mathcal{A}(U, \Psi^{-\infty}(E^0_{z_0} \oplus E^0_{z_0}))$. Similarly the second term in (3.18) is in $\mathcal{A}(U, \Psi^{-\infty}(E^0_{z_0} \oplus E^0_{z_0}))$. This proves (3.11) for $k = 1$. The general case now follows from the identity

$$\mathcal{R}_{1,2}^k - \mathcal{R}_{2,3}^k = \sum_{j=0}^{k-1} \mathcal{R}_{2,3}^j \, (\mathcal{R}_{1,2} - \mathcal{R}_{2,3}) \mathcal{R}_{1,2}^{k-j-1} \ . \tag{3.19}$$

The final assertion (3.12) follows by a similar argument. □

Since smoothing operators form an ideal in the bundle of ψdos $\Psi(\mathbb{E}^0 \oplus \mathbb{E}^0)$, (3.8) and the lemma therefore imply [1]. Equivalently, $(\mathbb{F}_t^{1,2} - \lambda \mathbb{I})^{-1}_{|\mathbb{E}^0} - (\mathbb{F}_t^{2,3} - \lambda \mathbb{I})^{-1}_{|\mathbb{E}^0}$ is a vertical integral operator with C^∞ kernel $K_{t,\lambda}^{1,2,3} \in \mathcal{A}(N \times_{\pi^N} N, (\mathbb{E}^0 \oplus \mathbb{E}^0) \boxtimes (\mathbb{E}^0 \oplus \mathbb{E}^0)^*)$. Hence $e^{-\mathbb{F}_t^{1,2}} - e^{-\mathbb{F}_t^{2,3}}$ has kernel

$$E_t^{1,2,3}(x,y) = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-\lambda} K_{t,\lambda}^{1,2,3}(x,y) \, d\lambda$$

which is therefore also in $\mathcal{A}(N \times_{\pi^N} N, (\mathbb{E}^0 \oplus \mathbb{E}^0) \boxtimes (\mathbb{E}^0 \oplus \mathbb{E}^0)^*)$. This completes the proof. □

3.2. **Chern character forms and transgression.** Defined on $\mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0))$ is the fibrewise supertrace

$$\mathrm{Str} : \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0)) \longrightarrow \mathcal{A}(B) \ , \qquad \mathrm{Str}(A)(z) = \int_{N_z} \mathrm{Str}(a_z(x,x)) \, dy \ , \tag{3.20}$$

where A has fibrewise kernel $a_z \in \mathcal{A}(N_z \times_{\pi^N} N_z, (E_z^0 \oplus E_z^0) \boxtimes (E_z^0 \oplus E_z^0)^*)$, and hence by Proposition 3.1 we have the relative Chern character form associated to the superconnections $\mathbb{B}_t^{1,2}, \mathbb{B}_t^{2,3}$ as the even degree element of $\mathcal{A}(B)$ defined by

$$\mathrm{ch}(\mathbb{B}_t^{1,2}, \mathbb{B}_t^{2,3}) = \mathrm{Str}(e^{-\mathbb{F}_t^{1,2}} - e^{-\mathbb{F}_t^{2,3}}) \ . \tag{3.21}$$

By Proposition 3.1[1]

$$\mathrm{ch}(\mathbb{B}_t^{1,2}, \mathbb{B}_t^{2,3}) = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-\lambda} \, \mathrm{Str} \left((\mathbb{F}_t^{1,2} - \lambda \mathbb{I})^{-1}_{|\mathbb{E}^0} - (\mathbb{F}_t^{2,3} - \lambda \mathbb{I})^{-1}_{|\mathbb{E}^0} \right) d\lambda \ . \tag{3.22}$$

The relative Chern character form defines a deRham cohomology class in $H^{2\bullet}(B)$:

Proposition 3.3. *The differential form* $\mathrm{ch}(\mathbb{B}_t^{1,2}, \mathbb{B}_t^{2,3})$ *is closed.*

Proof. We require the following lemma.

Lemma 3.4. *Let* $\mathcal{P}_\alpha, \mathcal{P}_\beta \in \mathrm{Gr}(\pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0))$ *be of even parity, and let* $a_\alpha = \mathcal{P}_\alpha a_\alpha \mathcal{P}_\alpha, a_\beta = \mathcal{P}_\beta a_\beta \mathcal{P}_\beta \in \mathcal{A}(B, \mathrm{End}(\pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0)))$ *be even parity such that* $a_\alpha - a_\beta \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0))$*. Let* $\mathbb{B}$ *be a superconnection on* $\pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0)$ *and let* $\mathbb{B}_\alpha = \mathcal{P}_\alpha \cdot \mathbb{B} \cdot \mathcal{P}_\alpha, \mathbb{B}_\beta = \mathcal{P}_\beta \cdot \mathbb{B} \cdot \mathcal{P}_\beta$*. Then in* $\mathcal{A}(B)$

$$d_B \cdot \mathrm{Str}(a_\alpha - a_\beta) = \mathrm{Str}(\mathbb{B}_\alpha a_\alpha - \mathbb{B}_\beta a_\beta) \ , \tag{3.23}$$

where $\mathbb{B}_\alpha a_\alpha = [\mathbb{B}_\alpha, a_\alpha]$*,* $\mathbb{B}_\beta a_\beta = [\mathbb{B}_\beta, a_\beta]$*.*

Proof. $a_\alpha - a_\beta \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0))$ is a consequence of $\mathcal{P}_\alpha - \mathcal{P}_\beta \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0))$. To see (3.23), we have from [BGV] Lem(9.14)

$$d_B \cdot \mathrm{Str}(a_\alpha - a_\beta) = \mathrm{Str}(\mathbb{B} a_\alpha - \mathbb{B} a_\beta) \ .$$

For $\sigma_1, \sigma_2 \in \mathcal{A}(B, \mathrm{End}(\pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0)))$ one has $\mathbb{B}(\sigma_1\sigma_2) = (\mathbb{B}\sigma_1) \cdot \sigma_2 + (-1)^{|\sigma_1|}\sigma_1 \cdot \mathbb{B}(\sigma_2)$. Hence $\mathbb{B} a_\alpha = \mathbb{B}(\mathcal{P}_\alpha \cdot a_\alpha) = (\mathbb{B}\mathcal{P}_\alpha) \cdot a_\alpha + \mathcal{P}_\alpha \mathbb{B} a_\alpha = (\mathbb{B}\mathcal{P}_\alpha) \cdot a_\alpha + \mathbb{B}_\alpha a_\alpha$ since $|\mathcal{P}_\alpha| = 0$. Thus

$$\mathrm{Str}(\mathbb{B} a_\alpha - \mathbb{B} a_\beta) = \mathrm{Str}((\mathbb{B}\mathcal{P}_\alpha) \cdot a_\alpha - (\mathbb{B}\mathcal{P}_\beta) \cdot a_\beta) + \mathrm{Str}(\mathbb{B}_\alpha a_\alpha - \mathbb{B}_\beta a_\beta) \ , \tag{3.24}$$

since $(\mathbb{B}\mathcal{P}_\alpha) \cdot a_\alpha - (\mathbb{B}\mathcal{P}_\beta) \cdot a_\beta$ is clearly a smooth family of smoothing operators.

The proof is thus to show that the first term on the right-side of (3.24) vanishes. We may work locally over an open set U in B, where $\mathbb{B} = d_U + \omega$ and $\omega = \sum_I D_I dz_I$. The term in question is

$$\mathrm{Str}((d\mathcal{P}_\alpha) \cdot a_\alpha - (d\mathcal{P}_\alpha) \cdot a_\beta) + \mathrm{Str}([\omega, \mathcal{P}_\alpha] \cdot a_\alpha - [\omega, \mathcal{P}_\beta] \cdot a_\beta)$$

$$= \mathrm{Str}((d\mathcal{P}_\alpha) \cdot a_\alpha \cdot \mathcal{P}_\alpha) - \mathrm{Str}((d\mathcal{P}_\alpha) \cdot a_\beta \cdot \mathcal{P}_\beta) + \mathrm{Str}(\mathcal{P}_\alpha^\perp \omega a_\alpha - \mathcal{P}_\beta^\perp \omega a_\beta) \ . \tag{3.25}$$

Locally the operators $(\mathcal{P}_\alpha)_{|U} = (P_{\alpha,z} \mid z \in U)$ can be taken in a fixed Grassmannian $\mathrm{Gr}_\infty(\partial_{z_0} \oplus \partial_{z_0})$ and hence relative to the trivialization $d\mathcal{P}_\alpha \in \mathcal{A}(U, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0))$. We have

$$\mathrm{Str}((d\mathcal{P}_\alpha) \cdot a_\alpha \cdot \mathcal{P}_\alpha) = \mathrm{Str}((d\mathcal{P}_\alpha) \cdot \mathcal{P}_\alpha a_\alpha \cdot \mathcal{P}_\alpha^2) = \mathrm{Str}((\mathcal{P}_\alpha(d\mathcal{P}_\alpha)\mathcal{P}_\alpha) a_\alpha \cdot \mathcal{P}_\alpha) = 0 \ ,$$

since $\mathcal{P}_\alpha^2 = \mathcal{P}_\alpha$ implies $\mathcal{P}_\alpha(d\mathcal{P}_\alpha)\mathcal{P}_\alpha = 0$. Since $\mathcal{P}_\alpha^\perp - \mathcal{P}_\beta^\perp, a_\alpha - a_\beta \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0))$, the remaining term in (3.25) is

$$\begin{aligned}
\mathrm{Str}(\mathcal{P}_\alpha^\perp \omega a_\alpha - \mathcal{P}_\beta^\perp \omega a_\beta) &= \mathrm{Str}((\mathcal{P}_\alpha^\perp - \mathcal{P}_\beta^\perp)\omega \mathcal{P}_\alpha a_\alpha \mathcal{P}_\alpha) + \mathrm{Str}(\mathcal{P}_\beta^\perp \omega(\mathcal{P}_\alpha a_\alpha \mathcal{P}_\alpha - \mathcal{P}_\beta a_\beta \mathcal{P}_\beta)) \\
&= -\mathrm{Str}(\mathcal{P}_\alpha \mathcal{P}_\beta^\perp \omega \mathcal{P}_\alpha a_\alpha \mathcal{P}_\alpha) + \mathrm{Str}(\mathcal{P}_\beta^\perp \omega \mathcal{P}_\alpha a_\alpha \mathcal{P}_\alpha \mathcal{P}_\beta^\perp) \\
&= \mathrm{Str}([\mathcal{P}_\beta^\perp \omega \mathcal{P}_\alpha a_\alpha \mathcal{P}_\alpha, \mathcal{P}_\alpha \mathcal{P}_\beta^\perp]) = 0 \ ,
\end{aligned}$$

where the second equality follows by cycling the trace and the third since $\mathcal{P}_\beta^\perp \omega \mathcal{P}_\alpha a_\alpha \mathcal{P}_\alpha$ and $\mathcal{P}_\alpha \mathcal{P}_\beta^\perp$ have odd and even parity respectively. □

Applying Lemma 3.4 with $\mathcal{P}_\alpha = \mathcal{P}_{1,2}, \mathcal{P}_\beta = \mathcal{P}_{2,3}$, so that $\mathbb{B}_\alpha = \mathbb{B}_t^{1,2}, \mathbb{B}_\beta = \mathbb{B}_t^{2,3}$, to (3.22)

$$d_B \mathrm{ch}(\mathbb{B}_t^{1,2}, \mathbb{B}_t^{2,3}) = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-\lambda} \mathrm{Str}\left([\mathbb{B}_t^{1,2}, (\mathbb{F}_t^{1,2} - \lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}] - [\mathbb{B}_t^{2,3}, (\mathbb{F}_t^{2,3} - \lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}] \right) d\lambda$$

which vanishes since $[\mathbb{B}_t^{i,j}, (\mathbb{F}_t^{i,j} - \lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}] = [\mathbb{B}_t^{i,j}, ((\mathbb{B}_t^{i,j})^2 - \lambda\mathbb{I})^{-1}_{|\mathbb{E}^0}]$ vanishes. □

Next, we have the transgression formula:

Proposition 3.5. *One has*

$$\frac{d}{dt}\mathrm{ch}(\mathbb{B}_t^{1,2},\mathbb{B}_t^{2,3}) = -d_B \mathrm{Str}(\dot{\mathbb{B}}_t^{1,2} e^{-\mathbb{F}_t^{1,2}} - \dot{\mathbb{B}}_t^{2,3} e^{-\mathbb{F}_t^{2,3}}) \ . \tag{3.26}$$

Equivalently, for $0 < t < T < \infty$

$$\mathrm{ch}(\mathbb{B}_T^{1,2},\mathbb{B}_T^{2,3}) - \mathrm{ch}(\mathbb{B}_t^{1,2},\mathbb{B}_t^{2,3}) = -d_B \int_t^T \mathrm{Str}(\dot{\mathbb{B}}_t^{1,2} e^{-\mathbb{F}_t^{1,2}} - \dot{\mathbb{B}}_t^{2,3} e^{-\mathbb{F}_t^{2,3}})\, dt \ . \tag{3.27}$$

Proof. From (3.6)

$$\partial_t \mathrm{ch}(\mathbb{B}_t^{1,2},\mathbb{B}_t^{2,3}) = \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\, \mathrm{Str}\left(\partial_t(\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1} - \partial_t(\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-1}\right) d\lambda \ ,$$

while since $\mathbb{B}_t^{i,j}$ commutes with $(\mathbb{F}_t^{i,j}-\lambda\mathbb{I})^{-1}$

$$\begin{aligned}\partial_t(\mathbb{F}_t^{i,j}-\lambda\mathbb{I})^{-1} &= -(\mathbb{F}_t^{i,j}-\lambda\mathbb{I})^{-1}\cdot(\mathbb{B}_t^{i,j}\dot{\mathbb{B}}_t^{i,j})\cdot(\mathbb{F}_t^{i,j}-\lambda\mathbb{I})^{-1} - (\mathbb{F}_t^{i,j}-\lambda\mathbb{I})^{-1}\cdot(\dot{\mathbb{B}}_t^{i,j}\mathbb{B}_t^{i,j})\cdot(\mathbb{F}_t^{i,j}-\lambda\mathbb{I})^{-1}\\ &= -[\mathbb{B}_t^{i,j},(\mathbb{F}_t^{i,j}-\lambda\mathbb{I})^{-1}\cdot\dot{\mathbb{B}}_t^{i,j}\cdot(\mathbb{F}_t^{i,j}-\lambda\mathbb{I})^{-1}] \ .\end{aligned}$$

By Lemma 3.4

$$\partial_t \mathrm{ch}(\mathbb{B}_t^{1,2},\mathbb{B}_t^{2,3}) = -d_B \tau_t \ ,$$

where

$$\tau_t = \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\, \mathrm{Str}\left((\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1}\dot{\mathbb{B}}_t^{1,2}(\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1} - (\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-1}\dot{\mathbb{B}}_t^{2,3}(\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-1}\right) d\lambda \ .$$

Writing the integrand as

$$\begin{aligned}\mathrm{Str}\Big(\Big((\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1} - (\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-1}\Big)\dot{\mathbb{B}}_t^{1,2}(\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1} + (\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1}\Big(\dot{\mathbb{B}}_t^{1,2}-\dot{\mathbb{B}}_t^{2,3}\Big)(\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-1}\\ +(\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1}\dot{\mathbb{B}}_t^{2,3}\Big((\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-1} - (\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-1}\Big)\Big)\end{aligned}$$

each of the differences is an element of $\mathcal{A}(B,\Psi^{-\infty}(\mathbb{E}^0\oplus\mathbb{E}^0))$ and hence we can cycle the trace on each term to get

$$\tau_t = \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\, \mathrm{Str}\left(\dot{\mathbb{B}}_t^{1,2}(\mathbb{F}_t^{1,2}-\lambda\mathbb{I})^{-2} - \dot{\mathbb{B}}_t^{2,3}(\mathbb{F}_t^{2,3}-\lambda\mathbb{I})^{-2}\right) d\lambda \ ,$$

and integrating by parts we obtain (3.26). □

3.3. **Computing** $\lim_{t\to 0}\mathrm{ch}(\mathbb{B}_t^{1,2},\mathbb{B}_t^{2,3})$**.** The curvature $\mathcal{R}_i$ of the bundle $\mathcal{W}_i$ is a pure 2-form with restricted differential operator coefficients. It is therefore nilpotent. The Chern character form

$$e^{-\mathcal{R}_i} = \frac{i}{2\pi}\int_{c_0} e^{-\lambda}\left((\mathcal{R}_i-\lambda\mathbb{I})^{-1} - \mathcal{P}_i^{\perp}(-\lambda)^{-1}\right) d\lambda \ , \tag{3.28}$$

where c_0 is any contour around the origin, is therefore well-defined and is equal to the sum with only finitely many non-zero terms $\mathcal{P}_i + \sum_{k\geq 1}(-1)^k \frac{1}{k!}\mathcal{R}_i^k$.

For two choices of Grassmann section $\mathcal{P}_i, \mathcal{P}_j$ we have $e^{-\mathcal{R}_i} - e^{-\mathcal{R}_j} \in \mathcal{A}(B,\Psi^{-\infty}(\mathbb{E}^0))$ by the same argument as Lemma 3.2. We therefore have the well-defined relative Eta-form

$$\eta(\mathcal{P}_i,\mathcal{P}_j) = \mathrm{Tr}\,(e^{-\mathcal{R}_i} - e^{-\mathsf{R}_j}) \in \mathcal{A}^{2\bullet}(B) \tag{3.29}$$

and since $\frac{i}{2\pi}\int_{c_0} e^{-\lambda}\,(-\lambda)^{-k-1}\, d\lambda = (k!)^{-1}$,

$$\eta(\mathcal{P}_i,\mathcal{P}_j) = \mathrm{Tr}\,(\mathcal{P}_i - \mathcal{P}_j) + \sum_{k=1}^{\infty}(-1)^k\frac{1}{k!}\,\mathrm{Tr}\left(\mathcal{R}_i^k - \mathcal{R}_i^k\right) \ . \tag{3.30}$$

In particular, the degree 0 term is the pointwise relative index:

$$\eta(\mathcal{P}_i,\mathcal{P}_j)_{[0]} = \mathrm{Tr}\,(\mathcal{P}_i - \mathcal{P}_j) = \mathrm{ind}\,(D_{P_j}) - \mathrm{ind}\,(D_{P_i})\ , \tag{3.31}$$

where D_{P_i} is any global boundary problem in the family $\mathsf{D}_{\mathcal{P}_i}$.
We have:

Proposition 3.6. *The differential form* $\mathrm{ch}(\mathbb{B}_t^{1,2},\mathbb{B}_t^{2,3})$ *has a limit as* $t \longrightarrow 0$ *in* $\mathcal{A}(B)$ *given by*

$$\lim_{t\to 0} \mathrm{ch}(\mathbb{B}_t^{1,2},\mathbb{B}_t^{2,3}) = \eta(\mathcal{P}_1,\mathcal{P}_3)\ . \tag{3.32}$$

Proof. From (3.8) we have

$$\mathrm{ch}(\mathbb{B}_t^{1,2},\mathbb{B}_t^{2,3})$$

$$= \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}(-\lambda)^{-1}\, d\lambda\ .\ \mathrm{Str}(\mathcal{P}_{1,2} - \mathcal{P}_{2,3}) \tag{3.33}$$

$$+ \sum_{k=0}^{\dim B}(-1)^k \sum_{i=0}^{k} \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\ \mathrm{Str}\left[\left\{(tL_{2,3}^2 - \lambda\mathbb{I})^{-1}(\mathcal{R}_{2,3} + t^{1/2}\nabla L_{2,3})\right\}^i \right. \tag{3.34}$$

$$\left. \times \left((tL_{1,2}^2 - \lambda\mathbb{I})^{-1} - (tL_{2,3}^2 - \lambda\mathbb{I})^{-1}\right)\left\{(\mathcal{R}_{1,2} + t^{1/2}\nabla L_{1,2})(tL_{1,2}^2 - \lambda\mathbb{I})^{-1}\right\}^{k-i}\right] d\lambda$$

(3.35)

$$+ \sum_{k=1}^{\dim B}(-1)^k \sum_{j=0}^{k-1} \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\ \mathrm{Str}\left[\left\{(tL_{2,3}^2 - \lambda\mathbb{I})^{-1}(\mathcal{R}_{2,3} + t^{1/2}\nabla L_{2,3})\right\}^j (tL_{2,3}^2 - \lambda\mathbb{I})^{-1}\right.$$

$$\times \Big((\mathcal{R}_{1,2} - \mathcal{R}_{2,3}) + t^{1/2}(\nabla L_{1,2} - \nabla L_{2,3})\Big)$$

$$\left. \times \left\{(tL_{1,2}^2 - \lambda\mathbb{I})^{-1}(\mathcal{R}_{1,2} + t^{1/2}\nabla L_{1,2})(tL_{1,2}^2 - \lambda\mathbb{I})^{-1}\right\}^{k-j-1}\right] d\lambda\ .$$

Clearly, the first term (3.33) is the degree zero term (3.31). Next, (3.13) implies

$$(tL_{1,2}^2 - \lambda\mathbb{I})^{-1} - (tL_{2,3}^2 - \lambda\mathbb{I})^{-1} = t(tL_{1,2}^2 - \lambda\mathbb{I})^{-1}(L_{2,3}^2 - L_{1,2}^2)(tL_{2,3}^2 - \lambda\mathbb{I})^{-1}$$

is $O(|\lambda|^{-2}).t$ in any C^l norm as $t \longrightarrow 0$. It follows using (3.5) that the second term (3.34) is $o(1)$ as $t \longrightarrow 0$ and hence makes no contribution to (3.32).

Similarly, asymptotically as $t \longrightarrow 0$ the third term (3.35) is

$$\sum_{k\geq 1}\sum_{j=0}^{k-1}(-1)^k \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\ \Big(\mathrm{Str}(\mathcal{R}_{2,3}^j(\mathcal{R}_{1,2} - \mathcal{R}_{2,3})\mathcal{R}_{1,2}^{k-j-1})(-\lambda)^{-k-1}t^0 + O(|\lambda|^{-k-2}).t\Big)\ d\lambda$$

$$= \sum_{k\geq 1}\sum_{j=0}^{k-1} \frac{(-1)^k}{k!}\ \mathrm{Str}(\mathcal{R}_{2,3}^j(\mathcal{R}_{1,2} - \mathcal{R}_{2,3})\mathcal{R}_{1,2}^{k-j-1}) + o(1)$$

$$= \sum_{k\geq 1} \frac{(-1)^k}{k!}\ \mathrm{Tr}\,(\mathcal{R}_1^k - \mathcal{R}_3^k) + o(1)\ .$$

This completes the proof. □

One can prove a somewhat stronger property of the transgression form

$$\tau(\mathbb{B}_t^{1,2}, \mathbb{B}_t^{2,3}) = \mathrm{Str}(\dot{\mathbb{B}}_t^{1,2} e^{-\mathbb{F}_t^{1,2}} - \dot{\mathbb{B}}_t^{2,3} e^{-\mathbb{F}_t^{2,3}}) \ .$$

Namely:

Proposition 3.7. *As $t \longrightarrow 0$ there is an asymptotic expansion*

$$\tau(\mathbb{B}_t^{1,2}, \mathbb{B}_t^{2,3}) \sim \sum_{j\geq 0} \tau_j t^{j-1/2} \ , \tag{3.36}$$

where $\tau_j \in \mathcal{A}(B)$.

It follows that the integral in (3.27) converges as $t \longrightarrow 0$. Combined with Proposition 3.6 this implies:

Corollary 3.8. *For any $t > 0$ one has in $\mathcal{A}(B)$*

$$\mathrm{ch}(\mathbb{B}_t^{1,2}, \mathbb{B}_t^{2,3}) = \eta(\mathcal{P}_1, \mathcal{P}_3) - d_B \mathcal{T}_t \ ,$$

where $\mathcal{T}_t = \lim_{s\to 0} \int_s^t \tau(\mathbb{B}_s^{1,2}, \mathbb{B}_s^{2,3})\ ds$.

3.4. **Computing $\lim_{t\to\infty} \mathrm{ch}(\mathbb{B}_t^{1,2}, \mathbb{B}_t^{2,3})$.** To complete the proof of Theorem(I) first observe by the linearity of the supertrace

$$\eta(\mathcal{P}_1, \mathcal{P}_2) + \eta(\mathcal{P}_2, \mathcal{P}_3) = \eta(\mathcal{P}_1, \mathcal{P}_3) \ .$$

Hence from Proposition 2.5, we see that (1.12) follows from (1.13) on setting $\mathcal{P}_3 = \mathcal{P}(\mathsf{D}), \mathcal{P}_1 = \mathcal{P}$, since $\mathrm{ch}(\mathrm{Ind}\,(\mathsf{D}_{\mathcal{P}(\mathsf{D})})) = 0$ in cohomology. From the transgression formula (3.27) and and the small t limit (3.32), and by Proposition 2.5, the remaining component in the proof of the theorem is the following identification.

Proposition 3.9. *The cohomology class of* $\mathrm{ch}(\mathbb{B}_t^{1,2}, \mathbb{B}_t^{2,3})$ *is equal to the Chern character of the index bundle* $\mathrm{Ind}\,(\mathcal{P}_3, \mathcal{P}_1)$*.*

Proof. Broadly the proof follows that of [BGV], but transferred to the relative context. We begin by assuming that the dimensions of the kernels of the operators $L_{i,j}^z$ on the fibres $W_{i,z} \oplus W_{j,z}$ of the bundles $\mathcal{W}_i \oplus \mathcal{W}_j$ are constant as z varies in B. Let $\Pi_{i,j}^z$ be the orthogonal projection onto $\mathrm{Ker}(L_{i,j}^z)$ considered as a subspace of $\pi_*^N(\mathbb{E}^0 \oplus \mathbb{E}^0)$. Then $\Pi_{i,j}^0 = \{\Pi_{i,j}^z \mid z \in B\} \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^0 \oplus \mathbb{E}^0))$ and it follows that the kernels of the $L_{i,j}^z$ form a smooth finite-rank super vector bundle $\mathrm{Ker}(L_{i,j}) = \mathrm{ran}\,(\Pi_{i,j})$ endowed with the connection $\nabla_0^{i,j} = \Pi_{i,j}^0 \cdot \nabla^{\pi_*^N(\mathbb{E}^0\oplus\mathbb{E}^0)} \cdot \Pi_{i,j}^0$. Clearly,

$$\mathrm{Ker}(L_{i,j}) = \mathrm{Ker}(\mathcal{P}_i\mathcal{P}_j) \oplus \mathrm{Ker}((\mathcal{P}_i\mathcal{P}_j)^*)$$

relative to the grading of $\pi_*^N(\mathbb{E}^0) \oplus \pi_*^N(\mathbb{E}^0)$, while

$$\mathrm{ch}(\nabla_0^{i,j}) = \mathrm{ch}(\mathrm{Ind}\,(\mathcal{P}_j, \mathcal{P}_i)) \ , \tag{3.37}$$

since $\mathrm{Ind}\,(\mathcal{P}_j, \mathcal{P}_i) = \mathrm{Ker}(\mathcal{P}_i\mathcal{P}_j) - \mathrm{Ker}(\mathcal{P}_j\mathcal{P}_i)$, in this case, and $\mathrm{ch}(\nabla_0^{i,j}) = \mathrm{Str}((\nabla_0^{i,j})^2)$ is defined since the bundle is finite-rank.

Lemma 3.10. *For each C^l norm on B one has*

$$\lim_{t\longrightarrow\infty} \operatorname{ch}(\mathbb{B}_t^{1,2},\mathbb{B}_t^{2,3}) = \operatorname{ch}(\nabla_0^{1,3}) \tag{3.38}$$

and a constant $C(l)$ such that

$$\|\tau(\mathbb{B}_t^{1,2},\mathbb{B}_t^{2,3})\|_l \leq C(l)t^{-3/2} \ . \tag{3.39}$$

Proof. Consider the spaces of ψdo valued forms

$$\mathcal{M}_{i,j} = \mathcal{A}(B,\Psi^*(\mathcal{W}_i\oplus\mathcal{W}_j)) \qquad \text{and} \qquad \mathcal{N}_{i,j} = \mathcal{A}(B,\Psi^{-\infty}(\mathcal{W}_i\oplus\mathcal{W}_j))$$

where sections of $\Psi^*(\mathcal{W}_i\oplus\mathcal{W}_j)$ are families $\mathcal{P}_{i,j}\cdot\mathbb{T}\cdot\mathcal{P}_{i,j}$, where $\mathbb{T}$ is a smooth family of ψdos on $\pi_*^N(\mathbb{E}^0\oplus\mathbb{E}^0)$, that is, a section of $\Psi^*(\mathbb{E}^0\oplus\mathbb{E}^0)$, and similarly for $\Psi^{-\infty}(\mathcal{W}_i\oplus\mathcal{W}_j)$ but with $\mathbb{T}$ a smooth family of smoothing operators.

Let $\Pi_{i,j}^0,\Pi_{i,j}^1 \in \mathcal{A}(B,\Psi^0(\mathcal{W}_i\oplus\mathcal{W}_j))$ be the smooth families of orthogonal projections onto $\operatorname{Ker}(L_{i,j})$, $\operatorname{Ker}(L_{i,j})^\perp$ as subspaces of $\mathbb{E}^0\oplus\mathbb{E}^0$, or of $\mathcal{W}_i\oplus\mathcal{W}_j$. Since $L_{i,j}$ is the restriction of the smooth family of elliptic operators $\mathcal{P}_i\cdot(\mathcal{P}_i\cdot\mathcal{P}_j+\mathcal{P}_i^\perp\cdot\mathcal{P}_j^\perp)^{-1}\cdot\mathcal{P}_j$, then by elliptic regularity on closed manifolds we have $\Pi_{i,j}^0\in\mathcal{N}_{i,j}$. Relative to the decomposition $\operatorname{Ker}(L_{i,j})\oplus\operatorname{Ker}(L_{i,j})^\perp$ of $\mathcal{W}_i\oplus\mathcal{W}_j$ we thus have, with $\mathbb{F}^{i,j}=\mathbb{F}_{t=1}^{i,j}$,

$$\mathbb{F}^{i,j} = \begin{bmatrix}\Pi_{i,j}^0\mathbb{F}^{i,j}\Pi_{i,j}^0 & \Pi_{i,j}^0\mathbb{F}^{i,j}\Pi_{i,j}^1\\ \Pi_{i,j}^1\mathbb{F}^{i,j}\Pi_{i,j}^0 & \Pi_{i,j}^1\mathbb{F}^{i,j}\Pi_{i,j}^1\end{bmatrix} \in \begin{bmatrix}\mathcal{N}_{i,j}^2 & \mathcal{N}_{i,j}^1\\ \mathcal{N}_{i,j}^1 & \mathcal{M}_{i,j}^0\end{bmatrix} \ , \tag{3.40}$$

where $\mathcal{M}_{i,j},\mathcal{N}_{i,j}$ are filtered by

$$\mathcal{M}_{i,j}^k = \sum_{r\geq k}\mathcal{A}^r(B,\Psi^*(\mathcal{W}_i\oplus\mathcal{W}_j)), \qquad \mathcal{N}_{i,j}^k = \sum_{r\geq k}\mathcal{A}^r(B,\Psi^{-\infty}(\mathcal{W}_i\oplus\mathcal{W}_j)) \ .$$

Write $R_{i,j}^0 := (\nabla^{i,j})^2$. It is not quite true that the curvature form $R_{i,j}^0$ coincides with the 2-form part of $\Pi_{i,j}^0\mathbb{F}^{i,j}\Pi_{i,j}^0$, due to the Second Fundamental Form defined by the off-diagonal terms in (3.40), rather

$$R_{i,j}^0 = (\Pi_{i,j}^0\mathbb{F}^{i,j}\Pi_{i,j}^0)_{[2]} - (\Pi_{i,j}^0\mathbb{F}^{i,j}\Pi_{i,j}^1)_{[1]}G_{i,j}(\Pi_{i,j}^1\mathbb{F}^{i,j}\Pi_{i,j}^0)_{[1]} \ ,$$

where $G_{i,j}$ is the inverse of $L_{i,j}$ restricted to $\operatorname{Ker}(L_{i,j})^\perp$, and zero elsewhere. However, by the Diagonalization Lemma of [BV] there is an invertible

$$g_{i,j}\in\mathcal{M}=\mathcal{A}(B,\Psi^*(\mathbb{E}^0\oplus\mathbb{E}^0))$$

with, since $\Pi_{i,j}^0\in\mathcal{N}_{i,j}$, $g_{i,j}-\mathbb{I}\in\mathcal{N}_1$, where $\mathcal{N}_k$ is the standard filtration on $\mathcal{N} = \mathcal{A}(B,\Psi^{-\infty}(\mathbb{E}^0\oplus\mathbb{E}^0))$, such that

$$\mathbb{F}^{i,j} = g_{i,j}^{-1}\begin{bmatrix}U_{i,j} & 0\\ 0 & V_{i,j}\end{bmatrix}g_{i,j} = g_{i,j}^{-1}\cdot\Pi_{i,j}^0\cdot U_{i,j}\cdot\Pi_{i,j}^0\cdot g_{i,j} + g_{i,j}^{-1}\cdot\Pi_{i,j}^1\cdot V_{i,j}\cdot\Pi_{i,j}^1\cdot g_{i,j}$$

where

$$U_{i,j} = R_{i,j}^0 \mod \mathcal{N}_{i,j}^3 \ , \qquad V_{i,j} = \Pi_{i,j}^1\cdot L_{i,j}\cdot\Pi_{i,j}^1 \mod \mathcal{N}_1 \ .$$

Moreover, since $\mathbb{F}_t^{i,j} = t\delta_t(\mathbb{F}^{i,j})$, we have

$$\mathbb{F}_t^{i,j} = \delta_t(g_{i,j})^{-1}\begin{bmatrix}t\delta_t(U_{i,j}) & 0\\ 0 & t\delta_t(V_{i,j})\end{bmatrix}\delta_t(g_{i,j})$$

$$= \delta_t(g_{i,j})^{-1}\cdot\Pi_{i,j}^0\cdot t\delta_t(U_{i,j})\cdot\Pi_{i,j}^0\cdot\delta_t(g_{i,j}) + \delta_t(g_{i,j})^{-1}\cdot\Pi_{i,j}^1\cdot t\delta_t(V_{i,j})\cdot\Pi_{i,j}^1\cdot\delta_t(g_{i,j}) \tag{3.41}$$

with

$$\delta_t(g_{i,j})^{\pm 1} = \mathbb{I}_{i,j} + O(t^{-1/2}) \tag{3.42}$$

$$t\delta_t(U_{i,j}) = R^0_{i,j} + O(t^{-1/2}) \tag{3.43}$$

$$t\delta_t(V_{i,j}) = tL^2_{i,j} + t^{1/2}\mathcal{M}^1_{i,j} \ , \tag{3.44}$$

$\mathbb{I}_{i,j}$ the identity operator on $\mathcal{W}_i \oplus \mathcal{W}_j$. Here, a 1-parameter family of $A_t \in \mathcal{N}$, $t > 0$, is $O(f(t))$ for some positive function $f(t)$ if for any $\varepsilon > 0, l \in \mathbb{N}$ and $\phi \in C^\infty_c(B)$, one has $\|(\pi^N)^*(\phi)(x)A_t(x,y)\|_l \leq C_{l,\varepsilon,\phi} f(t)$ for $t > \varepsilon$, while $B_t \in t^\delta \mathcal{M}^k_{i,j}$ if $B_t = \sum_{r\geq 0} t^{\delta_r}\omega_{[k+r]}$ with $\delta_0 = \delta > \delta_r \searrow -\infty$ and $\omega_{[k+r]} \in \mathcal{M}^k_{i,j}$.

It follows from (3.41) and (3.6) that

$$(\mathbb{F}^{1,2}_t - \lambda\mathbb{I})^{-1}_{|\mathbb{E}^0} - (\mathbb{F}^{2,3}_t - \lambda\mathbb{I})^{-1}_{|\mathbb{E}^0} \tag{3.45}$$

$$= \delta_t(g_{1,2})^{-1}\cdot\Pi^0_{1,2}\cdot(t\delta_t(U_{1,2}-\lambda\mathbb{I}^0_{1,2})^{-1}\cdot\Pi^0_{1,2}\cdot\delta_t(g_{1,2}) - \delta_t(g_{2,3})^{-1}\cdot\Pi^0_{2,3}\cdot(t\delta_t(U_{2,3}-\lambda\mathbb{I}^0_{2,3})^{-1}\cdot\Pi^0_{2,3}\cdot\delta_t(g_{2,3})$$
$$+\delta_t(g_{1,2})^{-1}\cdot\Pi^1_{1,2}\cdot(t\delta_t(V_{1,2}-\lambda\mathbb{I}^1_{1,2})^{-1}\cdot\Pi^1_{1,2}\cdot\delta_t(g_{1,2}) - \delta_t(g_{2,3})^{-1}\cdot\Pi^1_{2,3}\cdot(t\delta_t(V_{2,3}-\lambda\mathbb{I}^1_{2,3})^{-1}\cdot\Pi^1_{2,3}\cdot\delta_t(g_{2,3}) \ .$$

where $\mathbb{I}^0_{i,j}, \mathbb{I}^1_{i,j}$ are the vertical identity operators on $\mathrm{Ker}(L_{i,j})$ and $\mathrm{Ker}(L_{i,j})^\perp$, respectively, and zero elsewhere, and $\delta_t(g_{i,j}) = \delta_t(g_{i,j})_{|\mathbb{E}^0}$ is included as an element of $\mathcal{M}$. The first two terms, involving the resolvents of $U_{i,j}$, are equal to

(3.46)

$$\delta_t(g_{1,2})^{-1}\cdot\Pi^0_{1,2}\cdot(R^0_{1,2}-\lambda\mathbb{I}^0_{1,2})^{-1}\cdot\Pi^0_{1,2}\cdot\delta_t(g_{1,2}) - \delta_t(g_{2,3})^{-1}\cdot\Pi^0_{2,3}\cdot(R^0_{2,3}-\lambda\mathbb{I}^0_{2,3})^{-1}\cdot\Pi^0_{2,3}\cdot\delta_t(g_{2,3})$$
$$+\delta_t(g_{1,2})^{-1}\cdot\Pi^0_{1,2}\cdot F^{[3]}_{1,2}(t^{1/2},\lambda)\cdot\Pi^0_{1,2}\cdot\delta_t(g_{1,2}) - \delta_t(g_{2,3})^{-1}\cdot\Pi^0_{2,3}\cdot F^{[3]}_{2,3}(t^{1/2},\lambda)\cdot\Pi^0_{2,3}\cdot\delta_t(g_{2,3}) \ ,$$

where the notation $F^{[m]}_{i,j}(t^\varepsilon,\lambda)$ means a function of the form

$$F^{[m]}_{i,j}(t^\varepsilon,\lambda) = \sum_{k=1}^{\dim B} t^{-\varepsilon k}(A_{i,j} - \lambda\mathbb{I}^0_{i,j})^{-1} w_{i,j}[t]$$

for some differential form valued bundle endomorphism $A_{i,j}$, and $w_{i,j}[t] \in t^0\mathcal{N}^m_{i,j}$ polynomial in t. From (3.42)

$$\delta_t(g_{i,j})^{\pm 1}\cdot\Pi^0_{i,j} = (\Pi^0_{i,j} + \Pi^1_{i,j})\delta_t(g_{i,j})^{\pm 1}\cdot\Pi^0_{i,j} = \Pi^0_{i,j} + \sum_{k=0}^{1}\Pi^k_{i,j}\cdot O(t^{-1/2})\cdot\Pi^0_{i,j} \ ,$$

$$\Pi^0_{i,j}\cdot\delta_t(g_{i,j})^{\pm 1} = \Pi^0_{i,j} + \sum_{k=0}^{1}\Pi^0_{i,j}\cdot O(t^{-1/2})\cdot\Pi^k_{i,j} \ .$$

Hence (3.46) has the form

$$\Pi^0_{1,2}\cdot(R^0_{1,2} - \lambda\mathbb{I}^0_{1,2})^{-1}\cdot\Pi^0_{1,2} - \Pi^0_{2,3}\cdot(R^0_{2,3} - \lambda\mathbb{I}^0_{2,3})^{-1}\cdot\Pi^0_{2,3} \tag{3.47}$$
$$+\sum_{i=1}^{2}\sum_{k=0}^{1}\Pi^k_{i,i+1}\cdot F^{[0]}_{i,i+1}(t^{-1/2},\lambda)\cdot\Pi^{1-k}_{i,i+1} + \Pi^0_{1,2}\cdot B^{[0]}_{1,2}(t^{-1},\lambda)\cdot\Pi^0_{1,2} - \Pi^0_{2,3}\cdot B^{[0]}_{2,3}(t^{-1},\lambda)\cdot\Pi^0_{2,3}$$

From (3.44)

$$\Pi^1_{i,j}\cdot(t\delta_t(V_{i,j}-\lambda\mathbb{I}_{i,j})^{-1}\cdot\Pi^1_{i,j} = \Pi^1_{i,j}\cdot(tL^2_{i,j}-\lambda\mathbb{I}_{i,j})^{-1}\cdot\Pi^1_{i,j} + \Pi^1_{i,j}\cdot C^{[0]}_{i,j}(t^{-1/2},\lambda)\cdot\Pi^1_{i,j} \ . \tag{3.48}$$

Hence we obtain from (3.45), (3.46), (3.47), (3.48)

$$e^{-\mathbb{F}^{1,2}_t} - e^{-\mathbb{F}^{2,3}_t} \tag{3.49}$$

$$= \Pi^0_{1,2}\cdot\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\,(R^0_{1,2}-\lambda\mathbb{I}_{1,2}))\,d\lambda\cdot\Pi^0_{1,2} - \Pi^0_{2,3}\cdot\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\,(R^0_{2,3}-\lambda\mathbb{I}_{2,3}))\,d\lambda\cdot\Pi^0_{2,3}$$

$$+\sum_{i=1}^{2}\sum_{k=0}^{1}\Pi^k_{i,i+1}\cdot\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\,F^{[0]}_{i,i+1}(t^{-1/2},\lambda)\,d\lambda\cdot\Pi^{1-k}_{i,i+1}$$

$$+\sum_{i=1}^{2}(-1)^{i+1}\delta_t(g_{i,i+1})^{-1}\Bigg(\Pi^1_{i,i+1}\cdot\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\,(tL^2_{i,i+1}-\lambda\mathbb{I}_{i,i+1})^{-1}\,d\lambda\cdot\Pi^1_{i,i+1}$$

$$+\Pi^1_{i,i+1}\cdot\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\,A_{i,i+1}(t^{-1/2},\lambda)\,d\lambda\cdot\Pi^1_{i,i+1}\Bigg)\,\delta_t(g_{i,i+1})$$

$$= \Pi^0_{1,2}\cdot e^{-R^0_{1,2}}\Pi^0_{1,2} - \Pi^0_{2,3}\cdot e^{-R^0_{2,3}}\Pi^0_{2,3} \tag{3.50}$$

$$+\sum_{i=1}^{2}\sum_{k=0}^{1}\Pi^k_{i,i+1}\cdot O(t^{-1/2})\cdot\Pi^{1-k}_{i,i+1}$$

$$+\delta_t(g_{1,2})^{-1}\cdot\Pi^1_{1,2}\cdot e^{-tL^2_{1,2}}\cdot\Pi^1_{1,2}\cdot\delta_t(g_{1,2}) - \delta_t(g_{2,3})^{-1}\cdot\Pi^1_{2,3}\cdot e^{-tL^2_{2,3}}\cdot\Pi^1_{2,3}\cdot\delta_t(g_{2,3})$$

$$+\delta_t(g_{1,2})^{-1}\cdot\Pi^1_{1,2}\cdot O(t^{-1/2})\Pi^1_{1,2}\cdot\delta_t(g_{1,2}) - \delta_t(g_{2,3})^{-1}\cdot\Pi^1_{2,3}\cdot O(t^{-1/2})\cdot\Pi^1_{2,3}\cdot\delta_t(g_{2,3})$$

The operators in the first line of (3.50) are finite-rank with supertraces $\mathrm{Str}(e^{-R^0_{i,j}})$. The terms in the second line are also finite-rank but with vanishing traces by symmetry and $\Pi^0_{i,j}\Pi^1_{i,j}=0$. The terms in the third and fourth line are not trace class, but the differences are trace class and of order $O(e^{-ct})$, some $c>0$, and $O(t^{-1/2})$ as $t\longrightarrow\infty$.

Totally as an element of $\mathcal{A}(B)$ we therefore obtain

$$\mathrm{ch}(\mathbb{B}^{1,2}_t,\mathbb{B}^{2,3}_t) = \mathrm{Str}(e^{-R^0_{1,2}}) - \mathrm{Str}(e^{-R^0_{2,3}}) + O(t^{-1/2})\ ,$$

proving equation (3.38).

For (3.39) we can revert for simplicity to the case of the canonical superconnections (the proof of (3.38) holds for any induced superconnection). Then clearly

$$\partial_t\mathbb{B}^{1,2}_t = \begin{cases} 0\ , & \text{on } \mathrm{Ker}(L_{i,j}) = \mathrm{ran}\,(\Pi^0_{i,j}) \\ t^{-1/2}L_{i,j}\ , & \text{on } \mathrm{Ker}(L_{i,j})^{\perp} \end{cases}$$

It is then immediate from (3.50) that the estimate (3.39) holds. □

Notice that it follows from Proposition 3.7 and (3.39) that the integral $\int_0^\infty \tau(\mathbb{B}^{1,2}_t,\mathbb{B}^{2,3}_t)\,dt$ exists, and we have then the transgression formula:

Corollary 3.11. *In* $\mathcal{A}(B)$

$$\mathrm{ch}(\mathrm{Ind}\,(\mathcal{P}_1,\mathcal{P}_3)) = \eta(\mathcal{P}_3,\mathcal{P}_1) - d_B\int_0^\infty \tau(\mathbb{B}^{1,2}_t,\mathbb{B}^{2,3}_t)\ dt\ .$$

The extension to the general case is obtained in the manner of [BGV], applying the above argument to a perturbed family $L_{i,j}\oplus Z_{i,j} : \mathcal{W}_i\oplus\mathcal{W}_j\oplus\mathbb{C}^N \longrightarrow \mathcal{W}_i\oplus\mathcal{W}_j$ with constant kernel dimension and whose index bundle defines the same class in $K(B)$ as $\mathrm{Ind}\,(\mathcal{P}_i,\mathcal{P}_j)$. □

4. Chern Character Forms from the Interior

In this Section we construct a superconnection directly on the bundle $\pi_*(\mathbb{E}|\mathcal{P})$ whose Chern character represents $\mathrm{ch}(\mathrm{Ind}\,(\mathsf{D}_{\mathcal{P}}))$.

Let $\mathbb{A}$ be the canonical superconnection (2.27) on $\pi_*(\mathbb{E})$. Let $\mathcal{P}$ be a Grassmann section for D, and let $\mathsf{P} = \mathsf{P}(\mathcal{P}) \in \mathcal{A}^0(B, \mathrm{End}(\pi_*(\mathbb{E})))$ be the corresponding family of projections (2.17) onto $\pi_*(\mathbb{E}|\mathcal{P})$ The induced superconnection $\mathsf{P}\cdot\mathbb{A}_t\cdot\mathsf{P}$ on the subbundle $\pi_*(\mathbb{E}|\mathcal{P})$ of $\pi_*(\mathbb{E})$ takes the form

$$\mathsf{P}\cdot\mathbb{A}_t\cdot\mathsf{P} = \nabla^{\pi_*(\mathbb{E}|\mathcal{P})} + t^{1/2}\mathsf{P}\cdot\mathsf{D}\cdot\mathsf{P} \tag{4.1}$$

where $\nabla^{\pi_*(\mathbb{E}|\mathcal{P})} = \mathsf{P}\cdot\nabla^{\pi_*(\mathbb{E})}\cdot\mathsf{P}$ is the induced connection on $\pi_*(\mathbb{E}|\mathcal{P})$, in the usual sense. The curvature of $\mathsf{P}\cdot\mathbb{A}_t\cdot\mathsf{P}$ is the 2-form in $\mathcal{A}^2(B, \mathrm{End}(\pi_*(\mathbb{E}|\mathcal{P})))$

$$(\mathsf{P}\cdot\mathbb{A}_t\cdot\mathsf{P})^2 = \mathsf{R}^{\pi_*(\mathbb{E}|\mathcal{P})} + t^{1/2}\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}(\mathsf{P}\cdot\mathsf{D}\cdot\mathsf{P}) + t(\mathsf{P}\cdot\mathsf{D}\cdot\mathsf{P})^2 \ . \tag{4.2}$$

Here $\mathsf{R}^{\pi_*(\mathbb{E}|\mathcal{P})} = (\nabla^{\pi_*(\mathbb{E}|\mathcal{P})})^2$ is the curvature 2-form of $\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}$, given by

$$\mathsf{R}^{\pi_*(\mathbb{E}|\mathcal{P})} = -\mathsf{P}\cdot\nabla^{\pi_*(\mathbb{E})}_{\sigma}\cdot\mathsf{P} + \mathsf{P}\cdot R^{M/B}\cdot\mathsf{P} - II^*_{\mathcal{P}}\wedge II_{\mathcal{P}} \ , \tag{4.3}$$

where $II_{\mathcal{P}} = \nabla^{\pi_*(\mathbb{E})}_{|\pi_*(\mathbb{E}|\mathcal{P})} - \nabla^{\pi_*(\mathbb{E}|\mathcal{P})}$ is the second fundamental form.

The formal construction of a heat operator on $\pi_*(\mathbb{E}|\mathcal{P})$ from this superconnection is problematic as $(\mathsf{P}\cdot\mathsf{D}\cdot\mathsf{P})^2$ is neither a differential operator nor formally self-adjoint. However, its restriction to the dense subbundle $\pi_*(\mathbb{E}|\mathcal{P}^3)$ coincides with the Dirac Laplacian

$$(\mathsf{P}\cdot\mathsf{D}\cdot\mathsf{P})^2_{|\pi_*(\mathbb{E}|\mathcal{P}^3)} = (D^2_{\mathcal{P}})_{|_{\pi_*(\mathbb{E}|\mathcal{P}^3)}} \ , \tag{4.4}$$

which is a self-adjoint differential operator on $\pi_*(\mathbb{E}|\mathcal{P}^2)$. Here, for each integer $k \geq 1$ we have a dense subbundle of $\pi_*(\mathbb{E})$

$$\pi_*(\mathbb{E}|\mathcal{P}^k) = \{\psi \in \pi_*(\mathbb{E}) \mid \mathcal{P}\gamma\mathsf{D}^i\psi = 0 \ , \ 0 \leq i \leq k-1\} \ . \tag{4.5}$$

Thus for $k = 1$ this is $\pi_*(\mathbb{E}|\mathcal{P})$, the domain of $\mathsf{D}_{\mathcal{P}}$, while $k = 2$ is the domain of the family of the Laplacian boundary problems

$$\mathsf{D}^2_{\mathcal{P}} : \mathcal{A}(B, \pi_*(\mathbb{E}|\mathcal{P}^2)) \longrightarrow \mathcal{A}(B, \pi_*(\mathbb{E})) \ .$$

The restriction of the curvature of $\mathsf{P}\cdot\mathbb{A}_t\cdot\mathsf{P}$ to $\pi_*(\mathbb{E}|\mathcal{P}^3)$ has range in $\pi_*(\mathbb{E}|\mathcal{P})$ equal there to the curvature of the superconnection with domain $\pi_*(\mathbb{E}|\mathcal{P})$

$$\mathbb{A}_{t,\mathcal{P}} := \nabla^{\pi_*(\mathbb{E}|\mathcal{P})} + t^{1/2}\mathsf{D}_{\mathcal{P}} \tag{4.6}$$

adapted to the family of global Dirac boundary problems $\mathsf{D}_{\mathcal{P}}$. One has

$$\mathbb{A}_{t,\mathcal{P}} = (\mathsf{P}\cdot\mathbb{A}_t\cdot\mathsf{P})_{|\pi_*(\mathbb{E}|\mathcal{P}^2)} \ .$$

Thus $\mathbb{A}_{t,\mathcal{P}}$ extends $(\mathsf{P}\cdot\mathbb{A}_t\cdot\mathsf{P})_{|\pi_*(\mathbb{E}|\mathcal{P}^2)}$ to $\pi_*(\mathbb{E}|\mathcal{P})$ with range $\pi_*(\mathbb{E})$. The curvature of $\mathbb{A}_{t,\mathcal{P}}$, defined by

$$\mathcal{F}_{t,\mathcal{P}} = (\mathbb{A}^2_{t,\mathcal{P}})_{|\pi_*(\mathbb{E}|\mathcal{P}^2)} \in \mathcal{A}(B, \mathrm{Hom}(\pi_*(\mathbb{E}|\mathcal{P}^2), \pi_*(\mathbb{E})))$$

is the vertical differential operator, supercommuting with action of $\mathcal{A}(B)$,

$$\mathcal{F}_{t,\mathcal{P}} = \mathsf{R}^{\pi_*(\mathbb{E}|\widehat{\mathbb{P}})} + t^{1/2}\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}\mathsf{D}_{\mathcal{P}} + t\mathsf{D}^2_{\mathcal{P}} : \mathcal{A}(B, \pi_*(\mathbb{E}|\mathcal{P}^2)) \longrightarrow \mathcal{A}(B, \pi_*(\mathbb{E})) \ , \tag{4.7}$$

(Recall here that $\mathsf{D}_{\mathcal{P}} = \mathsf{D}$ as an operator, the subscript just refers to the domain restriction.)

Proposition 4.1. *For any $\lambda \in \mathbb{C}\backslash\mathbb{R}_+$, $\mathcal{F}_{t,\mathcal{P}}$ has a resolvent*

$$(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1} \in \mathcal{A}(B, \mathrm{End}(\pi_*(\mathbb{E}))) \ , \tag{4.8}$$

with range in $\mathcal{A}(B, \pi_*(\mathbb{E}|\mathcal{P}^2))$, that is,

$$\mathcal{P}\gamma(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1} = 0 \ , \qquad \mathcal{P}\gamma D(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1} = 0 \ , \tag{4.9}$$

where $\mathbb{I}$ is the vertical identity operator on $\mathcal{A}(B, \pi_*(\mathbb{E}))$. One has

$$(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1} = \sum_{k=0}^{\dim B} (-1)^k (tD^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1} \Big((R + t^{1/2}\nabla^{\pi_*(\mathbb{E}|\mathcal{P})} D_{\mathcal{P}})(tD^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1} \Big)^k \ . \tag{4.10}$$

Proof. The first statement is just the statement that $(\mathcal{F}_{t,\mathcal{P}})_{[0]} = \mathsf{D}^2_{\mathcal{P}}$ is a family of self-adjoint generalized Laplacians and hence has a resolvent $(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1}$ for $\lambda \in \mathbb{C}\backslash\mathbb{R}_+$. Then

$$\begin{aligned} (\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1} &= (\mathsf{R} + t^{1/2}\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}\mathsf{D}_{\mathcal{P}} + t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1} \\ &= (t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1} \Big((\mathsf{R} + t^{1/2}\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}\mathsf{D}_{\mathcal{P}})(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1} + \mathbb{I} \Big)^{-1} \ , \end{aligned}$$

and since $(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1}$ has range in $\mathcal{A}(B, \pi_*(\mathbb{E}|\mathcal{P}^2))$, (4.9) follows. Since

$$(\mathsf{R}^{\pi_*(\mathbb{E}|\mathcal{P})} + t^{1/2}\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}\mathsf{D}_{\mathcal{P}})(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1} \in \mathcal{A}^1(B, \mathrm{End}(\pi_*(\mathbb{E})))$$

it is nilpotent of degree $\dim B$, consisting of terms which raise form degree by 1 or 2, and (4.10) is the corresponding finite Neumann expansion. □

Now $(\mathsf{R}^{\pi_*(\mathbb{E}|\mathcal{P})} + t^{1/2}\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}\mathsf{D}_{\mathcal{P}})(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1}$ is bounded in the operator norm , since the first factor is a first-order vertical differential operator, while $(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1}$ has Sobolev order -2, then (4.10) implies the estimate

$$\|(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1}\| = O(|\lambda|^{-1/2})$$

as $|\lambda| \to \infty$ (for details, see the proof of Proposition 4.14 where a similar fact is used), as an operator on $\mathcal{A}(M, \pi^*(\wedge T * B) \otimes \mathbb{E})$. Hence we can define the heat operator:

Definition 4.2. *For $t > 0$ the heat operator is the convergent contour integral*

$$e^{-\mathcal{F}_{t,\mathcal{P}}} := \frac{i}{2\pi} \int_{\mathcal{C}} e^{-\lambda} (\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1} \, d\lambda \tag{4.11}$$

where $\mathcal{C}$ is a contour surrounding the positive real axis, as in the Introduction.

$e^{-\mathcal{F}_{t,\mathcal{P}}}$ is convergent as a vertical operator in $C^0(B, \pi_*(\mathrm{End}(\mathbb{E})))$. More precisely, the *sup*-norm is defined via compact subsets C of B by $\|T\|_C = \sup\{\|T_z\| \mid z \in B\}$ on a vertical family of operators in $C^0(B, \pi_*(\mathrm{End}(\mathbb{E})))$. Using identities such as

$$\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}_{\xi}(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1} = -t(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1}(\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}_{\xi}\mathsf{D}^2_{\mathcal{P}})(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1}$$

we obtain from (4.10) the estimate as $\lambda \to \infty$

$$\|\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}_{\xi_1} \dots \nabla^{\pi_*(\mathbb{E}|\mathcal{P})}_{\xi_k}(\mathsf{R} + t^{1/2}\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}\mathsf{D}_{\mathcal{P}} + t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1}\|_C \leq a(k, C)t^k|\lambda|^{-k} \ ,$$

for any $\xi_1, \dots \xi_k \in \mathrm{Vect}(B)$ and a constant $a(k, C)$. This proves convergence in each C^l norm:

Proposition 4.3.
$$e^{-\mathcal{F}_{t,\mathcal{P}}} \in \mathcal{A}(B, \mathrm{End}(\pi_*(\mathbb{E}))) \ .$$

To see $e^{-\mathcal{F}_{t,\mathcal{P}}}$ has a well defined trace in $\mathcal{A}(B)$ we show it is a vertical 'smoothing' operator, in the following generalized sense.

4.1. **Generalized Smoothing Operators.** A smooth family of (generalized) smoothing operators is an element of the space $\mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}'))$ of vertical smoothing operators acting on $\mathcal{A}(M, \mathbb{E})$. Vertical means supercommuting with the action of $\mathcal{A}(B)$. A section of $\Gamma(M, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}'))$ is an operator of the form

$$\mathsf{K} = \mathsf{K}_X + \mathsf{G}^{-\infty} \ , \tag{4.12}$$

where K_X is a smooth family of interior smoothing operators, a family of ψdo in the usual sense, and $\mathsf{G}^{-\infty}$ is a smooth family of singular Green's operators (sgo) of order $-\infty$. Thus, K_X is defined by a kernel

$$< x_1|\mathsf{K}_X|x_2 > \in \Gamma(M \times_\pi M, \pi^*(\wedge T^*B) \otimes \mathbb{E} \boxtimes_\pi \mathbb{E}), \tag{4.13}$$

C^∞ up to the boundary ∂M–equivalently, it is the restriction of a vertical smoothing operator from the closed double of M.

A vertical sgo is a vertical operator of the form

$$\mathsf{G}^{-\infty} = \mathcal{K}\mathcal{T}\mathcal{G}_X \in \mathcal{A}(B, \mathrm{End}(\pi_*(\mathbb{E}))) \tag{4.14}$$

where

$$\mathcal{K} \in \mathcal{A}(B, \mathrm{Hom}(\pi_*^N(\mathbb{E}'), \pi_*(\mathbb{E}))) \tag{4.15}$$

$$\mathcal{T} = \sum_{0 \leq j \leq r} \mathsf{S}_j \gamma_j + \widetilde{\mathcal{T}} \in \mathcal{A}(B, \mathrm{Hom}(\pi_*(\mathbb{E})), \pi_*^N(\mathbb{E}')) \tag{4.16}$$

$$\mathcal{G}_X \in \mathcal{A}(B, \mathrm{End}(\pi_*(\mathbb{E}))) \tag{4.17}$$

are, respectively, smooth vertical families of *Poisson, trace*, and *interior singular Green's operators* of order m_1, m_2, m_3 and of total order $-\infty$. This means the following. For any local trivialization of $\pi_*(\mathbb{E})$ we have an identification

$$\mathcal{A}(U, \mathrm{End}(\pi_*(\mathbb{E}))) = \mathcal{A}(U) \times \mathrm{End}(\Gamma(X_{z_0}, E_{z_0})) \ . \tag{4.18}$$

Then $\mathsf{G}^{-\infty}$ is a smooth family of sgo if with respect to each such trivialization

$$\mathsf{G}^{-\infty}_{|U} = \sum_I G_I^z dz_I \tag{4.19}$$

where $G_I^z \in \mathrm{End}(\Gamma(X_{z_0}, E_{z_0}))$ is a sgo of order $-\infty$ in the classical single operator sense [G1] depending smoothly on $z \in U$. Similarly, $\mathcal{K}$ is a smooth family of Poisson operators if locally $\mathcal{K}_{|U} = \sum_I K_I^z dz_I$, where $K_I^z : \Gamma(\partial X_{z_0}, E^0_{z_0}) \longrightarrow \Gamma(X_{z_0}, E_{z_0})$ is a classical Poisson operator [G1] of order $m_1 \leq 0$, defined by a symbol kernel satisfying the estimates in [G4] and depending smoothly on z. The trace operator $\mathcal{T}$ takes the local form

$$\mathcal{T}_{|U} = \sum_I \sum_0^r (S^z_{j,I} \gamma_j^{z_0} + \widetilde{T}_I^z) dz_I \ , \tag{4.20}$$

where $\gamma_j^{z_0} s = \frac{\partial^j}{\partial r^j} s_{|\partial X_{z_0}}$ is the restriction map assigning the j^{th} normal derivative at ∂X_{z_0}, $S^z_{j,I}$ is the local form of a smooth family of boundary pseudodifferential operators $\mathsf{S}_j \in \mathcal{A}(B, \Psi(\mathbb{E}^{'}))$, and $\widetilde{T}^z_I : \Gamma(X_{z_0}, E_{z_0}) \longrightarrow \Gamma(\partial X_{z_0}, E^0_{z_0})$ a trace operator of order $m_2 \leq 0$ defined by a classical symbol kernel [G1] smooth in z. Finally, $\mathcal{G}_X = \sum_I G^z_{X,I} dz_I$ with $G^z_{X,I}$ a sgo operator of order $m_3 \leq 0$ defined by a classical symbol kernel [G2] depending smoothly on z.

If the orders of $\mathcal{K}, \mathsf{S}_j, \widetilde{\mathcal{T}}, \mathcal{G}_X$ are sufficiently negative they are given by integral operators with continuous kernels; thus if $m_1 \leq -\dim(X) - k$ then $\mathcal{K}$ is given by a kernel

$$< x|\mathcal{K}|y > \in \Gamma^k(M \times_\pi \partial M, \pi^*(\wedge T^*B) \otimes \mathbb{E} \boxtimes_\pi \mathbb{E}^{'}) \ ,$$

where $M \times_\pi \partial M = \{(x,y) \in M \times \partial M \mid \pi(x) = \pi^N(y)\}$, and so on. Since $\mathsf{G}^{-\infty}$ is of order $-\infty$ it extends to a bounded operator $\mathcal{A}^s(B, \pi_*(\mathbb{E})) \rightarrow \mathcal{A}^{s+d}(B, \pi_*(\mathbb{E}))$ on the Sobolev completions (Appendix 1) for any $d > 0$. It follows that $\mathsf{G}^{-\infty}$ is a vertical trace class operator. Hence there is a trace on $\Psi^{-\infty}(\mathbb{E}, \mathbb{E}^{'})$ over each fibre X_z of M defining a supertrace

$$\text{Str} : \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}^{'})) \longrightarrow \mathcal{A}(B) \ . \tag{4.21}$$

More precisely, by the boundary pseudodifferential operator calculus of [G1, G2] the cycled operator

$$\mathsf{K}_{\partial X} = \widetilde{\mathcal{T}} \mathcal{G}_X \mathcal{K} \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}^{'})) \tag{4.22}$$

is a smooth family of smoothing operators over the family of closed manifolds N in the usual sense [BGV] given by a C^∞ kernel

$$< y_1|\mathsf{K}_{\partial X}|y_2 > \in \Gamma^k(N \times_{\pi^N} N, (\pi^N)^*(\wedge T^*B) \otimes \mathbb{E}^{'} \boxtimes_{\pi^N} \mathbb{E}^{'}) \ .$$

Using the invariance of the trace with respect to cyclic permutation of products of bounded and trace-class operators, and Lidskii's Theorem, we therefore obtain:

Lemma 4.4. *The supertrace* (4.21) *of an operator $\mathsf{K} = \mathsf{K}_X + \mathsf{G}^{-\infty} \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}^{'}))$ is the differential form on B*

$$z \longmapsto \int_{X_z} \text{Str} < x|\mathsf{K}^z_X|x > + \int_{\partial X_z} \text{Str} < y|\mathsf{K}^z_{\partial X}|y > \ , \tag{4.23}$$

where $\mathsf{K}_{\partial X}$ is the cycled boundary operator (4.22).

The boundary trace of the singular Green's operator term in (4.23) has the consequence that the trace of the commutator of a differential operator with a smoothing operator is non-zero, in contrast to closed manifolds [BGV]:

Lemma 4.5. *Let $\mathbb{D} \in \mathcal{A}(B, \mathcal{D}(\mathbb{E}))$ be a vertical differential operator of order r and let $K \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}^{'}))$. Then*

[1] *$\mathbb{D}K$ and $K\mathbb{D}$ are elements of $\mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}^{'}))$.*

[2] *$[\mathbb{D}, K] = A\mathcal{K}\rho$, where $\mathcal{K}$ is a Poisson operator of order $-\infty$, $\rho = (\gamma_0, \gamma_1, \ldots, \gamma_{r-1})$, and A is a boundary Green's form.* Tr $([\mathbb{D}, K])$ *is the differential form on B*

$$z \longmapsto \int_{\partial X_z} \text{Tr}\,(A_z < y|\mathcal{K}|y >) \ dy \ . \tag{4.24}$$

For certain preferred classes of vertical smoothing operators the vanishing of the trace on such commutators can nevertheless be restored by imposing APS-boundary conditions $\mathcal{P}$ on D . The trangression formula for the Chern character, below, depends on this fact. The proof of Lemma 4.5 of this lemma is a straightforward application of Green's formula – see [DF] for the case of a Dirac operator.

4.2. **The Chern Character.**

Proposition 4.6. *The integral* (4.11) *converges in the* C^∞ *topology to an element of* $\mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}'))$. *In particular,* $e^{-\mathcal{F}_{t,\mathcal{P}}}$ *is trace class for each* $t > 0$. *Moreover, with* $\mathcal{F}_\mathcal{P} := \mathcal{F}_{1,\mathcal{P}}$, $\mathbb{H}(t) = e^{-t\mathcal{F}_\mathcal{P}}$ *is the unique solution to the heat equation along the fibre* X_z*:*

$$\left(\frac{d}{dt} + (\mathbb{A}^z_\mathcal{P})^2\right)\mathbb{H}(t)^z = 0 \tag{4.25}$$

subject to $\lim_{t\to 0}\mathbb{H}(t)^z\psi^z = \psi^z$ *for* $\psi^z \in \pi^*(\wedge T^*_z B)\otimes\Gamma(X_z, E_z)$. *One has*

$$e^{-\mathcal{F}_{t,\mathcal{P}}} = \delta_t(e^{-t\mathcal{F}_\mathcal{P}}) \ . \tag{4.26}$$

First, we require the following Lemma. For $k = 1, 2$ let $\mathsf{P}_k = \mathsf{P}(\mathcal{P}_k)$ be the families of projection operators associated to Grassmann sections $\mathcal{P}_k \in \mathcal{A}(B, \mathrm{End}(\pi^N_*(\mathbb{E}')))$ defining the subbundles $\pi_*(\mathbb{E}|\mathcal{P}_k)$ of $\pi_*(\mathbb{E})$. Let $\mathsf{R}_k = \mathsf{R}^{\pi_*(\mathbb{E}|\mathcal{P}_k)}$ be the vertical curvature forms.

Lemma 4.7.

$$P_1 - P_2 \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}')) \tag{4.27}$$

with supertrace

$$\mathrm{Str}(P_1 - P_2) = 2\,\mathrm{ind}\,(\mathcal{P}_2\mathcal{P}_1) \ . \tag{4.28}$$

$$(tD^2_{\mathcal{P}_1} - \lambda\mathbb{I})^{-1} - (tD^2_{\mathcal{P}_2} - \lambda\mathbb{I})^{-1} \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}')) \ . \tag{4.29}$$

For $k \in \mathbb{N}$

$$R^k_1 - R^k_2 \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}')) \ . \tag{4.30}$$

$$\nabla^{\pi_*(\mathbb{E}|\mathcal{P}_1)}\cdot D - \nabla^{\pi_*(\mathbb{E}|\mathcal{P}_2)}\cdot D \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}')) \ . \tag{4.31}$$

Proof. From (2.17)

$$\mathsf{P}_1 - \mathsf{P}_2 = \mathsf{K}_\chi(\mathcal{P}_2 - \mathcal{P}_1)\gamma \ , \tag{4.32}$$

where K_χ is a vertical Poisson operator, while $\mathcal{P}_2 - \mathcal{P}_1 \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}'))$. Hence $\mathsf{P}_1 - \mathsf{P}_2$ is a sgo operator of order $-\infty$. Cycling the trace we have

$$\begin{aligned} \mathrm{Str}(\mathsf{P}_1 - \mathsf{P}_2) &= \mathrm{Str}\left(\mathsf{K}_\chi \begin{bmatrix} \mathcal{P}_2 - \mathcal{P}_1 & 0 \\ & \Psi^{-1}(\mathcal{P}_2 - \mathcal{P}_1)\Psi \end{bmatrix}\gamma\right) \\ &= \mathrm{Tr}\,(\mathcal{P}_2 - \mathcal{P}_1) - \mathrm{Tr}\,(\Psi^{-1}(\mathcal{P}_2 - \mathcal{P}_1)\Psi) = 2\mathrm{Tr}\,(\mathcal{P}_2 - \mathcal{P}_1) \ . \end{aligned}$$

To see (4.29) we have from [G2]

$$(t\mathsf{D}^2_{\mathcal{P}_i} - \lambda\mathbb{I})^{-1} - (t\mathsf{D}^2 - \lambda\mathbb{I})^{-1}_+ = \mathcal{K}_\lambda S_{i,\lambda}\mathcal{P}_i\gamma_1(t\mathsf{D}^2 - \lambda\mathbb{I})^{-1}_+ \ , \tag{4.33}$$

where $(t\mathsf{D}^2 - \lambda\mathbb{I})^{-1}_+$ is the restriction to X of the inverse of the invertible double operator of $t\mathsf{D}^2 - \lambda\mathbb{I}$, $\mathcal{K}_\lambda$ is the classical vertical Poisson operator for $t\mathsf{D}^2 - \lambda\mathbb{I}$, and $S_{i,\lambda} \in \mathcal{A}(B, \Psi^0(\mathbb{E}'))$

is a vertical right inverse to $\mathcal{P}\circ P(\mathsf{D}^2)$. Hence the relative resolvent is a vertical sgo. Since $\mathcal{P}\circ P(\mathsf{D}^2)^\perp \in \mathcal{A}(B,\Psi^{-\infty}(\mathbb{E}'))$, and

$$\mathcal{P}\gamma_1(t\mathsf{D}^2-\lambda\mathbb{I})_+^{-1} = \mathcal{P}(\mathsf{D}^2))^\perp\gamma_1(t\mathsf{D}^2-\lambda\mathbb{I})_+^{-1}$$

it is of order $-\infty$, and (4.29) follows.

It is enough to prove (4.30) locally over $U\subset B$. Since $\nabla^{\pi_*(\mathbb{E}|\mathcal{P}_i)} = \mathsf{P}_i\cdot\nabla^{\pi_*(\mathbb{E})}\cdot\mathsf{P}_i$ we have

$$\nabla^{\pi_*(\mathbb{E}|\mathcal{P}_i)}_{|U} = \mathsf{P}_i^z\cdot d_U\cdot\mathsf{P}_i^z + \sum_{j=1}^{\dim B}\mathsf{P}_i^z\cdot\mathcal{D}_j^z\cdot\mathsf{P}_i^z dz_j\ .$$

$\mathcal{D}_j^z$ are first-order differential operators on $\Gamma(X_{z_0},E_{z_0})$. The P_i^z are defined with respect to elements of the fixed Grassmannian $\mathrm{Gr}_\infty(\partial_{z_0})$ depending smoothly on z and defining $\mathsf{P}_i^z : U\longrightarrow\mathcal{A}(U)\otimes\Gamma(\partial X_{z_0},\mathrm{End}(E^0_{z_0}))$. Since $\mathsf{P}_i^z = I-\mathsf{K}_i^z\mathcal{P}_i^z\gamma$

$$d\mathsf{P}_i^z = -(d\mathsf{K}_i^z)\mathcal{P}_i^z\gamma - \mathsf{K}_i^z(d\mathcal{P}_i^z)\gamma\in\mathcal{A}^1(U,\Psi^{-\infty}(E_{z_0},E^0_{z_0}))\ ,$$

is a smooth family of (generalized) smoothing operators. For,

$$d\mathsf{K}_i^z = \chi(u)\sum_{j=0}^{\dim B}\frac{\partial}{\partial z_j}e^{-u(\partial^z)^2}dz_i$$

and by Duhamel's formula $\partial_j e^{-u(\partial^z)^2}\in\Gamma(U,\Psi^{-\infty}(E^0_{z_0}\oplus E^0_{z_0})))$, while for fixed $\mathsf{P}_0=\mathsf{P}(\mathcal{P}_0)$ we have

$$d\mathsf{P}_i^z = d(\mathsf{P}_i^z-\mathsf{P}_0)\in\mathcal{A}(U,\Psi^{-\infty}(\mathbb{E},\mathbb{E}'))\ ,$$

since $\mathsf{P}_i^z-\mathsf{P}_0\in\mathcal{A}(U,\Psi^{-\infty}(\mathbb{E},\mathbb{E}'))$.

The curvature can be computed by a local formula similar to that in Lemma 3.2, consisting of elements in $\mathcal{A}(U,\Psi^{-\infty}(\mathbb{E},\mathbb{E}'))$ and non-smoothing terms but whose difference in $\mathsf{R}_1-\mathsf{R}_2$ are in $\mathcal{A}(U,\Psi^{-\infty}(\mathbb{E},\mathbb{E}'))$. The details are similar to Lemma 3.2. Equation (4.31) follows by an analogous argument. □

Proof of Proposition 4.6:

Proof. Integrating by parts in (4.11) we have

$$e^{-\mathcal{F}_{t,\mathcal{P}}} := \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\partial_\lambda^m(\mathcal{F}_{t,\mathcal{P}}-\lambda\mathbb{I})^{-1}\,d\lambda\ . \tag{4.34}$$

By (4.10), $\partial_\lambda^m(\mathcal{F}_{t,\mathcal{P}}-\lambda\mathbb{I})^{-1}$ is a sum of terms of the form

$$\partial_\lambda^{j_1}G_{t,\lambda}^{-1}F\partial_\lambda^{j_2}G_{t,\lambda}\dots F\partial_\lambda^{j_k}G_{t,\lambda}\ ,$$

where $G_{t,\lambda}^{-1} = (t\mathsf{D}^2_{\mathcal{P}}-\lambda\mathbb{I})$, and $F\in\mathcal{A}^2(B,\mathcal{D}(\mathbb{E}))$ is a 2-form with first-order differential operator coefficients. From sgo theory on manifolds with boundary [G1, G2] and equations (4.10) and (4.33), and Lemma 4.5 [1], Lemma 4.7, we find that $\partial_\lambda^m(\mathcal{F}_{t,\mathcal{P}}-\lambda\mathbb{I})^{-1}$ is a vertical sgo consisting of terms of order $-2m-2-k$ with $k\geq 0$. Since this is so for any m, from (4.34) $e^{-\mathcal{F}_{t,\mathcal{P}}}$ is a vertical sgo of order $-\infty$. In particular it is trace class with a generalized smoothing kernel. A detailed study of form degree zero term is given in §4.2 of [G1].

Since $(t\mathcal{F}_{\mathcal{P}}-\lambda\mathbb{I})^{-1}$ has range in $\mathcal{A}(B,\pi_*(\mathbb{E}|\mathcal{P}^2))$, then $\mathcal{F}(t\mathcal{F}_{\mathcal{P}}-\lambda\mathbb{I})^{-1} = \mathcal{F}_{\mathcal{P}}(t\mathcal{F}_{\mathcal{P}}-\lambda\mathbb{I})^{-1}$ and hence

$$\begin{aligned}\partial_t(t\mathcal{F}_{\mathcal{P}}-\lambda\mathbb{I})^{-1} &= -t(t\mathcal{F}_{\mathcal{P}}-\lambda\mathbb{I})^{-1}\mathcal{F}_{\mathcal{P}}(t\mathcal{F}_{\mathcal{P}}-\lambda\mathbb{I})^{-1}\\ &= -\mathcal{F}_{\mathcal{P}}(t\mathcal{F}_{\mathcal{P}}-\lambda\mathbb{I})^{-2}\ .\end{aligned}$$

Hence

$$\partial_t e^{-t\mathcal{F}_{\mathcal{P}}} = -\int_C e^{-\lambda}\mathcal{F}_{\mathcal{P}}(t\mathcal{F}_{\mathcal{P}} - \lambda\mathbb{I})^{-2}\, d\lambda = -\mathcal{F}_{\mathcal{P}} e^{-t\mathcal{F}_{\mathcal{P}}}$$

integrating by parts for the second equality. Since $\mathcal{F}_{\mathcal{P}} = \mathbb{A}^2_{\mathcal{P}}$ on $\mathcal{A}(B, \pi_*(\mathbb{E}|\mathcal{P}^2))$, this proves (4.25). A similar argument shows that $\|(e^{-t\mathcal{F}_{\mathcal{P}}} - e^{-t})\psi\|_s \to 0$ as $t \to 0$ in each Sobolev norm and hence the boundary condition at $t = 0$ holds. This completes the proof. □

We can now define the *Chern character form* of the superconnection $\mathbb{A}_{t,\mathcal{P}}$ adapted to the family of APS-type boundary problems $\mathsf{D}_{\mathcal{P}}$ to be the differential form in $\mathcal{A}^{\text{even}}(B)$

$$\text{(4.35)} \qquad \operatorname{ch}(\mathbb{A}_{t,\mathcal{P}}) = \operatorname{Str}\left(e^{-\mathcal{F}_{t,\mathcal{P}}}\right) \ .$$

For later use, observe that for large m, $\partial^m_\lambda(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1}$ is trace class and hence from (4.10), (4.34)

$$\text{(4.36)} \qquad \operatorname{ch}(\mathbb{A}_{t,\mathcal{P}}) = \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\operatorname{Str}(\partial^m_\lambda(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1})\, d\lambda$$

$$= \sum_{k=0}^{\dim B}(-1)^k\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\operatorname{Str}\left[\partial^m_\lambda\left((t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1}\left((\mathsf{R}^{\pi_*(\mathbb{E}|\mathcal{P})} + t^{1/2}\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}\mathsf{D}_{\mathcal{P}})(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1}\right)^k\right)\right]\, d\lambda\ .$$

4.3. **Transgression Formula.** Because we are considering Dirac operators over a manifold with boundary with domain restricted by the Grassmann section the supertrace still vanishes on commutators of APS-type boundary problems with smoothing operators which have the correct range. This gives us a direct analogue of the transgression formula for closed manifolds [BV, BGV, B].

Theorem 4.8.

[1] *The differential form* $\operatorname{ch}(\mathbb{A}_{t,\mathcal{P}})$ *on* B *is closed.*

[2] *The cohomology class defined by* $\operatorname{ch}(\mathbb{A}_{t,\mathcal{P}})$ *is independent of the parameter* t*. One has*

$$\text{(4.37)} \qquad \frac{d}{dt}\operatorname{ch}(\mathbb{A}_{t,\mathcal{P}}) = -d_B\operatorname{Str}\left(\dot{\mathbb{A}}_{t,\mathcal{P}}\, e^{-\mathcal{F}_{t,\mathcal{P}}}\right)\ ,$$

where $\dot{\mathbb{A}}_{t,\mathcal{P}} = (d/dt)\mathbb{A}_{t,\mathcal{P}}$*. Equivalently, for* $0 < t < T < \infty$,

$$\text{(4.38)} \qquad \operatorname{ch}(\mathbb{A}_{T,\mathcal{P}}) - \operatorname{ch}(\mathbb{A}_{t,\mathcal{P}}) = -d_B\int_t^T \operatorname{Str}\left(\dot{\mathbb{A}}_{s,\mathcal{P}}\, e^{-\mathcal{F}_{s,\mathcal{P}}}\right) ds\ .$$

More generally, $\operatorname{ch}(\mathbb{A}_{\sigma,\mathcal{P}})$ *is invariant for a general family* $\mathbb{A}_{\sigma,\mathcal{P}}$ *for variations that leave unchanged the symbol of the boundary family* D_Y *of Dirac operators, but it is not a general homotopy invariant.*

First, we need the following property.

Lemma 4.9. *Let* X *be a compact manifold with boundary and let* D *be an elliptic operator of Dirac type acting on* $\Gamma(X, E)$*. Let* $P \in \operatorname{Gr}_\infty(D_{|\partial X})$ *be an APS-type boundary condition* *for D. Then if there exists a generalized smoothing operator K_P on $\Gamma(X,E)$ such that for all non-negative integers $k = 0, 1, \ldots$*

$$P\gamma D^k K_P = 0 \ , \tag{4.39}$$

then for any differential operator A acting on $\Gamma(X,E)$

$$\mathrm{Str}\,([A, K_P]) = 0 \ . \tag{4.40}$$

Proof. The reason for this property is that D_P has both a left and a right parametrix. Let us briefly explain this further. Suppose to begin with that D_P is invertible. Then one has

$$D\cdot D_P^{-1} = I, \qquad D_P^{-1}\cdot D = I - \mathcal{K}^{-\infty}S(P)^{-1}P\gamma \ . \tag{4.41}$$

The second identity means D has no left inverse, here $\mathcal{K}^{-\infty}$ is the Poisson operator for D, which is the source of Green's theorem and the non-commutativity of the trace in general for compositions of differential and smoothing operators. On the other hand, $I = D{\cdot}D_P^{-1} = D_P\cdot D_P^{-1}$, so D_P^{-1} is also a right inverse for D_P. Further, since $\mathcal{K}^{-\infty}S(P)^{-1}P\gamma = 0$ on $\mathrm{dom}\,(D_P)$ then $D_P^{-1}\cdot D_P = I_{|\mathrm{dom}\,(D_P)}$, and so it is a two-sided inverse. The sufficient condition (4.39) allows this to be repeated this with all higher powers D_P^k (with domain $\mathrm{dom}\,(D_P^k) = \{\psi\in\Gamma(X,E) \mid P\gamma D^i\psi = 0,\ 0\leq i\leq k-1\}$) and hence sufficiently bound the Sobolev norm of A. Precisely, choosing $N > \mathrm{ord}(A)$ we have that AD_P^{-N} is bounded in the operator norm on $\Gamma(X,E)$ defined by metrics on E and TX (– more is true: by taking N sufficiently large we can ensure a C^r kernel). Now we have

$$I_{|\mathrm{dom}\,(D_P^N)} = D_P^{-N}D^N_{|\mathrm{dom}\,(D_P^N)} = D_P^{-N}D_P^N \ , \tag{4.42}$$

while taking $k = N-1$ in (4.39) we have $\mathrm{ran}\,(K_P)\subset\mathrm{dom}\,(D_P^N)$, and hence

$$D_P^{-N}D^N K_P = K_P \ .$$

Since by Lemma 4.5 [1] $D^N K_P$ is a generalized smoothing operator, we can now follow the same argument as [BGV] Prop(2.48)

$$\mathrm{Str}(AK_P) = \mathrm{Str}\left((AD_P^{-N})D^N K_P\right) = \mathrm{Str}\left((D^N K_P A)D_P^{-N}\right)$$

$$= \mathrm{Str}\left((D_P^{-N}D^N)K_P A\right) = \mathrm{Str}(K_P A),$$

using the commutativity of the trace for combinations of bounded and smoothing operators.

If D_P is not invertible, then the argument obviously extends by replacing the inverse by a parametrix Q_P (so $\mathrm{ran}\,(Q_P)\subset\mathrm{dom}\,(D_P^N)$ and $Q_P D_P - I$ and $I - DQ_P$ are generalized smoothing operators). □

We prove Theorem 4.8 for *any* superconnection adapted to $\mathsf{D}_{\mathcal{P}}$:

Proof. The essential point is to show that for $k = 0, 1, \ldots$

$$\mathcal{P}\gamma\mathsf{D}^{2k}e^{-\mathcal{F}_{t,\mathcal{P}}} = 0 \ . \tag{4.43}$$

We have $t\mathsf{D}^2(t\mathsf{D}^2 - \lambda\mathbb{I})^{-1} = \mathbb{I} + \lambda(t\mathsf{D}^2 - \lambda\mathbb{I})^{-1}$ on $\mathcal{A}(B,\pi_*(\mathbb{E}))$ and hence

$$\mathsf{D}^2 e^{-t\mathsf{D}^2_{\mathcal{P}}} = \frac{i}{2\pi}\int_{\mathcal{C}} t^{-1}\lambda e^{-\lambda}(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1}\,d\lambda \ ,$$

and iteratively

$$\mathsf{D}^{2k}e^{-t\mathsf{D}^2_{\mathcal{P}}} = \frac{i}{2\pi}\int_{\mathcal{C}} t^{-k}\lambda^k e^{-\lambda}(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1}\, d\lambda\ .$$

Hence, by (4.9), for any $t > 0$ and non-negative integer k

$$\mathcal{P}\gamma\mathsf{D}^{2k}e^{-t\mathsf{D}^2_{\mathcal{P}}} = \frac{i}{2\pi}\int_{\mathcal{C}} t^{-k}\lambda^k e^{-\lambda}\mathcal{P}\gamma(t\mathsf{D}^2_{\mathcal{P}} - \lambda\mathbb{I})^{-1}\, d\lambda = 0\ . \tag{4.44}$$

On the other hand Duhamel's formula gives

$$e^{-t\mathcal{F}_{\mathcal{P}}} = e^{-t\mathsf{D}^2_{\mathcal{P}}} + \sum_{k=1}^{\dim B}(-t)^k I_{k,t}\ , \tag{4.45}$$

where

$$I_{k,t} = \int_{\sigma_k} e^{-s_0\mathsf{D}^2_{\mathcal{P}}}(\mathcal{F}_{\mathcal{P}} - \mathsf{D}^2_{\mathcal{P}})e^{-s_1\mathsf{D}^2_{\mathcal{P}}}(\mathcal{F}_{\mathcal{P}} - \mathsf{D}^2_{\mathcal{P}})\ldots e^{-s_{k-1}\mathsf{D}^2_{\mathcal{P}}}(\mathcal{F}_{\mathcal{P}} - \mathsf{D}^2_{\mathcal{P}})e^{-s_k\mathsf{D}^2_{\mathcal{P}}}\, d\sigma_k\ ,$$

and σ_k is the standard k-simplex. From (4.44), (4.45) we therefore have $\mathcal{P}\gamma\mathsf{D}^{2k}e^{-t\mathcal{F}_{\mathcal{P}}} = 0$, and (4.43) then follows from (4.26). (Equivalently, one may use (4.44) and (4.10).)

Now let $\mathbb{A}$ be any superconnection on $\pi_*(\mathbb{E})$. Locally, $\mathbb{A}_{|U} = d_U + \sum_I D^z_I dz_I$, while $e^{-\mathcal{F}_{t,\mathcal{P}}} = \sum_J K(\mathcal{P})^z_J dz_J$, with $K(\mathcal{P})^z_J$ a generalized smoothing operator satisfying

$$\mathcal{P}\gamma D^{2k}K(\mathcal{P})^z_J = 0\ . \tag{4.46}$$

Hence by Lemma 4.9

$$\mathrm{Str}([\sum_I D^z_I dz_I, \sum_J K(\mathcal{P})^z_J dz_J]) = \sum_{I,J}\mathrm{Str}([D^z_I, K(\mathcal{P})^z_J])\, dz_I\wedge dz_J = 0\ . \tag{4.47}$$

Thus $\mathrm{Str}([\mathbb{A}_{|U}, e^{-\mathcal{F}_{t,\mathcal{P}}}]) = \mathrm{Str}([d_U, e^{-\mathcal{F}_{t,\mathcal{P}}}]) = d_U\mathrm{Str}(e^{-\mathcal{F}_{t,\mathcal{P}}})$, implying the global formula

$$d_B\mathrm{Str}(e^{-\mathcal{F}_{t,\mathcal{P}}}) = \mathrm{Str}([\mathbb{A}, e^{-\mathcal{F}_{t,\mathcal{P}}}])\ . \tag{4.48}$$

To see the right-side vanishes we show it is equal to $\mathrm{Str}([\mathbb{A}_{t,\mathcal{P}}, e^{-\mathcal{F}_{t,\mathcal{P}}}])$ – the potential obstruction to this comes from the second fundamental form $II_{\mathcal{P}} = \nabla^{\pi_*(\mathbb{E})}_{|\pi_*(\mathbb{E}|\mathcal{P})} - \nabla^{\pi_*(\mathbb{E}|\mathcal{P})}$. It is enough to establish this locally, and by the above discussion it is sufficient to prove:

Lemma 4.10.

$$\mathrm{Str}([\mathsf{P}\cdot d\cdot \mathsf{P}, e^{-\mathcal{F}_{t,\mathcal{P}}}]) = \mathrm{Str}([d, e^{-\mathcal{F}_{t,\mathcal{P}}}])\ .$$

Proof. For any $\alpha\in\mathcal{A}(B,\Psi^{-\infty}(\mathbb{E},\mathbb{E}'))$ one has $\mathrm{Str}(\alpha) = \mathrm{Str}(\mathsf{P}\alpha\mathsf{P}) + \mathrm{Str}(\mathsf{P}^\perp\alpha\mathsf{P}^\perp)$. Hence, since $e^{-\mathcal{F}_{t,\mathcal{P}}} = \mathsf{P}e^{-\mathcal{F}_{t,\mathcal{P}}}$ is even, with $d_U^{\pi_*(\mathbb{E}|\mathcal{P})} := \mathsf{P}\cdot d_U\cdot\mathsf{P}$,

$$\begin{aligned}
\mathrm{Str}([d_U, e^{-\mathcal{F}_{t,\mathcal{P}}}]) &= \mathrm{Str}(d_U\cdot\mathsf{P}\cdot e^{-\mathcal{F}_{t,\mathcal{P}}} - \mathsf{P}\cdot e^{-\mathcal{F}_{t,\mathcal{P}}}\cdot d_U)\\
&= \mathrm{Str}(\mathsf{P}\cdot d_U\cdot \mathsf{P}e^{-\mathcal{F}_{t,\mathcal{P}}}\cdot\mathsf{P} - \mathsf{P}\cdot e^{-\mathcal{F}_{t,\mathcal{P}}}(P+P^\perp)d_U\cdot\mathsf{P}\\
&\quad + \mathsf{P}^\perp\cdot d_U\cdot\mathsf{P}e^{-\mathcal{F}_{t,\mathcal{P}}}\cdot\mathsf{P}^\perp - \mathsf{P}^\perp\mathsf{P}\cdot e^{-\mathcal{F}_{t,\mathcal{P}}}(P+P^\perp)d_U\cdot\mathsf{P}^\perp)\\
&= \mathrm{Str}([d_U^{\pi_*(\mathbb{E}|\mathcal{P})}, e^{-\mathcal{F}_{t,\mathcal{P}}}]) + \mathrm{Str}([\mathsf{P}^\perp d\mathsf{P}, \mathsf{P}e^{-\mathcal{F}_{t,\mathcal{P}}}\mathsf{P}^\perp])\\
&= \mathrm{Str}([d_U^{\pi_*(\mathbb{E}|\mathcal{P})}, e^{-\mathcal{F}_{t,\mathcal{P}}}])\ ,
\end{aligned}$$

since Str vanishes on supercommutators in $\mathcal{A}(B,\mathrm{End}(\pi_*(\mathbb{E}))$. □

We conclude
(4.49)
$$d_B\mathrm{Str}(e^{-\mathcal{F}_{t,\mathcal{P}}}) = \mathrm{Str}([\mathbb{A}_{t,\mathcal{P}}, e^{-\mathcal{F}_{t,\mathcal{P}}}]) = \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\,\mathrm{Str}\left((\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1}[\mathbb{A}_{t,\mathcal{P}}, \mathcal{F}_{t,\mathcal{P}}](\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1}\right)\,d\lambda$$

which vanishes since $\mathcal{F}_{t,\mathcal{P}} = \mathbb{A}^2_{t,\mathcal{P}}$ on the dense subbundle $\pi_*(\mathbb{E}|\mathcal{P}^2)$ of $\pi_*(\mathbb{E})$.

To prove [2], from (4.34) we have that

$$\begin{aligned}\frac{d}{dt}e^{-\mathcal{F}_{t,\mathcal{P}}} &= \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\,\partial_t\partial^m_\lambda(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1}\,d\lambda\\ &= -\frac{i}{2\pi}m!\int_{\mathcal{C}} e^{-\lambda}\,(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-m-1}\partial_t((\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{m+1})(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-m-1}\,d\lambda\ ,\end{aligned}$$

noting that the domains match-up; that is, $\mathcal{F}^m_t(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-m-1} = \mathcal{F}^m_{t,\mathcal{P}}(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-m-1}$. Similarly, since the variation is interior $\mathcal{P}\gamma\dot{\mathcal{F}}_t = 0$, so $\dot{\mathcal{F}}_t = \dot{\mathcal{F}}_{t,\mathcal{P}}$. Hence

$$\partial_t(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{m+1} = \sum_{k=0}^{m}(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{m-k}\dot{\mathcal{F}}_{t,\mathcal{P}}(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^k\ ,$$

and so

$$\frac{d}{dt}e^{-\mathcal{F}_{t,\mathcal{P}}} = -\frac{i}{2\pi}m!\sum_{k=0}^{m}\int_{\mathcal{C}} e^{-\lambda}\,(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-k-1}\dot{\mathcal{F}}_{t,\mathcal{P}}(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{k-m-1}\,d\lambda\ .$$

For $m > 4n$, at least one of the vertical operators $(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-k-1}$ or $\dot{\mathcal{F}}_{t,\mathcal{P}}(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{k-m-1}$ is trace class, while the other is bounded in the operator norm. Hence we can swap the order of the operators in the trace to obtain

$$\begin{aligned}\frac{d}{dt}\mathrm{Str}(e^{-\mathcal{F}_{t,\mathcal{P}}}) &= -\frac{i}{2\pi}m!\sum_{k=0}^{m}\int_{\mathcal{C}} e^{-\lambda}\,\mathrm{Str}(\dot{\mathcal{F}}_{t,\mathcal{P}}(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-m-2})\,d\lambda\\ &= -\frac{i}{2\pi}m!\sum_{k=0}^{m}\int_{\mathcal{C}} e^{-\lambda}\,\mathrm{Str}(\dot{\mathcal{F}}_{t,\mathcal{P}}\,\partial^{m+1}_\lambda(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1})\,d\lambda\\ &= -\,\mathrm{Str}(\dot{\mathcal{F}}_{t,\mathcal{P}}\,e^{-\mathcal{F}_{t,\mathcal{P}}})\ .\end{aligned}$$

Since $\mathcal{F}_{t,\mathcal{P}} = \mathbb{A}^2_{t,\mathcal{P}}$ on the dense subbundle, we have

(4.50) $$\dot{\mathcal{F}}_{t,\mathcal{P}}\,e^{-\mathcal{F}_{t,\mathcal{P}}} = \mathbb{A}_{t,\mathcal{P}}\dot{\mathbb{A}}_{t,\mathcal{P}}\,e^{-\mathcal{F}_{t,\mathcal{P}}} + \dot{\mathbb{A}}_{t,\mathcal{P}}\mathbb{A}_{t,\mathcal{P}}\,e^{-\mathcal{F}_{t,\mathcal{P}}} = \mathbb{A}_t\dot{\mathbb{A}}_{t,\mathcal{P}}\,e^{-\mathcal{F}_{t,\mathcal{P}}} + \dot{\mathbb{A}}_{t,\mathcal{P}}\,e^{-\mathcal{F}_{t,\mathcal{P}}}\mathbb{A}_t\ .$$

Here we commute $\mathbb{A}_{t,\mathcal{P}}$ and $e^{-\mathcal{F}_{t,\mathcal{P}}}$ since $\mathbb{A}_{t,\mathcal{P}}(\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1} = (\mathcal{F}_{t,\mathcal{P}} - \lambda\mathbb{I})^{-1}\mathbb{A}_{t,\mathcal{P}}$ on $\pi_*(\mathbb{E}|\mathcal{P})$. (4.50) now gives us

$$\mathrm{Str}(\dot{\mathcal{F}}_{t,\mathcal{P}}\,e^{-\mathcal{F}_{t,\mathcal{P}}}) = \mathrm{Str}([\mathbb{A}_t, \dot{\mathbb{A}}_{t,\mathcal{P}}\,e^{-\mathcal{F}_{t,\mathcal{P}}}]) = d_B\mathrm{tr}\,(\dot{\mathbb{A}}_{t,\mathcal{P}}\,e^{-\mathcal{F}_{t,\mathcal{P}}})$$

as in part [1].

The above argument depends on the variation leaving the boundary condition invariant, but otherwise holds for a general 1-parameter family of superconnections $\mathbb{A}_t$. In particular, given two superconnections $\mathbb{A}_1, \mathbb{A}_2$ it holds for $\mathbb{A}_t = t\mathbb{A}_1 + (1-t)\mathbb{A}_2$ adapted to $\mathsf{D}_{\mathcal{P}}$. Hence the statement on interior homotopy invariance follows. This completes the proof. □

4.4. **Computing** $\lim_{t\to 0}(\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_1}) - \mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_2}))$**.** Next we show that $\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_1}) - \mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_2})$ has a limit as $t \to 0$. Notice that $\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_i})$ do not have limits as $t \to 0$; for closed manifolds the content of the local families index theorem is that such a limit does exist for the Bismut superconnection.

First, let us point out that the limit

$$\lim_{t\to 0}(\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_1}) - \mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_2})) \tag{4.51}$$

is not (quite) the generalized relative Eta-form

$$\eta^{[M]}(\mathsf{P}_1, \mathsf{P}_2) = \mathrm{Str}(e^{-\mathsf{R}_1} - e^{-\mathsf{R}_2}) \ , \tag{4.52}$$

but also includes a regularized trace term. The right-side of (4.52) exists for the same reason as in finite-dimensions explained in the introduction. More precisely:

Proposition 4.11.

$$e^{-R_1} - e^{-R_2} \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}')) \ .$$

One has

$$\eta^{[M]}(P_1, P_2) = \mathrm{Str}(P_1 - P_2) + \sum_{k=1}^{\infty}(-1)^k \frac{1}{k!}\,\mathrm{Str}\left(R_1^k - R_2^k\right) \tag{4.53}$$

Proof. The essential point is that since the curvature $\mathsf{R}_i = (\mathsf{P}_i \cdot \nabla^{\pi_*(\mathbb{E})} \cdot \mathsf{P}_i)^2$ is a 2-form, with restricted differential operator coefficients, it is nilpotent. Thus

$$(\mathsf{R}_1 - \lambda\mathbb{I})^{-1} - (\mathsf{R}_2 - \lambda\mathbb{I})^{-1} = \sum_{k=1}^{\infty}(-1)^k(-\lambda)^{-k}(\mathsf{R}_1^k - \mathsf{R}_2^k)$$

is a finite sum, and hence so is

$$\begin{aligned} e^{-\mathsf{R}_1} - e^{-\mathsf{R}_2} &= \frac{i}{2\pi}\int_c e^{-\lambda}((\mathsf{R}_1 - \lambda\mathbb{I})^{-1} - (\mathsf{R}_2 - \lambda\mathbb{I})^{-1} + (\mathsf{P}_2^{\perp} - \mathsf{P}_1^{\perp})(-\lambda)^{-1})\, d\lambda \\ &= (\mathsf{P}_1 - \mathsf{P}_2) + \sum_{k=1}^{\infty}(-1)^k\frac{1}{k!}\left(\mathsf{R}_1^k - \mathsf{R}_2^k\right) \ , \end{aligned}$$

where c can be any simple closed contour around the origin. The Proposition now follows from (4.30). □

Consider now the limit (4.51).

Lemma 4.12.

$$(\mathcal{F}_{t,\mathcal{P}_1} - \lambda\mathbb{I})^{-1} - (\mathcal{F}_{t,\mathcal{P}_2} - \lambda\mathbb{I})^{-1} \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}')) \ . \tag{4.54}$$

Proof. Let $Z_{t,i} = t\mathsf{R}_i + t^{1/2}\nabla^{\pi_*(\mathbb{E}|\mathcal{P})}D_i$. From (4.10) we have

$$(\mathcal{F}_{t,\mathcal{P}_1} - \lambda\mathbb{I})^{-1} - (\mathcal{F}_{t,\mathcal{P}_2} - \lambda\mathbb{I})^{-1} =$$

$$\left\{(t\mathsf{D}_{\mathcal{P}_1}^2 - \lambda\mathbb{I})^{-1} - (t\mathsf{D}_{\mathcal{P}_2}^2 - \lambda\mathbb{I})^{-1}\right\}\sum_{k=0}^{\dim B}(-1)^k(Z_{t,1}(t\mathsf{D}_{\mathcal{P}_1}^2 - \lambda\mathbb{I})^{-1})^k \tag{4.55}$$

$$+\ (t\mathsf{D}_{\mathcal{P}}^2 - \lambda\mathbb{I})^{-1}\sum_{k=0}^{\dim B}(-1)^k\left\{Z_{t,1}(t\mathsf{D}_{\mathcal{P}_1}^2 - \lambda\mathbb{I})^{-1})^k - (Z_{t,2}(t\mathsf{D}_{\mathcal{P}_2}^2 - \lambda\mathbb{I})^{-1})^k\right\} \ . \tag{4.56}$$

From Lemma 4.7 the first factor in (4.55) is smooth family of smoothing operators. Since the second factor consists of terms which are smooth families of first-order differential operator valued forms , coming from $Z_{t,1}$, and sgo operators of order -1, coming from $(t\mathsf{D}^2_{\mathcal{P}_1}-\lambda\mathbb{I})^{-1}$ then the composition is in $\mathcal{A}(B,\Psi^{-\infty}(\mathbb{E},\mathbb{E}^{'}))$ by the sgo calculus and Lemma 4.5[1].

Similarly, (4.56) is in $\mathcal{A}(B,\Psi^{-\infty}(\mathbb{E},\mathbb{E}^{'}))$ since the first factor is a vertical family sgo operators of order -1, while it follows from Lemma 4.7 and Lemma 4.5[1] that the second factor is a smooth family of smoothing operators. This can be checked term by term. When $k=0$ this is trivial, the $k=1$ term can be expanded

$$(Z_{t,1}-Z_{t,2})(t\mathsf{D}^2_{\mathcal{P}_1}-\lambda\mathbb{I})^{-1}+Z_{t,2}((t\mathsf{D}^2_{\mathcal{P}_1}-\lambda\mathbb{I})^{-1}-(t\mathsf{D}^2_{\mathcal{P}_2}-\lambda\mathbb{I})^{-1})\ ,$$

which is in $\mathcal{A}(B,\Psi^{-\infty}(\mathbb{E},\mathbb{E}^{'}))$, as above. The general term is given by a similar formula, and we hence reach the conclusion. □

By the linearity of the supertrace we have:

Corollary 4.13. *For $t>0$*

$$\begin{aligned}(4.57)\qquad & \mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_1})-\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_2})\\ &=\frac{i}{2\pi}\int_{\mathcal{C}}e^{-\lambda}\,\mathrm{Str}\left\{(R_1+t^{1/2}\nabla^1 D_{\mathcal{P}_1}+tD^2_{\mathcal{P}_1}-\lambda\mathbb{I})^{-1}-(R_2+t^{1/2}\nabla^2 D_{\mathcal{P}_2}+tD^2_{\mathcal{P}_2}-\lambda\mathbb{I})^{-1}\right\}\,d\lambda\\ &=\sum_{k=0}^{\dim B}\frac{i}{2\pi}\int_{\mathcal{C}}e^{-\lambda}\,\mathrm{Str}\left\{(tD^2_{\mathcal{P}_1}-\lambda\mathbb{I})^{-1}(Z_{t,1}(tD^2_{\mathcal{P}_1}-\lambda\mathbb{I})^{-1})^k-(tD^2_{\mathcal{P}_2}-\lambda\mathbb{I})^{-1}(Z_{t,2}(tD^2_{\mathcal{P}_2}-\lambda\mathbb{I})^{-1})^k\right\}\,d\lambda\ .\end{aligned}$$

The crucial fact in the proof of Theorem (II) is that the limit (4.51) has no contribution coming from the term $t^{1/2}\nabla^i\mathsf{D}_{\mathcal{P}_i}$. This eliminates complicated Wodzicki residue terms in the regularized limit $\mathrm{LIM}_{t\to 0}\mathrm{ch}(\mathbb{A}_t)$–see [S3] for precise formulas on closed manifolds.

Proposition 4.14. *The limit $\lim_{t\to 0}(\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_1})-\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_2}))$ exists. One has*

$$\begin{aligned}(4.58)\qquad & \lim_{t\to 0}(\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_1})-\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_2}))\\ &=\lim_{t\to 0}\frac{i}{2\pi}\int_{\mathcal{C}}e^{-\lambda}\,\mathrm{Str}\left\{(R_1+tD^2_{\mathcal{P}_1}-\lambda\mathbb{I})^{-1}-(R_2+tD^2_{\mathcal{P}_2}-\lambda\mathbb{I})^{-1}\right\}\,d\lambda\end{aligned}$$

in each C^l norm.

Proof. Notice that the right-side of (4.57) at $t=0$ is just $\eta(\mathsf{P}_1,\mathsf{P}_2)$, but this is not the limit (4.58) — $\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_i})$ do not have limits as $t\to 0$.

The supertrace of the relative vertical resolvent $(\mathcal{F}_{t,\mathcal{P}_1}-\lambda\mathbb{I})^{-1}-(\mathcal{F}_{t,\mathcal{P}_2}-\lambda\mathbb{I})^{-1}$ is equal to the sum of the supertraces of the expressions in (4.55) and (4.56). For the first of these we have using (4.33), with $\|.\|,\|.\|_1$ respectively the operator and trace norms on $\mathcal{A}(B,\mathrm{End}(\pi_*(\mathbb{E})))$,

$$\begin{aligned}(4.59)\qquad & |\mathrm{Str}\left(\{(t\mathsf{D}^2_{\mathcal{P}_1}-\lambda\mathbb{I})^{-1}-(t\mathsf{D}^2_{\mathcal{P}_2}-\lambda\mathbb{I})^{-1}\}\sum_{k=0}^{\dim B}(-1)^k(Z_{t,1}(t\mathsf{D}^2_{\mathcal{P}_1}-\lambda\mathbb{I})^{-1})^k\right)|\\ &\le t^{-1}\|(\mathsf{D}^2_{\mathcal{P}_1}-t^{-1}\lambda\mathbb{I})^{-1}-(\mathsf{D}^2_{\mathcal{P}_2}-t^{-1}\lambda\mathbb{I})^{-1}\|_1\sum_{k=0}^{\dim B}t^{-1}\|(Z_{t,1}(\mathsf{D}^2_{\mathcal{P}_1}-t^{-1}\lambda\mathbb{I})^{-1})\|^k\\ &\le t^{-1}(\|\mathcal{K}_{t^{-1}\lambda}S_{1,t^{-1}\lambda}\|.\|\mathcal{P}_1P(\mathsf{D}^2)^{\perp}\gamma_1\|_1+\|\mathcal{K}_{t^{-1}\lambda}S_{2,t^{-1}\lambda}\|.\|\mathcal{P}_2P(\mathsf{D}^2))^{\perp}\gamma_1\|_1)\|(\mathsf{D}^2-t^{-1}\lambda\mathbb{I})^{-1}_{+}\|\end{aligned}$$

$$\times \sum_{k=0}^{\dim B} t^{-1}\|(Z_{t,1}(\mathsf{D}^2_{\mathcal{P}_1} - t^{-1}\lambda\mathbb{I})^{-1})\|^k \ . \tag{4.60}$$

From standard estimates in elliptic theory on manifolds with boundary [G1] and closed manifolds respectively, we have as $t \longrightarrow 0$

$$\|\mathcal{K}_{t^{-1}\lambda}S_{1,t^{-1}\lambda}\| = O(1), \qquad \|(\mathsf{D}^2 - t^{-1}\lambda\mathbb{I})^{-1}_{+}\| = O(t|\lambda|^{-1}) \ . \tag{4.61}$$

To estimate the sum, let $Q_{\mathcal{P}_1} \in \mathcal{A}(B, \mathrm{End}(\pi_*(\mathbb{E})))$ be a vertical parametrix for $\mathsf{D}_{\mathcal{P}_1}$, so that $Q_{\mathcal{P}_1}\mathsf{D}_{\mathcal{P}_1} = \mathbb{I} + S$ with $S \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}^{'}))$. Then

$$t^{-1}Z_{t,1}(\mathsf{D}^2_{\mathcal{P}_1} - t^{-1}\lambda\mathbb{I})^{-1} = t^{-1}Z_{t,1}Q_{\mathcal{P}_1}\mathsf{D}(\mathsf{D}^2_{\mathcal{P}_1} - t^{-1}\lambda\mathbb{I})^{-1} - t^{-1}Z_{t,1}S(\mathsf{D}^2_{\mathcal{P}_1} - t^{-1}\lambda\mathbb{I})^{-1}$$

$$= \frac{t^{-1}}{2}Z_{t,1}Q_{\mathcal{P}_1}\left\{(\mathsf{D}_{\mathcal{P}_1} - t^{-1/2}\lambda^{1/2}\mathbb{I})^{-1} + (\mathsf{D}_{\mathcal{P}_1} + t^{-1/2}\lambda^{1/2}\mathbb{I})^{-1}\right\} - t^{-1}Z_{t,1}S(\mathsf{D}^2_{\mathcal{P}_1} - t^{-1}\lambda\mathbb{I})^{-1} \ ,$$

since $\mathcal{D}_{\mathcal{P}_1}$ is a family of self-adjoint operators. Since $Z_{t,1}$ is a vertical family of first-order differential operators it follows that $Z_{t,1}Q_{\mathcal{P}_1}$ is bounded in the operator norm. Hence

$$\|(Z_{t,1}(\mathsf{D}^2_{\mathcal{P}_1} - t^{-1}\lambda\mathbb{I})^{-1})\| \leq \frac{t^{-1}}{2}\|Z_{t,1}Q_{\mathcal{P}_1}\|(\|(\mathsf{D}_{\mathcal{P}_1} - t^{-1/2}\lambda^{1/2}\mathbb{I})^{-1})\| + \|(\mathsf{D}_{\mathcal{P}_1} + t^{-1/2}\lambda^{1/2}\mathbb{I})^{-1}\|)$$
$$- t^{-1}\|Z_{t,1}S\|\|(\mathsf{D}^2_{\mathcal{P}_1} - t^{-1}\lambda\mathbb{I})^{-1})\| \ . \tag{4.62}$$

We have

$$\|Z_{t,1}Q_{\mathcal{P}_1}\| = O(t^{1/2}) \ ,$$
$$\|(\mathsf{D}_{\mathcal{P}_1} \pm t^{-1/2}\lambda^{1/2}\mathbb{I})^{-1}\| = O(|\lambda|^{-1/2}).t^{1/2} \qquad \text{as } t \longrightarrow 0 \ ,$$
$$\|Z_{t,1}S\| = O(t^{1/2}) \ ,$$
$$\|(\mathsf{D}^2_{\mathcal{P}_1} - t^{-1}\lambda\mathbb{I})^{-1}\| = O(t|\lambda|^{-1}) \qquad \text{as } t \longrightarrow 0 \ ,$$

and hence that

$$\|(Z_{t,1}(\mathsf{D}^2_{\mathcal{P}_1} - t^{-1}\lambda\mathbb{I})^{-1})\| = O(t^0|\lambda|^{-1}) \qquad \text{as } t \longrightarrow 0 \ . \tag{4.63}$$

From (4.60), (4.61), (4.63) we have that (4.59) is $O(t^0|\lambda|^{-1})$ as $t \longrightarrow 0$. An entirely similar analysis gives the same result for the supertrace of (4.56). Hence we have

$$|\mathrm{Str}((\mathcal{F}_{t,\mathcal{P}_1} - \lambda\mathbb{I})^{-1} - (\mathcal{F}_{t,\mathcal{P}_2} - \lambda\mathbb{I})^{-1})| = O(t^0|\lambda|^{-1})$$

as $t \longrightarrow 0$, which by Corollary 4.13 proves the first sentence of the Proposition.

To prove (4.58), we expand the resolvent

$$(\mathsf{R}_i + t^{1/2}\nabla^i\mathsf{D}_{\mathcal{P}_i} + t\mathsf{D}^2_{\mathcal{P}_i} - \lambda\mathbb{I})^{-1} = (\mathsf{R}_i + t\mathsf{D}^2_{\mathcal{P}_i} - \lambda\mathbb{I})^{-1}$$
$$+ \sum_{k=1}^{\dim B} t^{k/2}(\mathsf{R}_i + t\mathsf{D}_{\mathcal{P}_i} - \lambda\mathbb{I})^{-1}\left((\nabla^i\mathsf{D}_{\mathcal{P}_i})(\mathsf{R}_i + t\mathsf{D}^2_{\mathcal{P}_i} - \lambda\mathbb{I})^{-1}\right)^k \ .$$

Hence

$$\left\{(\mathcal{F}_{t,1} - \lambda\mathbb{I})^{-1} - (\mathcal{F}_{t,1} - \lambda\mathbb{I})^{-1}\right\} - \left\{(\mathsf{R}_1 + t\mathsf{D}^2_{\mathcal{P}_1} - \lambda\mathbb{I})^{-1} - (\mathsf{R}_2 + t\mathsf{D}^2_{\mathcal{P}_2} - \lambda\mathbb{I})^{-1}\right\}$$
$$= \sum_{k=1}^{\dim B} t^{k/2}\left\{(\mathsf{R}_1 + t\mathsf{D}^2_{\mathcal{P}_1} - \lambda\mathbb{I})^{-1}\left((\nabla^1\mathsf{D}_{\mathcal{P}_1})(\mathsf{R}_1 + t\mathsf{D}^2_{\mathcal{P}_1} - \lambda\mathbb{I})^{-1}\right)^k \ .\right.$$
$$\left. - (\mathsf{R}_2 + t\mathsf{D}^2_{\mathcal{P}_2} - \lambda\mathbb{I})^{-1}\left((\nabla^2\mathsf{D}_{\mathcal{P}_2})(\mathsf{R}_2 + t\mathsf{D}^2_{\mathcal{P}_2} - \lambda\mathbb{I})^{-1}\right)^k\right\} \ . \tag{4.64}$$

Since

$$(\mathsf{R}_i + t\mathsf{D}^2_{\mathcal{P}_i} - \lambda\mathbb{I})^{-1}(\nabla^i\mathsf{D}_{\mathcal{P}_i})(\mathsf{R}_i + t\mathsf{D}^2_{\mathcal{P}_i} - \lambda\mathbb{I})^{-1}$$
$$= (t\mathsf{D}^2_{\mathcal{P}_i} - \lambda\mathbb{I})^{-1}.\sum_{j\geq 0}(\mathsf{R}_i(t\mathsf{D}^2_{\mathcal{P}_i} - \lambda\mathbb{I})^{-1})^j.(\nabla^i\mathsf{D}_{\mathcal{P}_i}).\sum_{j\geq 0}(\mathsf{R}_i(t\mathsf{D}^2_{\mathcal{P}_i} - \lambda\mathbb{I})^{-1})^j$$

we obtain using Lemma 4.7 that the terms in the sum in (4.64) are smooth families of smoothing operators, and by a exactly similar argument to the first part that

$$|\mathrm{Str}\left\{(\mathsf{R}_1 + t\mathsf{D}^2_{\mathcal{P}_1} - \lambda\mathbb{I})^{-1}\left((\nabla^1\mathsf{D}_{\mathcal{P}_1})(\mathsf{R}_1 + t\mathsf{D}^2_{\mathcal{P}_1} - \lambda\mathbb{I})^{-1}\right)^k\right.$$
$$\left.-(\mathsf{R}_2 + t\mathsf{D}^2_{\mathcal{P}_2} - \lambda\mathbb{I})^{-1}\left((\nabla^2\mathsf{D}_{\mathcal{P}_2})(\mathsf{R}_2 + t\mathsf{D}^2_{\mathcal{P}_2} - \lambda\mathbb{I})^{-1}\right)^k\right\}| = O(1)$$

as $t \longrightarrow 0$. It follows that the supertrace of (4.64) is $O(t^{1/2})$ as $t \longrightarrow 0$, and (4.58) follows.

Taking sums of derivatives with respect to the base B produces similar terms and hence the assertions hold in each C^l norm. □

It follows, then, that the object of interest is the asymptotic behavior as $t \longrightarrow 0$ of

$$\mathrm{Str}\left\{(\mathsf{R}_1 + t\mathsf{D}^2_{\mathcal{P}_1} - \lambda\mathbb{I})^{-1} - (\mathsf{R}_2 + t\mathsf{D}^2_{\mathcal{P}_2} - \lambda\mathbb{I})^{-1}\right\} \tag{4.65}$$
$$= \sum_{k=0}^{\infty}(-1)^k\mathrm{Str}\left(\mathsf{G}^{-1}_{t,\lambda}(\mathsf{R}_1\mathsf{G}^{-1}_{t,\lambda})^k - \mathsf{H}^{-1}_{t,\lambda}(\mathsf{R}_1\mathsf{H}^{-1}_{t,\lambda})^k\right)$$

where

$$\mathsf{G}^{-1}_{t,\lambda} = (t\mathsf{D}^2_{\mathcal{P}_1} - \lambda\mathbb{I})^{-1}\ , \qquad \mathsf{H}^{-1}_{t,\lambda} = (t\mathsf{D}^2_{\mathcal{P}_2} - \lambda\mathbb{I})^{-1}\ .$$

Proposition 4.15. *As $\lambda \longrightarrow \infty$ and for all C^l norms*

$$\mathrm{Str}\left(G^{-1}_{t,\lambda}(R_1G^{-1}_{t,\lambda})^k - H^{-1}_{t,\lambda}(R_2H^{-1}_{t,\lambda})^k\right) \tag{4.66}$$

$$= \mathrm{Str}(R_1^k - R_2^k)(-\lambda)^{-k-1} + O(|\lambda|^{-k-2}).t \tag{4.67}$$

$$+\ \mathrm{Str}(R(D)^k((G^{-1}_{t,\lambda} - (H^{-1}_{t,\lambda}))(-\lambda)^{-k} + O(|\lambda|^{-k-1}).t\ . \tag{4.68}$$

In (4.77), the curvature form $R(D)$ on the bundle defined by the Calderon section $P(D) \in \mathrm{Gr}(\mathbb{E}^0)$, can be replaced by $R(D) + \mathcal{S}$, where $\mathcal{S}$ is any vertical smoothing operator in $\mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}'))$, and hence by any curvature form $R^{\pi_*(\mathbb{E}|\mathcal{P})}$ with $\mathcal{P} \in \mathrm{Gr}(\mathbb{E}^0)$, without affecting the constant order t^0 term.

Proof. For $k = 0$ both sides are equal to $\mathrm{Str}(\mathsf{G}^{-1}_{t,\lambda} - \mathsf{H}^{-1}_{t,\lambda})$. To explain the proof for $k > 0$ we prove first the $k = 1$ case. We have

$$\mathrm{Str}\left(\mathsf{G}^{-1}_{t,\lambda}\mathsf{R}_1\mathsf{G}^{-1}_{t,\lambda} - \mathsf{H}^{-1}_{t,\lambda}\mathsf{R}_2\mathsf{H}^{-1}_{t,\lambda}\right)$$
$$= \mathrm{Str}\left(\mathsf{G}^{-1}_{t,\lambda}(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{G}^{-1}_{t,\lambda} + (\mathsf{G}^{-1}_{t,\lambda} - \mathsf{H}^{-1}_{t,\lambda})\mathsf{R}_2\mathsf{G}^{-1}_{t,\lambda} + \mathsf{H}^{-1}_{t,\lambda}\mathsf{R}_2(\mathsf{G}^{-1}_{t,\lambda} - \mathsf{H}^{-1}_{t,\lambda})\right)$$
$$= \mathrm{Str}\left(\mathsf{G}^{-1}_{t,\lambda}(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{G}^{-1}_{t,\lambda}\right) + \mathrm{Str}\left((\mathsf{G}^{-1}_{t,\lambda} - \mathsf{H}^{-1}_{t,\lambda})\mathsf{R}_2\mathsf{G}^{-1}_{t,\lambda}\right) + \mathrm{Str}\left(\mathsf{H}^{-1}_{t,\lambda}\mathsf{R}_2(\mathsf{G}^{-1}_{t,\lambda} - \mathsf{H}^{-1}_{t,\lambda})\right)$$

(4.69)
$$= \mathrm{Str}\left(\mathsf{G}^{-1}_{t,\lambda}(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{G}^{-1}_{t,\lambda}\right) + \mathrm{Str}\left(\mathsf{R}_2\mathsf{G}^{-1}_{t,\lambda}(\mathsf{G}^{-1}_{t,\lambda} - \mathsf{H}^{-1}_{t,\lambda})\right) + \mathrm{Str}\left(\mathsf{H}^{-1}_{t,\lambda}\mathsf{R}_2(\mathsf{G}^{-1}_{t,\lambda} - \mathsf{H}^{-1}_{t,\lambda})\right)\ ,$$

using the norm boundedness of $\mathsf{R}_2\mathsf{G}^{-1}_{t,\lambda}$ and (4.29) to cycle the trace.

From the resolvent formulae on $\mathcal{A}(B, \mathrm{End}(\pi_*(\mathbb{E}))$

$$\mathsf{G}_{t,\lambda}^{-1} = (-\lambda)^{-1} - t\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1}(-\lambda)^{-1} , \qquad \mathsf{H}_{t,\lambda}^{-1} = (-\lambda)^{-1} - t\mathsf{D}_2^2\mathsf{H}_{t,\lambda}^{-1}(-\lambda)^{-1} , \tag{4.70}$$

where $\mathsf{D}_i := \mathsf{D}_{\mathcal{P}_i}$, we have

$$\begin{aligned} &\mathrm{Str}\left(\mathsf{G}_{t,\lambda}^{-1}(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{G}_{t,\lambda}^{-1}\right) \\ &= \mathrm{Str}\,(\mathsf{R}_1 - \mathsf{R}_2)\,(-\lambda)^2 - \mathrm{Str}\left((\mathsf{R}_1 - \mathsf{R}_2)\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1}\right)\,t(-\lambda)^2 \\ &- \mathrm{Str}\left(\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1}(\mathsf{R}_1 - \mathsf{R}_2)\right)\,t(-\lambda)^2 + \mathrm{Str}\left(\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1}(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1}\right)\,t(-\lambda)^2 . \end{aligned} \tag{4.71}$$

Since $(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{D}_1^2 \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}'))$ we have

$$\begin{aligned} |\mathrm{Str}((\mathsf{R}_1 - \mathsf{R}_2)\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1})| &\leq \|(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1}\|_1 \\ &\leq \|(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{D}_1^2\|_1\,\|\mathsf{G}_{t,\lambda}^{-1}\| \\ &= O(t^0|\lambda|^{-1})) \quad \text{as} \ \ \lambda \longrightarrow \infty , \end{aligned}$$

where $\|.\|_1$ is the trace norm, and hence

$$|\mathrm{Str}((\mathsf{R}_1 - \mathsf{R}_2)\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1})\,t(-\lambda)^2| = O(|\lambda|^{-3}).t \quad \text{as} \ \ \lambda \longrightarrow \infty .$$

Similar estimates hold for the third and fourth terms on the right-side of (4.71), and hence

$$|\mathrm{Str}(\mathsf{G}_{t,\lambda}^{-1}(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{G}_{t,\lambda}^{-1}) - \mathrm{Str}(\mathsf{R}_1 - \mathsf{R}_2)\,(-\lambda)^2| = O(|\lambda|^{-3}).t \quad \text{as} \ \ \lambda \longrightarrow \infty . \tag{4.72}$$

For the second term in (4.69) we have using (4.70)

$$\begin{aligned} &\mathrm{Str}\left(\mathsf{R}_2\mathsf{G}_{t,\lambda}^{-1}(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1})\right) \\ &= \mathrm{Str}(\mathsf{R}_2(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1}))(-\lambda)^{-1} - \mathrm{Str}(\mathsf{R}_2\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1}(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1})\,t(-\lambda)^{-1} . \end{aligned} \tag{4.73}$$

The second term of (4.73) is equal to

$$\mathrm{Str}(\mathsf{R}_2\mathsf{D}_1^2(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1}))\,t(-\lambda)^{-2} - \mathrm{Str}(\mathsf{R}_2\mathsf{D}_1^4\mathsf{G}_{t,\lambda}^{-1}(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1}))\,t^2(-\lambda)^{-2}$$

which is $O(|\lambda|^{-3}).t$ as $\lambda \longrightarrow \infty$. Thus the second and, by a similar analysis, the third terms of (4.69) are equal to

$$\mathrm{Str}(\mathsf{R}_2(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1}))(-\lambda)^{-1} + O(|\lambda|^{-3}).t \quad \text{as} \ \ \lambda \longrightarrow \infty . \tag{4.74}$$

Equations (4.72) and (4.74) prove (4.76) for the sup-norm. Taking derivatives with respect to B gives similar estimates proving (4.76) for each C^l norm.

To explain in this case the statement on the curvature form in (4.77), let $\mathcal{S} \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}'))$. Then

$$\mathrm{Str}(\mathcal{S}(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1})) = t(-\lambda)^{-1}\,(\mathcal{S}\mathsf{D}_2^2\mathsf{H}_{t,\lambda}^{-1} - \mathcal{S}\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1}) , \tag{4.75}$$

and hence

$$|\mathrm{Str}(\mathcal{S}(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1}))| \leq t(-\lambda)^{-1}\,(\|\mathcal{S}\mathsf{D}_2^2\|_1\,\|\mathsf{H}_{t,\lambda}^{-1}\| + \|\mathcal{S}\mathsf{D}_1^2\|_1\,\|\mathsf{G}_{t,\lambda}^{-1}\|) = O(|\lambda|^{-2}).t .$$

This proves that

$$\begin{aligned} \mathrm{Str}\left(\mathsf{G}_{t,\lambda}^{-1}\mathsf{R}_1\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1}\mathsf{R}_2\mathsf{H}_{t,\lambda}^{-1}\right) &= \mathrm{Str}(\mathsf{R}_1 - \mathsf{R}_2)(-\lambda)^{-2} + O(|\lambda|^{-3}).t \\ &+2\,\mathrm{Str}(\mathsf{R}(\mathsf{D})(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1}))(-\lambda)^{-2} + O(t|\lambda|^{-2}) . \end{aligned} \tag{4.76}$$

For the general case, we have inductively for $k \geq 1$, by (4.29) and (4.30),

$$\operatorname{Str}\left(\mathsf{G}_{t,\lambda}^{-1}(\mathsf{R}_1\mathsf{G}_{t,\lambda}^{-1})^k - \mathsf{H}_{t,\lambda}^{-1}(\mathsf{R}_2\mathsf{H}_{t,\lambda}^{-1})^k\right) = \tag{4.77}$$

$$\sum_{i=0}^{k}\operatorname{Str}\left((\mathsf{H}_{t,\lambda}^{-1}\mathsf{R}_2)^i(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1})(\mathsf{R}_1\mathsf{G}_{t,\lambda}^{-1})^{k-i}\right)$$

$$+$$

$$\sum_{j=0}^{k-1}\operatorname{Str}\left((\mathsf{H}_{t,\lambda}^{-1}\mathsf{R}_2)^j\mathsf{H}_{t,\lambda}^{-1}(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{G}_{t,\lambda}^{-1}(\mathsf{R}_1\mathsf{G}_{t,\lambda}^{-1})^{k-j-1}\right) .$$

For the second sum in (4.77)

$$\operatorname{Str}\left((\mathsf{H}_{t,\lambda}^{-1}\mathsf{R}_2)^j\mathsf{H}_{t,\lambda}^{-1}(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{G}_{t,\lambda}^{-1}(\mathsf{R}_1\mathsf{G}_{t,\lambda}^{-1})^{k-j-1}\right)$$

$$= \operatorname{Str}\Big(((-\lambda)^{-j}(\mathsf{R}_2 - t\mathsf{D}_2^2\mathsf{H}_{t,\lambda}^{-1}\mathsf{R}_2)^j).\,(-\lambda)^{-1}(\mathbb{I} - t\mathsf{D}_1^2\mathsf{H}_{t,\lambda}^{-1}).(\mathsf{R}_1 - \mathsf{R}_2)$$

$$\times\,(-\lambda)^{-1}(\mathbb{I} - t\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1}).(-\lambda)^{-k+j+1}(\mathsf{R}_1 - t\mathsf{R}_1\mathsf{D}_1^2\mathsf{G}_{t,\lambda})^{k-j-1}\Big)$$

$$= (-\lambda)^{-k-1}\operatorname{Str}\left((\mathsf{R}_2 - t\mathsf{D}_2^2\mathsf{H}_{t,\lambda}^{-1}\mathsf{R}_2)^j(\mathbb{I} - t\mathsf{D}_1^2\mathsf{H}_{t,\lambda}^{-1})(\mathsf{R}_1 - \mathsf{R}_2)(\mathbb{I} - t\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1})(\mathsf{R}_1 - t\mathsf{R}_1\mathsf{D}_1^2\mathsf{G}_{t,\lambda})^{k-j-1}\right)$$

$$= (-\lambda)^{-k-1}\operatorname{Str}\Big((\mathsf{R}_2 - t\mathsf{D}_2^2\mathsf{H}_{t,\lambda}^{-1}\mathsf{R}_2)^j((\mathsf{R}_1 - \mathsf{R}_2) - t\mathsf{D}_2^2\mathsf{H}_{t,\lambda}^{-1}(\mathsf{R}_1 - \mathsf{R}_2)$$

$$-t(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1} + t^2\mathsf{D}_2^2\mathsf{H}_{t,\lambda}^{-1}(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1}(\mathsf{R}_1 - t\mathsf{R}_1\mathsf{D}_1^2\mathsf{G}_{t,\lambda})^{k-j-1}\Big) .$$

—-expanding the terms $(\mathsf{R}_2 - t\mathsf{D}_2^2\mathsf{H}_{t,\lambda}^{-1}\mathsf{R}_2)^j$ and $(\mathsf{R}_1 - t\mathsf{R}_1\mathsf{D}_1^2\mathsf{G}_{t,\lambda})^{k-j-1}$—-

$$= \operatorname{Str}\left(\mathsf{R}_2^j\mathsf{R}_1^{k-j} - \mathsf{R}_2^{j+1}\mathsf{R}_1^{k-j-1}\right)(-\lambda)^{-k-1} + \operatorname{Str}(\mathrm{pol}(t, \mathsf{R}_1, \mathsf{R}_2, \mathsf{G}_{t,\lambda}^{-1}, \mathsf{H}_{t,\lambda}^{-1}))\ , \tag{4.78}$$

where $\mathrm{pol}(t, \mathsf{R}_1, \mathsf{R}_2, \mathsf{G}_{t,\lambda}^{-1}, \mathsf{H}_{t,\lambda}^{-1})$ is a finite polynomial expression consisting of terms like

$$\text{const.}\, t^{j-p}\mathsf{R}_2^p(\mathsf{D}_2^2\mathsf{H}_{t,\lambda}^{-1}\mathsf{R}_2)^{j-p}.t(\mathsf{R}_1 - \mathsf{R}_2)\mathsf{R}_1\mathsf{G}_{t,\lambda}^{-1}.\mathsf{R}_1^q(\mathsf{R}_1\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1})^{k-j-1-q}\ ,$$

where $0 \leq p \leq j-1$, $0 \leq q \leq k-j-2$. By an estimate akin to the $k = 1$ case, we obtain

$$|\operatorname{Str}(\mathrm{pol}(t, \mathsf{R}_1, \mathsf{R}_2, \mathsf{G}_{t,\lambda}^{-1}, \mathsf{H}_{t,\lambda}^{-1}))| = O(|\lambda|^{-k-2}).t \quad \text{as} \quad \lambda \longrightarrow \infty\ . \tag{4.79}$$

From (4.78), (4.79), the second sum in (4.77) hence telescopes down to

$$\sum_{j=0}^{k-1}\operatorname{Str}\left(\mathsf{R}_2^j\mathsf{R}_1^{k-j} - \mathsf{R}_2^{j+1}\mathsf{R}_1^{k-j-1}\right)(-\lambda)^{-k-1} + O(|\lambda|^{-k-2}).t \tag{4.80}$$

$$= \operatorname{Str}\left(\mathsf{R}_1^k - \mathsf{R}_2^k\right)(-\lambda)^{-k-1} + O(|\lambda|^{-k-2}).t\ ,$$

which is (4.67).

For the first sum in (4.77), since $(\mathsf{R}_1\mathsf{G}_{t,\lambda}^{-1})^{k-i}$ is a vertical family of bounded operators, we have cycling the trace

$$\operatorname{Str}\left((\mathsf{H}_{t,\lambda}^{-1}\mathsf{R}_2)^i(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1})(\mathsf{R}_1\mathsf{G}_{t,\lambda}^{-1})^{k-i}\right)$$

$$= \operatorname{Str}\left((\mathsf{R}_1\mathsf{G}_{t,\lambda}^{-1})^{k-i}(\mathsf{H}_{t,\lambda}^{-1}\mathsf{R}_2)^i(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1})\right)$$

$$= (-\lambda)^{-k}\operatorname{Str}\left((\mathsf{R}_1 - t\mathsf{R}_1\mathsf{D}_1^2\mathsf{G}_{t,\lambda}^{-1})^{k-i}(\mathsf{R}_2 - t\mathsf{D}_2^2\mathsf{H}_{t,\lambda}^{-1}\mathsf{R}_2)^i(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda})^{-1}\right)$$

$$= (-\lambda)^{-k}\mathrm{Str}\left(\mathsf{R}_1^{k-i}\mathsf{R}_2^{i}(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda})^{-1}\right) + O(|\lambda|^{-k-1}).t$$
$$= (-\lambda)^{-k}\mathrm{Str}\left(\mathsf{R}(\mathsf{D})^{k}(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda})^{-1}\right) + O(|\lambda|^{-k-1}).t\ ,$$

for since $\mathsf{R}_1^{k-i}\mathsf{R}_2^{i} - \mathsf{R}(\mathsf{D})^k \in \mathcal{A}(B, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}'))$ we can apply the argument of (4.75) to make the replacement.

Since there are $k+1$ such terms in the first sum of (4.77), we get (4.68). □

Remark 4.16. *Essentially the point here is that if $Q_\lambda^{-1} = (\Delta - \lambda)^{-1}$ is a resolvent for a differential operator Δ, and $\mathcal{S}$ a smoothing operator, there is an asymptotic expansion*

$$\mathrm{Str}(\mathcal{S}Q_\lambda^{-1}) \sim \mathrm{Str}(\mathcal{S})(-\lambda)^{-1} + \sum_{j\geq 2}\mathrm{Str}(\mathcal{S}\Delta^{j-1})(-\lambda)^{-j}t^{j-1}\ , \tag{4.81}$$

as $\lambda \to \infty$, which if Δ is a positive operator has the heat trace realization as $t \to 0$

$$\mathrm{Str}(\mathcal{S}e^{-t\Delta}) \sim \mathrm{Str}(\mathcal{S}) + \sum_{j\geq 1}\frac{1}{(j-1)!}\mathrm{Str}(\mathcal{S}\Delta^{j-1})t^j\ ,$$

which follows from (4.81) *via*

$$\mathrm{Str}(\mathcal{S}e^{-t\Delta}) = \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\mathrm{Str}(\mathcal{S}Q_\lambda^{-1})\ d\lambda\ .$$

We need the following result in order to deal with the term (4.68) in the Chern character. For convenience write $\mathsf{R} := \mathsf{R}(\mathsf{D})$ and $\mathsf{G}_\lambda^{-1} = \mathsf{G}_{1,\lambda}^{-1}$, $\mathsf{H}_\lambda^{-1} = \mathsf{H}_{1,\lambda}^{-1}$, so that

$$\mathrm{Str}(\mathsf{R}^k(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1})) = t^{-1}\mathrm{Str}(\mathsf{R}^k(\mathsf{G}_{t^{-1}\lambda}^{-1} - \mathsf{H}_{t^{-1}\lambda}^{-1}))\ . \tag{4.82}$$

Proposition 4.17. *As $\lambda \longrightarrow \infty$ there is an asymptotic expansion of the differential form $\mathrm{Str}(R^k(G_{t,\lambda}^{-1} - H_{t,\lambda}^{-1}))$ on B*

$$\mathrm{Str}(R^k(G_{t,\lambda}^{-1} - H_{t,\lambda}^{-1}))(z) \sim \sum_{j\geq 0}c_{j,k}(z)(-\lambda)^{-\frac{j}{2}-1} + \sum_{i\geq 1}d_{i,k}(z)\log(-\lambda).(-\lambda)^{-\frac{i}{2}-1}\ . \tag{4.83}$$

Proof. For $k=0$ this is the asymptotic expansion as $\lambda \longrightarrow \infty$ of [G2, G3]

$$\mathrm{Str}(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1}) \sim (s\zeta_1(0) - s\zeta_2(0))(-\lambda)^{-1} + \sum_{j=1}^{\infty}a_j(-\lambda)^{-1-k/2}t^{j/2}\ , \tag{4.84}$$

where $s\zeta_i(0) = \mathrm{Str}(D_{\mathcal{P}_1}^{-z})|_{z=0}^{\mathrm{mer}}$ is the super-zeta function.

The method of Grubb [G2, G3] extends also to the more general case here. (The existence of a trace expansion for $\mathrm{Str}(\mathsf{R}^k\mathsf{G}_{t,\lambda}^{-1})$ is a harder problem.) We refer to [G1] for a comprehensive account of the symbol spaces below.

Write $\lambda = \mu^2, \mu \in \mathbb{C}\backslash\mathbb{R}$. From (4.33) or [G3] we obtain

$$\mathsf{R}^k(\mathsf{G}_{t,\mu^2}^{-1} - \mathsf{H}_{t,\mu^2}^{-1}) = \mu^{-1}\mathsf{R}^k\mathsf{K}_\mu\gamma\mathsf{S}_\mu\mathsf{T}_\mu\ , \tag{4.85}$$

where in the sgo calculus of [G1, G2, G3] K_μ is a vertical strongly polyhomogeneous Poisson operator of order 0, T_μ is a vertical trace operator of order -1, and S_μ is a vertical weakly polyhomogeneous ψdo acting on $\mathcal{A}(B, \pi_*(\mathbb{E}'))$ with symbol pointwise in $S^{-\infty,0}$. Since $F_{k,\mu} = \mathsf{R}^k(\mathsf{G}_{t,\lambda}^{-1} - \mathsf{H}_{t,\lambda}^{-1})$ is the combination of a vertical differential-operator of order k valued form, vertical Poisson operators and singular Green's operators, and since the relative inverse is a smooth family of smoothing operators (4.29), then by the symbol calculus and Lemma 4.5[1] $F_{k,\mu} \in \mathcal{A}(S, \Psi^{-\infty}(\mathbb{E}, \mathbb{E}'))$ with cycled trace equal to the trace

$\mu^{-1}\mathrm{Str}(\mathsf{S}_\mu\mathsf{T}_\mu\mathsf{R}^k\mathsf{K}_\mu\gamma)$ on $\mathcal{A}(B,\Psi^{-\infty}(\mathbb{E}'))$. Since $\mathsf{T}_\mu\mathsf{R}^k\mathsf{K}_\mu$ is a strongly polyhomogeneous vertical pdo in $\mathcal{A}(B,\Psi^{k-1}(\mathbb{E}'))$ of order $k-1$ and with symbol in $S^{k-1,0}\cap S^{0,k-1}$, the composed operator is a weakly polyhomogeneous vertical pdo with symbol pointwise in $S^{-\infty,k-1}\cap S^{-\infty,k-2}$, . Now by [GS] there is an asymptotic expansion

$$\mathrm{Str}(F_{k,\mu})(z)\sim\sum_{-\infty<j<0}\alpha_{j,k}(z)\mu^{k-2-j}+\sum_{j\geq 0}(\beta_{j,k}(z)\log\mu+\beta'_{j,k}(z))\mu^{k-2-j}\ .$$

Next since $F_{k,\mu}\in\mathcal{A}(S,\Psi^{-\infty}(\mathbb{E},\mathbb{E}'))$ we have $|\mathrm{Str}(F_{k,\mu})(z)|=O(|\mu^{-2}|)$ and hence all terms $\alpha_{j,k},\beta_{j,k}$ with $j>k$ and terms $\beta'_{j,k}$ with $j>k+1$ vanish. This completes the proof. Finally, since the log coefficients come entirely from the homogeneous symbol of S_μ one has as in [GS] Th.2.1 that the terms with, respectively, $j=k$ and $j=k+1$ vanish. □

Let $\Delta_i=D^2_{\mathcal{P}_i}$. Then the operator $\mathsf{R}^k(\Delta_1^{-z}-\Delta_2^{-z})$ is trace class for $\mathrm{Re}(z)>>0$. The relative superzeta function trace $\mathrm{Str}(\mathsf{R}^k(\Delta_1^{-z}-\Delta_2^{-z}))$ is holomorphic for such z and has a meromorphic continuation to all of $\mathbb{C}$ with singularity structure determined by the asymptotic expansion (4.83). Since there is no term $(-\lambda)^{-1}\log(-\lambda)$ there is a well-defined regularized trace

$$\mathrm{TR}_{[\Delta_1,\Delta_2]}(\mathsf{R}^k)=\mathrm{Str}(\mathsf{R}^k(\Delta_1^{-s}-\Delta_2^{-s}))|^{\mathrm{mer}}_{z=0}\ . \tag{4.86}$$

The equivalence between the resolvent trace expansion and the meromorphic extension to $\mathbb{C}$ is standard, see for example [GS] Prop.2.1.

Theorem 4.18. *As forms in* $\mathcal{A}^{2\bullet}(B)$

$$\lim_{t\longrightarrow 0}(\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_1})-\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_2}))=\eta^{[M]}(P_1,P_2)+\sum_{k\geq 0}\frac{k+1}{k!}\mathrm{sTR}_{[\Delta_1,\Delta_2]}(R^k)\ . \tag{4.87}$$

Here $R^0_i:=P_i$ *and the degree zero part of the right-side of* (4.87) *is the pointwise relative index* $\mathrm{ind}\,(D_{P_1})-\mathrm{ind}\,(D_{P_2})$.

Proof. From (4.65), (4.58), (4.36) and Proposition 4.15 we have as $t\to 0$

$$\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_1})-\mathrm{ch}(\mathbb{A}_{t,\mathcal{P}_2}) \tag{4.88}$$

$$=\sum_{k=0}^{\infty}(-1)^k\frac{i}{2\pi}\int_{\mathcal{C}}e^{-\lambda}\mathrm{Str}\left(\mathsf{G}^{-1}_{t,\lambda}(\mathsf{R}_1\mathsf{G}^{-1}_{t,\lambda})^k-\mathsf{H}^{-1}_{t,\lambda}(\mathsf{R}_1\mathsf{H}^{-1}_{t,\lambda})^k\right)\,d\lambda+o(1)$$

$$=\sum_{k=1}^{\infty}(-1)^k\frac{i}{2\pi}\int_{\mathcal{C}}e^{-\lambda}(-\lambda)^{-k}\,d\lambda\,.\,\mathrm{Str}(\mathsf{R}_1^k-\mathsf{R}_2^k) \tag{4.89}$$

$$+\sum_{k=0}^{\infty}(-1)^k(k+1)\frac{i}{2\pi}\int_{\mathcal{C}}e^{-\lambda}(-\lambda)^{-k}\mathrm{Str}(\mathsf{R}^k(\mathsf{G}^{-1}_{t,\lambda}-\mathsf{H}^{-1}_{t,\lambda}))\,d\lambda\,.+o(1)\ . \tag{4.90}$$

(Specifically the $o(1)$ term is given by an asymptotic expansion $\sum_{j\geq 1}a_k t^{k/2}$ as $t\longrightarrow 0$.) Since $\frac{i}{2\pi}\int_{\mathcal{C}}e^{-\lambda}(-\lambda)^{-k}\,d\lambda=(k!)^{-1}$ the first sum (4.89) determines the η-form part of (4.87). Apart, that is, from the degree zero term which arises as the $k=0$ term of (4.90), as follows. From (4.82) and (4.83) there is an asymptotic expansion as $t\to 0$

$$\mathrm{Str}(\mathsf{R}^k((\mathsf{G}^{-1}_{t,\lambda}-\mathsf{H}^{-1}_{t,\lambda}))\sim\frac{1}{t}\sum_{j\geq 0}c_{j,k}\left(-\frac{\lambda}{t}\right)^{-\frac{j}{2}-1}+\frac{1}{t}\sum_{i\geq 1}d_{i,k}\log\left(-\frac{\lambda}{t}\right).\left(-\frac{\lambda}{t}\right)^{-\frac{i}{2}-1}\ .$$

Hence for any $\varepsilon > 0$ the $k^{\rm th}$ term of (4.90) is

$$(k+1)\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}(-\lambda)^{-k}{\rm Str}(\mathsf{R}^k((\mathsf{G}_{t,\lambda}^{-1}-\mathsf{H}_{t,\lambda}^{-1}))\,d\lambda$$

$$= (k+1)\sum_{j=0}^{N-1} c_{j,k}\left(\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}(-\lambda)^{-\frac{j}{2}-k-1}\,d\lambda\right) t^{\frac{j}{2}} + O(t^{N/2})$$

$$+(k+1)\sum_{j=0}^{N-1} d_{j,k}\left(\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}(-\lambda)^{-\frac{j}{2}-k-1}\log(-\frac{\lambda}{t})\,d\lambda\right) t^{\frac{j}{2}}\log t + O(t^{N/2-\varepsilon})$$

$$= \frac{(k+1)}{\Gamma(k+1)}c_{0,k} + (k+1)\sum_{j=1}^{N-1}\Gamma(\frac{j}{2}+k+1)^{-1}c_{j,k}\,t^{\frac{j}{2}}$$

$$+(k+1)\sum_{j=1}^{N-1}\Gamma'(\frac{j}{2}+k+1)^{-1}d_{j,k}\,t^{\frac{j}{2}} - (k+1)\sum_{j=1}^{N-1}\Gamma(\frac{j}{2}+k+1)^{-1}d_{j,k}\,t^{\frac{j}{2}}\log t + o(1)$$

$$= \frac{(k+1)}{\Gamma(k+1)}c_{0,k} + o(1)$$

$$= \frac{(k+1)}{\Gamma(k+1)}{\rm sTR}_{[\Delta_1,\Delta_2]}(\mathsf{R}^k) + o(1)\ .$$

For the degree 0 pointwise index, observe that when $k=0$ and the usual zeta function formula for the index we have from (4.84)

$$c_{0,0} = s\zeta_1(0) - s\zeta_2(0) = {\rm ind}\,(D_{P_1}) - {\rm ind}\,(D_{P_2})\ .$$

On the other hand, the degree zero part of the right-side of (4.87) is

$$\begin{aligned}
{\rm Str}(\mathsf{P}_1-\mathsf{P}_2) + {\rm sTR}_{[\Delta_1,\Delta_2]}(\mathsf{P}) &= {\rm Str}\left((\mathsf{P}_1-\mathsf{P}_2)\Delta_1^{-z}\right)|_{z=0}^{\rm mer} + {\rm Str}\left(\mathsf{P}_2(\Delta_1^{-z}-\Delta_2^{-z})\right)|_{z=0}^{\rm mer}\\
&= {\rm Str}\left(\mathsf{P}_1\Delta_1^{-z}-\mathsf{P}_2\Delta_2^{-z})\right)|_{z=0}^{\rm mer}\\
&= {\rm Str}\left(\Delta_1^{-z}-\Delta_2^{-z})\right)|_{z=0}^{\rm mer}\\
&= s\zeta_1(0) - s\zeta_2(0)\ .
\end{aligned}$$

This completes the proof. □

4.5. **Computing** $\lim_{t\to\infty}({\rm ch}(\mathbb{A}_{t,\mathcal{P}})$**.** The proof of Theorem (II) is completed by the following identification.

Proposition 4.19. *The cohomology class of the Chern character form* ${\rm ch}(\mathbb{A}_{t,\mathcal{P}})$ *is equal to* ${\rm ch}({\rm Ind}\,(D_{\mathcal{P}}))$.

Proof. The proof follows closely [BGV] and so we shall make only brief comments. We assume the kernels ${\rm Ker}(D^z_{P_z})$ form a super vector bundle ${\rm Ker}(\mathsf{D}_{\mathcal{P}})$ over B, identified with a smooth family of finite-rank projections Π_0 – the general case follows by standard arguments from this one. We then have ${\rm ch}(\nabla^0) = {\rm ch}({\rm Ind}\,(\mathsf{D}_{\mathcal{P}}))$, where $\nabla^0 = \Pi_0\cdot\nabla^{\pi_*(\mathbb{E})}\cdot\Pi_0$. By the Diagonalization Lemma of [BGV] there is an invertible $g\in\mathcal{A}(B,{\rm End}(\pi_*(\mathbb{E}))$ with $g-\mathbb{I}\in\sum_{r\geq 1}\mathcal{A}^r(B,\Psi^{-\infty}(\mathbb{E},\mathbb{E}^{'}))$ such that with respect to the decomposition ${\rm Ker}(\mathsf{D}_{\mathcal{P}})\oplus{\rm Ker}(\mathsf{D}_{\mathcal{P}})^{\perp}$

$$(\mathcal{F}_{t,\mathcal{P}}-\lambda\mathbb{I})^{-1} = \delta_t(g)^{-1}\left(\begin{bmatrix} t\delta_t(\mathcal{U}) & 0\\ 0 & t\delta_t(\mathcal{V})\end{bmatrix} - \lambda\mathbb{I}\right)^{-1}\delta_t(g)\ , \tag{4.91}$$

with

$$\begin{array}{llll} (4.92) & \delta_t(g)^{\pm} & = & \mathbb{I} + O(t^{-1/2}) \ , \\ (4.93) & t\delta_t(\mathcal{U}) & = & (\nabla^0)^2 + O(t^{-1/2}) \ , \\ (4.94) & t\delta_t(\mathcal{V}) & = & t\mathsf{D}^2_{\mathcal{P}} \quad \mod (\sum_{r\geq 1} \mathcal{A}^r(B, \mathrm{End}(\pi_*(\mathbb{E}^{'})))) \ . \end{array}$$

It follows from (4.91), (4.93),(4.94) that $e^{-\mathcal{F}_{t,\mathcal{P}}}$ is of the form

$$\delta_t(g)^{-1}(\frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\begin{bmatrix}\Pi_0((\nabla^0)^2-\lambda\mathbb{I})^{-1}\Pi_0 & 0 \\ 0 & \Pi_0^{\perp}(t\mathsf{D}^2_{\mathcal{P}}-\lambda\mathbb{I})^{-1}\Pi_0^{\perp}\end{bmatrix} d\lambda)\begin{bmatrix}\mathbb{I}+O(t^{-1/2}) & 0 \\ 0 & O(t^{1/2})\end{bmatrix}\delta_t(g)$$

and hence from (4.92) that

$$\mathrm{Str}(e^{-\mathcal{F}_{t,\mathcal{P}}}) = \mathrm{Str}(e^{-(\nabla^0)^2}) + O(t^{-1/2}) \ .$$

This implies that $\lim_{t\longrightarrow\infty} \mathrm{ch}(\mathbb{A}_{t,\mathcal{P}}) = \mathrm{ch}(\nabla^0)$, which completes the proof. □

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