# ZETA FORMS AND THE LOCAL FAMILY INDEX THEOREM

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## 0. Introduction and Preliminaries

Let $X$ be a $C^\infty$ $n$-dimensional compact Riemannian manifold without boundary and let $E = E^+ \oplus E^-$ be a $\mathbb{Z}_2$-graded (super) complex vector bundle over $X$. We write $\Gamma(X, E)$ for the space of $C^\infty$ sections of $E$ and $\tau$ for the involution defining the induced $\mathbb{Z}_2$-grading. The super (or $\mathbb{Z}_2$-graded) trace of a trace class operator $a$ on $\Gamma(X, E)$ is defined by $\mathrm{Str}(a) = \mathrm{Tr}\,(\tau a)$. Let $A$ and $F$ be classical (one-step polyhomogeneous) pseudodifferential operators ($\psi$dos) acting on $\Gamma(X, E)$. Suppose that $A$ is of order $\nu \in \mathbb{R}$ and that $F$ is elliptic of positive integer order $k$ and such that there is an angle $\theta$ for which the principal symbol $\sigma_k(F)(x, \xi)$ has no eigenvalues on $R_\theta = \{re^{i\theta} \mid r > 0\}$. In this situation, Grubb-Seeley [GS1] show that as $\lambda \longrightarrow \infty$ in an open sub-sector of $\mathbb{C}$ around $R_\theta$ there is an asymptotic expansion of the resolvent supertrace for $m > (n + \nu)/k$

$$(0.1)\quad \mathrm{Str}(A(F - \lambda I)^{-m}) \sim \sum_{j=0}^{\infty} \alpha_j (-\lambda)^{\frac{\nu+n-j}{k} - m} + \sum_{k=0}^{\infty} (\alpha_k^{'} \log(-\lambda) + \alpha_k^{''})(-\lambda)^{-k-m} \ .$$

On the other hand, for $\mathrm{Re}(s) > (n + \nu)/k$ the complex powers $AF_\theta^{-s}$ are trace class and a generalized (super) zeta function can be defined by

$$\zeta_\theta(A, F, s) = \mathrm{Str}(AF_\theta^{-s}) \ .$$

When $A$ is the identity we write $\zeta_\theta(F, s) := \zeta_\theta(I, F, s)$. It is well known [GS1, GS2, S] that the expansion (0.1) is essentially equivalent to the meromorphic extension $\zeta(A, F, s)|^{\mathrm{mer}}$ of $\zeta(A, F, s)$ (omitting the $\theta$ subscript) to all of $\mathbb{C}$ with the singularity structure

$$(0.2)\quad \Gamma(s)\, \zeta(A, F, s)|^{\mathrm{mer}} \sim \sum_{j=0}^{\infty} \frac{a_j}{s + \frac{j+\nu-n}{k}} - \frac{\mathrm{Str}(A\Pi_0(F))}{s} + \sum_{k=0}^{\infty} \frac{a_k^{'}}{(s+k)^2} + \frac{a_k^{''}}{s+k} \ ,$$

where $\Pi_0(F)$ is the orthogonal projection onto the kernel $\mathrm{Ker}(F)$ of $F$. The coefficients in (0.2) differ from those in (0.1) by universal multiplicative constants. The $a_j, a_j^{'}$ are local, being determined by finitely many homogeneous terms of the local symbol expansions, while the $a_j^{''}$ depend globally on $A, F$ and the bundle $E$.

Since $\Gamma(s)^{-1} = s + o(s)$ around $s = 0$, (0.2) implies that $\zeta_\theta(A, F, s)|^{\mathrm{mer}}$ is holomorphic at $s = 0$ provided $a_0^{'} = 0$. In particular, this holds for the zeta function $\zeta_\theta(P^2, s)|^{\mathrm{mer}}$ associated to the odd parity operator $P = \begin{bmatrix} 0 & P^- \\ P^+ & 0 \end{bmatrix}$, where $P^+ : \Gamma(X, E^+) \longrightarrow \Gamma(X, E^-)$ is a classical elliptic $\psi$do of positive order, and $P^-$ its formal adjoint. Since $P^+P^-$ and $P^-P^+$ have identical non-zero spectrum while

$P^{-2s}$ vanishes on $\mathrm{Ker}(P)$ for $\mathrm{Re}(s) > 0$, it follows that $\zeta_\theta(P^2, s)|^{\mathrm{mer}} = 0$. Evaluating at $s = 0$ gives the Atiyah-Bott-Seeley zeta function formula for the index

$$\zeta_\pi(P^2, 0)|^{\mathrm{mer}} = 0 \; ; \tag{0.3}$$

for, $\mathrm{Str}(\Pi_0(P^2)) = \dim \mathrm{Ker}(P^+) - \dim \mathrm{Ker}(P^-) := \mathrm{ind}\,(P)$ and hence from (0.2) (and (0.1)) equation (0.3) is the identity

$$\mathrm{ind}\,(P) = a_n + a_0^{''} = \alpha_n + \alpha_0^{''} \; . \tag{0.4}$$

When $P$ is a differential operator, then $a_0^{''} = 0$ and (0.4) gives a formula for the index as the integral over $X$ of a locally determined density.

Since $P^2$ is positive, (0.3) and (0.4) are further equivalent for $t > 0$ to the heat trace formula $\mathrm{ind}\,(P) = \mathrm{Str}(e^{-tP^2})$; — if $F$ is positive (0.1) and (0.2) are equivalent [GS2] to a heat trace expansion as $t \to 0+$ (with the same coefficients as (0.2))

$$\mathrm{Str}(Ae^{-tF}) \sim \sum_{j=0}^{\infty} a_j t^{\frac{j-\nu-n}{k}} + \sum_{k=0}^{\infty} (-a_k^{'} \log t + a_k^{''}) t^k \; . \tag{0.5}$$

If $a_0^{'} = 0$, the next term up in the Laurent expansion of $\zeta_\theta(F, s)|^{\mathrm{mer}}$ around $s = 0$ of a classical $\psi$do $F$ is the logarithm of the regularized (graded- or super-) zeta-determinant $\det_{\zeta,\theta} F$. Thus

$$\log \det_{\zeta,\theta} F = -\frac{d}{ds}\zeta_\theta(F, s)|^{\mathrm{mer}}_{s=0} = \zeta_\theta(\log F, F, s)|^{\mathrm{mer}}_{s=0} \; . \tag{0.6}$$

It is consistent to also write $\mathrm{sdet}_{\,\zeta,\theta}F$ for the super zeta-determinant, but here we prefer to retain the usual notation unless we need to emphasize the grading. Notice that in the case of the trivial grading the supertrace reduces to the usual operator trace and so $\det_{\zeta,\theta} F$ then coincides with the usual ungraded zeta determinant, while, for example, for even-parity $F = F^+ \oplus F^-$ one has

$$\mathrm{sdet}_{\zeta,\theta} F = \frac{\det_{\zeta,\theta} F^+}{\det_{\zeta,\theta} F^-}$$

with $\det_{\zeta,\theta} F^{\pm}$ ungraded zeta determinants.

In this paper we extend these constructions to geometric families of $\psi$dos. We consider a $C^\infty$ fibration $\pi : M \longrightarrow B$ of finite-dimensional manifolds with closed boundaryless fibre $M_z = \pi^{-1}(z)$ equipped with a Riemannian metric $g_{M/B}$ on the vertical tangent bundle $T(M/B)$. Let $| \wedge_\pi | = | \wedge (T^*(M/B))|$ be the line bundle of vertical densities, restricting on each fibre to the usual bundle of densities $| \wedge_{M_z} |$ along $M_z$. Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a vertical Hermitian super bundle over $M$ and let $\pi_*(\mathcal{E}) = \pi_*(\mathcal{E}^+) \oplus \pi_*(\mathcal{E}^-)$ the graded infinite-dimensional Frechet bundle with fibre $\Gamma(M_z, \mathcal{E}^z \otimes | \wedge_{M_z} |^{1/2})$ at $z \in B$, where $\mathcal{E}^z$ is the super bundle over $M_z$ obtained by restriction of $\mathcal{E}$. By definition, a $C^\infty$ section of $\pi_*(\mathcal{E})$ over $B$ is a $C^\infty$ section of $\mathcal{E} \otimes | \wedge_\pi |^{1/2}$ over $M$, and more generally the de-Rham complex of $C^\infty$ forms on $B$ with values in $\pi_*(\mathcal{E})$ is defined by

$$\mathcal{A}(B, \pi_*(\mathcal{E})) = \Gamma(M, \pi^*(\wedge T^*B) \otimes \mathcal{E} \otimes | \wedge_\pi |^{1/2})$$

with $\otimes$ the graded tensor product. We write $\Psi(\mathcal{E})$ for the infinite-dimensional bundle of algebras with fibre $\Psi(\mathcal{E}^z)$ the space of classical $\psi$dos on $\Gamma(M_z, \mathcal{E}^z \otimes | \wedge_{M_z} |^{1/2})$.

A section $\mathsf{F} \in \mathcal{A}(B, \Psi(\mathcal{E}))$ defines a smooth family of $\psi$dos with differential form coefficients parameterized by $B$. If the smooth family of $\psi$dos

$$\mathsf{P} := \mathsf{F}_{[0]} \in \Gamma(B, \Psi(\mathcal{E})) \tag{0.7}$$

defined by the form degree zero component of $\mathsf{F}$ has positive order and admits a spectral cut $R_\theta$ then, for an auxiliary family of $\psi$dos $\mathsf{A} \in \mathcal{A}(B, \Psi(\mathcal{E}))$, we use the fibrewise supertrace to define for $\mathrm{Re}(s) >> 0$ a mixed degree differential form

$$\zeta_\theta(\mathsf{A}, \mathsf{F}, s) = \mathrm{Str}(\mathsf{A}\mathsf{F}_\theta^{-s}) \in \mathcal{A}(B). \tag{0.8}$$

When $\mathsf{A} = \mathsf{I}$ is the vertical family of identity operators we write $\zeta_\theta(\mathsf{F}, s) := \zeta_\theta(\mathsf{I}, \mathsf{F}, s)$. In a similar way to the single operator case, an analysis of the asymptotic behavior of the corresponding resolvent trace differential form defined for sufficiently large $m$ and $|\lambda|$

$$\mathrm{Str}(\mathsf{A}(\mathsf{F} - \lambda\mathsf{I})^{-m}) \in \mathcal{A}(B, \Psi(\mathcal{E})) \ ,$$

shows that if the kernels $\mathrm{Ker}P_z$ of the family (0.7) have constant dimension, then the zeta trace form (0.8) extends meromorphically on $\mathbb{C}$ to a form $\zeta_\theta(\mathsf{A}, \mathsf{F}, s)|^{\mathrm{mer}}$.

For a family of strictly positive operators $\mathsf{F}$ one has additionally the heat trace form

$$\mathrm{Str}(\mathsf{A}e^{-\mathsf{F}}) \in \mathcal{A}(B) \ , \tag{0.9}$$

which, in the case when $\mathsf{A} = \mathsf{I}$ and $\mathsf{F}$ is the curvature form of a superconnection, is the object of interest in Bismut's heat equation proof of the local Atiyah-Singer index theorem for families of Dirac operators. However, it is the resolvent trace form, or equivalently the zeta trace form, which is the more fundamental geometric invariant in so far as the heat trace forms are computed from these by simple transition formulae, and, moreover, the resolvent and zeta trace forms are defined for families of non-positive operators; for example, the zeta form for a family of self-adjoint first-order elliptic differential operators over an odd-dimensional manifold. This is concordant with [GS1, GS2] where the resolvent trace and power operators were shown to provide a considerably more powerful tool for $\psi$do analysis, than heat kernel methods alone – a principle which applies equally well to families of $\psi$dos. These constructions are by no means restricted only to Dirac operators, the latter are only really of interest if one is after explicit formulas for locally determined coefficients in the trace expansions, such as in the case of the local index density, for example.

It is of interest, then, to use these methods to compute higher geometric invariants as generalized $\zeta$-form invariants of special interest such as Wodzicki residue trace forms (see [SP]) which extends the usual residue trace functional in so far as it vanishes on super commutators of families of $\psi$dos, the Kontsevich-Vishik trace form, eta-forms, analytic torsion forms, and the extension of the corresponding zeta form invariants to families of singular manifolds. Here we illustrate the methods with an alternative view point onto the Atiyah-Singer family index theorem.

The following formulas generalize the Atiyah-Bott-Seeley formula (0.3).

**Theorem 0.1.** *Let* $\mathbb{A}^2 \in \mathcal{A}(B, \Psi(\mathcal{E}))$ *be the curvature form of a superconnection* $\mathbb{A}$ *on* $\pi_*(\mathcal{E})$ *adapted to a smooth family of elliptic* $\psi$dos $\mathsf{P} = (P_z \mid z \in B) \in \Gamma(B, \Psi(\mathcal{E}))$.

*Suppose that the family of kernels* $\mathrm{Ker}P_z = \mathrm{Ker}P_z^+ \oplus \mathrm{Ker}P_z^-$ *have constant dimension as* $z$ *varies in* $B$*, forming a super bundle* $\mathrm{Ker}\mathsf{P}$ *over* $B$*. Then* $\zeta_\pi(\mathbb{A}^2, s)|^{\mathrm{mer}}$ *is canonically exact in* $\mathcal{A}(B)$*. There is a canonical transgression form* $\tau_{\mathbb{A}^2} \in \mathcal{A}(B)$ *such that the following formula holds in* $\mathcal{A}(B)$

$$\sum_{k=0}^{\dim B} \frac{1}{k!}\, \zeta_\pi(\mathbb{A}^2, -k)|^{\mathrm{mer}} = d\tau_{\mathbb{A}^2} \tag{0.10}$$

*and implies the Family Index Theorem transgression formula for the Chern character form. Precisely, replacing* $\mathbb{A}$ *by the* $t$*-rescaled superconnection* $\mathbb{A}_t$*, the regularized limit as* $t \to 0+$ *of* (0.10) *is the formula*

$$\begin{aligned} \mathrm{ch}(\mathrm{Ker}\mathsf{P}, \nabla^0) &= \mathrm{LIM}_{t\to 0}\left(\mathrm{ch}(\mathbb{A}_t) - d\int_t^\infty \mathrm{Str}(\dot{\mathbb{A}}_\sigma e^{-\mathbb{A}_\sigma^2})\, d\sigma\right) \\ &= \int_{M/B} \mathrm{Str}(A(x)) - d\,\mathrm{LIM}_{t\to 0}\left(\int_t^\infty \mathrm{Str}(\dot{\mathbb{A}}_\sigma e^{-\mathbb{A}_\sigma^2})\, d\sigma\right) , \end{aligned} \tag{0.11}$$

*where* $A$ *is a section of the bundle* $\wedge T^*M \otimes \mathrm{End}(\mathcal{E})$ *over* $M$ *whose* $d$*-form component* $A_{[d]}$ *is the coefficient of* $\lambda^{-m-1-d/2}$ *in the asymptotic expansion of the kernel of the resolvent* $(\mathbb{A}^2 - \lambda \mathsf{I})^{-m} \in \mathcal{A}(B, \Psi(\mathcal{E}))$ *as* $\lambda \to \infty$ *in a subsector of* $\mathbb{C}$.

In (0.11), $\mathbb{A}_t$ is defined by multiplying the form degree $i$ component $\mathbb{A}_{[i]}$ of $\mathbb{A}$ by $t^{1-i/2}$. The scaled super Chern character form is defined by $\mathrm{ch}(\mathbb{A}_t) = \mathrm{Str}(e^{-\mathbb{A}_t^2})$, while for a finite rank super bundle $V$ with connection $\nabla$, $\mathrm{ch}(V, \nabla) = \mathrm{Str}(e^{-\nabla^2})$ is the classical graded Chern character form. If $\Pi_0$ is the smooth family of smoothing operator projections onto $\mathrm{Ker}\mathsf{P}$, $\nabla^0 = \Pi_0 \cdot \mathbb{A}_{[1]} \cdot \Pi_0$ is the induced classical connection on $\mathrm{Ker}\mathsf{P}$. $\int_{M/B} : \mathcal{A}^k(M) \longrightarrow \mathcal{A}^{k-n}(B)$ denotes integration over the fibre, while the regularized limit $\mathrm{LIM}_{t\to 0}$ picks out the $t^0$ term in the asymptotic expansion as $t \to 0+$.

Notice from (0.10) that it is now not just the value of the zeta function at zero that determines the index, but also its meromorphically continued value at a finite number of negative integers.

By standard index theory arguments Theorem 0.1 implies the following general cohomological formula for any smooth family of elliptic $\psi$dos.

**Corollary 0.2.** *For any elliptic family of* $\psi$dos $\mathsf{P} \in \Gamma(B, \Psi^{>0}(\mathcal{E}))$ *there exists a smooth family of smoothing operators* $\mathsf{K} \in \Gamma(B, \Psi^{-\infty}(\mathcal{E}))$ *such that* $\mathsf{P}+\mathsf{K}$ *has constant kernel dimension and* $\mathrm{Ind}\,(\mathsf{P}+\mathsf{K}) = \mathrm{Ind}\,(\mathsf{P})$ *in* $K(B)$*. The following formula holds in* $H^*(B)$

$$\sum_{k=0}^{\dim B} \frac{1}{k!}\, \zeta_\pi(\mathbb{A}^2 + \mathsf{K}, -k)|^{\mathrm{mer}} = 0 \tag{0.12}$$

*and implies the cohomological family index theorem in* $H^*(B)$

$$\mathrm{ch}(\mathrm{Ind}\,\mathsf{P}) = \mathrm{LIM}_{t\to 0}\mathrm{ch}(\mathbb{A}_t) = \int_{M/B} \mathrm{Str}(A(x)) \ . \tag{0.13}$$

Equivalently, this zeta form may be viewed as a map $K(B) \longrightarrow H^*(B)$ which vanishes identically – naturally generalizing (0.3). More generally, the meromorphically continued zeta form for a smooth family of $\psi$dos is a K-theoretic invariant.

Following the single operator case, for a general smooth family of $\zeta$-admissible $\psi$dos $\mathsf{F} \in \mathcal{A}(B, \Psi(E))$ with spectral cut $R_\theta$ the next term up in the Laurent expansion around $s = 0$ of $\zeta_\theta(\mathsf{F}, s)$ defines the logarithm of the super zeta determinant form

$$\det_{\zeta,\theta}\mathsf{F} \in \mathcal{A}(B) \ . \tag{0.14}$$

This is a non-local mixed-degree differential form invariant which extends the classical operator zeta determinant. We may on occasion write $\mathrm{sdet}_{\zeta,\theta}\mathsf{F}$ for (0.14) to emphasize the grading.

**Lemma 0.3.** *Let $P = F_{[0]} \in \Gamma(B, \Psi^{>0}(\mathcal{E}))$ be the degree zero component of $F$. Then $P$ is $\zeta$-admissible and*

$$\det_{\zeta,\theta} F = \det_{\zeta,\theta} P + \omega_{\zeta,\theta}(F) \ , \tag{0.15}$$

*where $\omega_{\zeta,\theta}(F) \in \mathcal{A}^{>0}(B) = \sum_{k\geq 1} \mathcal{A}^k(B)$.*

Hence the degree zero part of $\det_{\zeta,\theta}\mathsf{F}$ coincides with the classical zeta determinant function $z \longmapsto \det_{\zeta,\theta} P_z$. Lemma 0.3 is proved in Section 2.

We use these methods to give the following geometric application of the $\zeta$-determinant form. The total Chern class on the semi-group $\mathrm{Vect}(X)$ of finite rank complex vector bundles defines a stable characteristic class and so descends to a ring homomorphism

$$\mathsf{c} : K(X) \longrightarrow H^*(B) \ ,$$

on the K-ring of virtual bundles, which is an isomorphism on the rational coefficient ring. For an element $[V^+] - [V^-] \in K(B)$, represented by $V^{\pm} \in \mathrm{Vect}(X)$, one has

$$\mathsf{c}([V^+] - [V^-]) = \frac{\mathsf{c}(V^+)}{\mathsf{c}(V^-)} \ . \tag{0.16}$$

To construct a de-Rham representative for the cohomology identity (0.16) we may use Quillen's observation [Q] that just as form representatives for the characteristic classes of a vector bundle can be computed from a connection, so the characteristic class forms of a virtual bundle can be computed from a superconnection, extending Chern-Weil theory from $\mathrm{Vect}(X)$ to $K(X)$. This applies to the Chern class in the infinite-dimensional setting in the following way.

**Theorem 0.4.** *Let $\mathbb{A}$ be a superconnection adapted to a family of self-adjoint Dirac operators $D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$ associated to a Clifford connection over a Riemannian fibration of spin manifolds $\pi : M \to B$. The zeta-Chern form defined by the super zeta determinant form*

$$c_\zeta(\mathbb{A}) = \mathrm{sdet}_{\zeta,\pi}(I + \mathbb{A}^2) \in \mathcal{A}(B) \tag{0.17}$$

*is a closed differential form and a homotopy invariant of $\mathbb{A}$ representing the Chern class $c(\mathrm{Ind}\,(D)) \in H^*(B)$ of the index bundle.*

*For $t>0$, let $\mathbb{A}_t$ be the scaled superconnection. If $\mathsf{D}$ has constant kernel dimension, defining a super bundle $\mathrm{Ker}(\mathsf{D})\to B$, then there is a Chern-Simons generalized zeta form $\omega_{t,\infty}(\mathbb{A})\in\mathcal{A}(B)$ such that for all $t>0$ the transgression formula*

$$\mathsf{c}(\mathrm{Ker}(\mathsf{D}),\nabla^0)=\mathsf{c}_\zeta(\mathbb{A}_t)+d\omega_{t,\infty}(\mathbb{A}) \tag{0.18}$$

*holds in $\mathcal{A}(B)$, where $\nabla^0$ is defined in Theorem 0.1. The left-side of (0.18) is the classical (super) Chern form, defined for a finite-rank complex graded vector bundle $V$ with connection $\nabla$ by $\mathsf{c}(V,\nabla)=\mathrm{sdet}(I+\nabla^2):=\exp(\mathrm{Str}(\log(I+\nabla^2)))$. (A precise formula for $\omega_{t,\infty}(\mathbb{A})$ is given in Section 5.)*

*When $\mathbb{A}_t$ is the Bismut superconnection then the form $\mathsf{c}_\zeta(\mathbb{A}_t)$ has a limit in $\mathcal{A}(B)$ as $t\to 0+$ with*

$$\lim_{t\to 0+}\mathsf{c}_\zeta(\mathbb{A}_t)=\prod_{j=1}^{[\dim B/2]} e^{(-1)^{j-1}(j-1)!\left((2\pi i)^{-n}\int_{M/B}\widehat{A}(M/B)\ \mathrm{ch}'(\mathcal{E})\right)_{[2j]}}\ , \tag{0.19}$$

*where $\widehat{A}(M/B)=\det^{1/2}\left(\frac{R^{M/B}/2}{\sinh(R^{M/B}/2)}\right)$ is the vertical $\widehat{A}$-genus form and $\mathrm{ch}'(\mathcal{E})$ is the twisted Chern character form for $\mathcal{E}$. Here $\tau_{[j]}$ is the $j$-form component of $\tau\in\mathcal{A}(B)$.*

By standard index theory arguments the constant kernel condition can be dropped for the cohomological formula for $\mathsf{c}(\mathrm{Ind}\,(\mathsf{D}))$: Theorem 0.4 implies the following Chern class Families Index Theorem.

**Corollary 0.5.** *For any smooth family of Dirac operators $\mathsf{D}$ associated to a Clifford connection one has in $H^*(B)$*

$$\mathsf{c}(\mathrm{Ind}\,(\mathsf{D}))=\prod_{j=1}^{[\dim B/2]} e^{(-1)^{j-1}(j-1)!\left((2\pi)^{-n}\int_{M/B}\widehat{A}(M/B)\ \mathrm{ch}'(\mathcal{E})\right)_{[2j]}}\ . \tag{0.20}$$

**Remarks.**
[1] As a generalized zeta-determinant, the form $\mathsf{c}_\zeta(\mathbb{A})$ is a highly non-local invariant. That the identities (0.19), (0.20) hold is due to a localization of $\mathsf{c}_\zeta(\mathbb{A}_t)$ when $\mathbb{A}_t$ is the scaled Bismut superconnection

$$\mathsf{c}_\zeta(\mathbb{A}_t)=\mathsf{c}_{\mathrm{local},\zeta}(\mathbb{A})+\mathsf{c}_{\mathrm{global},\zeta}(\mathbb{A})(t)+d\omega_t$$

into a local term plus a global term $\mathsf{c}_{\mathrm{global},\zeta}(\mathbb{A})(t)$ which is $O(t^{1/2})$ and an exact global term which is $O(1)$ as $t\to 0+$. For a general superconnection the form $\mathsf{c}_\zeta(\mathbb{A}_t)$ does not converge as $t\to 0+$.

[2] The determinant line bundle $\mathcal{L}\to B$ of a family of elliptic $\psi$dos $\mathsf{P}\in\Gamma(B,\Psi(\mathcal{E}))$ of odd parity is the complex line bundle with fibre $\wedge^{\max}\mathrm{Ker}(P_z^+)^*\otimes\wedge^{\max}\mathrm{Ker}(P_z^-)$. The Quillen-Bismut-Freed connection is defined on $\mathcal{L}$ via a meromorphically continued zeta trace [Q, BF] and one has:

**Corollary 0.6.** *The curvature of the Quillen-Bismut-Freed connection on $\mathcal{L}$ is*

$$R^{(\mathcal{L})}=\mathrm{LIM}_{t\to 0+}\mathsf{c}_\zeta(\mathbb{A}_t)_{[2]}\ .$$

## 1. Asymptotic Expansion of the Resolvent Trace Form

Let $\pi : M \longrightarrow B$ be a smooth family of closed Riemannian manifolds with Hermitian vector bundle $\mathcal{E} \longrightarrow M$, as in the introduction. Let $\Psi(\mathcal{E})$ be the bundle of subalgebras of $\mathrm{End}(\pi_*(\mathcal{E}))$ of classical $\psi$do s, and let $\Psi^\nu(\mathcal{E})$ (resp. $\Psi^{<\nu}(\mathcal{E})$) be the subbundle of operators of order $\nu \in \Gamma(B)$ (resp. less than $\nu$). Thus the fibre of $\Psi^\nu(\mathcal{E})$ at $z \in B$ is the algebra $\Psi^{\nu(z)}(M_z, \mathcal{E}_z)$ of classical $\psi$dos on $\Gamma(M_z, \mathcal{E}_z)$ of order $\nu(z) \in \mathbb{R}$, and its sections are families of $\psi$dos which in any local trivialization of $M$ and $\mathcal{E}$ over an open subset $U \subset B$ depend smoothly on the local coordinates.

The fibre product $M \times_\pi M$ is the fibration over $B$ with fibre $M_z \times M_z$ and vertical bundle $\mathcal{E} \boxtimes \mathcal{E} := p_1^*(\mathcal{E}) \otimes p_2^*(\mathcal{E})^*$, where $p_1, p_2 : M \times_\pi M \to M$ are the canonical projection maps. For a smooth family of $\psi$dos with differential form coefficients $\mathsf{Q} \in \mathcal{A}(B, \Psi(\mathcal{E}))$, if $x \in M$ is not in the support of $\psi \in \mathcal{A}(B, \pi(\mathcal{E}))$ then there is a smooth family of smooth kernels on $M \times_\pi M \backslash \Delta(M)$, where $\Delta(M)$ is the diagonal $\{(x,x) \mid x \in M\}$ in $M \times_\pi M$,

$$K(\mathsf{Q}) \in \Gamma(M \times_\pi M \backslash \Delta(M), \pi^*(\wedge T^*B) \otimes (\mathcal{E} \otimes |\Lambda_\pi|^{1/2}) \otimes (\mathcal{E}^* \otimes |\Lambda_\pi|^{1/2}))$$

with $|\Lambda_\pi|^{1/2}$ the line bundle of half-densities along the fibres of $M$, such that

$$(\mathsf{Q}\psi)(x) = \int_{M/B} K(\mathsf{Q})(x,y)\psi(y) \ . \tag{1.1}$$

Restricted to the fibre $M_z$ (1.1) reduces to the usual pointwise kernel formula; for $x \in M_z$ not in the support of $\psi \in \Gamma(M_z, \mathcal{E}_z)$

$$(Q^z\psi)(x) = \int_{M_z} K(Q^z)(x,y)\psi(y) \ . \tag{1.2}$$

For a family $\mathsf{Q} \in \mathcal{A}(B, \Psi^{<-n}(\mathcal{E}))$ of order less than the fibre dimension $K(\mathsf{Q})$ extends continuously across $\Delta(M)$, and hence applying to $\mathsf{Q}$ the $\mathcal{A}(B)$ valued supertrace

$$\mathrm{Str} : \mathcal{A}(B, \Psi^{<-n}(\mathcal{E})) \longrightarrow \mathcal{A}(B) \ ,$$

defined fibrewise for $z \in B$ by the operator supertrace on $\Psi^{<-n}(\mathcal{E}^z)$, defines a differential form $\mathrm{Str}(\mathsf{Q}) \in \mathcal{A}(B)$. On the other hand, the restriction of $K(\mathsf{Q})$ to the diagonal $M \subset M \times_\pi M$ defines a continuous section of $\pi^*(\wedge T^*B) \otimes \mathrm{End}(\mathcal{E}) \otimes |\Lambda_\pi|$ depending smoothly on the base parameters, and so the $\mathrm{End}(\mathcal{E})$-supertrace defines a section $\mathrm{Str}(K(\mathsf{Q})(x,x))$ in $\Gamma(M, \pi^*(\wedge T^*B) \otimes |\Lambda_\pi|)$ which can be integrated over the fibres and we have

$$\mathrm{Str}(\mathsf{Q}) = \int_{M/B} \mathrm{Str}\,(K(\mathsf{Q})(x,x)) \in \mathcal{A}(B) \ . \tag{1.3}$$

For the general case, $\mathsf{Q} \in \mathcal{A}(B, \Psi(\mathcal{E}))$ means that in any local trivialization of $M$ and $\mathcal{E}$ the operator $\mathsf{Q}$ is represented by a *vertical polyhomogeneous symbol.* This means the following. Assume a local trivialization $M_{|U_B} \cong U_B \times M_{z_0}$ over an open subset $U_B \subset B$ with $z_0 \in U_B$, and a trivialization $\mathcal{E}_{U_M} \cong U_M \times \mathbb{R}^N$, where $\mathbb{R}^N$ inherits a grading from $\mathcal{E}_{U_M}$, over an open subset $U_M \subset \pi^{-1}(U_B)$. $U_M$ may be identified as a product $U_M \cong U_B \times U_{z_0} \cong \mathbb{R}^{\dim B} \times \mathbb{R}^n$ with $U_{z_0} = U_M \cap M_{z_0}$. A vertical symbol may be written according to form degree as $\mathbf{q} = \mathbf{q}_{[0]} + \ldots + \mathbf{q}_{[\dim B]}$ with

$$\mathbf{q}_{[k]}(z,x,\xi) \in \Gamma\left(U_B \times U_{z_0} \times \mathbb{R}^n\backslash\{0\},\ \pi^*(\wedge^k T^* U_B) \otimes \mathbb{R}^N \times (\mathbb{R}^N)^*\right) \ , \tag{1.4}$$

where $\xi$ may be identified with an element of the vertical (or fibre) cotangent space $T^*_x(M/B)$. Each $\mathbf{q}_{[k]}$ can be written (locally) as a finite sum of terms of the form $\sum_{j=0}^J \omega_j \otimes q_{[k],j}$ with $\omega_j \in \mathcal{A}^k(U_B)$ a basis of local $k$-forms and $q_{[k],j}$ is a symbol (in the usual single manifold sense) of form degree zero. For clarity of exposition we shall assume for the moment that the vertical symbols are *simple*, meaning that they have the local form $\mathbf{q}_{[k]} = \omega_k \otimes q_{[k]}$ with just one term in each form degree ($J = 0$). That $\mathbf{q}_{[k]}$ be a vertical (simple) symbol of order $\nu \in \Gamma(B, \mathbb{R}^{\dim B+1})$ is the growth requirement in fibre direction that for $k = 0, \ldots, \dim B$ and for all multi-indices $\alpha, \beta, \gamma$

$$|\partial_z^\gamma \partial_x^\alpha \partial_\xi^\beta \mathbf{q}_{[k]}(z,x,\xi)| < C(1+|\xi|)^{\nu_k(z)-|\beta|} \ , \tag{1.5}$$

where $x \in U_{z_0}$, $z \in U_B$, and $\nu(z) = (\nu_1(z), \ldots, \nu_{\dim(B)+1}(z))$, while on the left-side $|\,.\,|$ is a choice of norm on $\pi^*(\wedge^k T^* U_B) \otimes \mathbb{R}^N \times (\mathbb{R}^N)^*$. The estimate (1.5) holds uniformly in $\xi$, and on compact subsets of $U_B \times U_{z_0}$ uniformly in $(z, x)$. A vertical symbol $\mathbf{q}$ is *classical* (1-step polyhomogeneous) of order $\nu \in \Gamma(B, \mathbb{R}^{\dim B+1})$ if there is an asymptotic expansion

$$\mathbf{q}_{[k]}(z,x,\xi) \sim \sum_{j\geq 0} \mathbf{q}_{[k],j}(z,x,\xi) \tag{1.6}$$

as $|\xi| \to \infty$, meaning $\mathbf{q}_{[k]}(z,x,\xi) - \sum_{j=0}^{N-1} \mathbf{q}_{[k],j}(z,x,\xi)$ is a symbol of order $\nu_k(z) - N$, and where $\mathbf{q}_{[k],j}(z,x,\xi)$ is positively homogeneous of degree $\nu_k(z) - j$ in $\xi$, meaning that for $t > 0$

$$\mathbf{q}_{[k],j}(z,x,t\xi) = t^{\nu_k(z)-j}\mathbf{q}_{[k],j}(z,x,\xi) \ .$$

Then $\mathsf{Q} : \mathcal{A}(B, \pi_*(\mathcal{E})) \to \mathcal{A}(B, \pi_*(\mathcal{E}))$ is a *simple* vertical classical family of $\psi$dos parameterized by $B$ and of order $\nu \in \Gamma(B, \mathbb{R}^{\dim B+1})$, and we write $\mathsf{Q} \in \mathcal{A}(B, \pi_*(\mathcal{E}))$, if in any local trivialization, as above, there is a simple vertical classical symbol $\mathbf{q} = \mathbf{q}_{[0]} + \ldots + \mathbf{q}_{[\dim B]}$ of order $\nu$ such that for $\psi$ with support in a compact subset of $U_M$

$$\mathsf{Q}\psi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{U_{z_0}} e^{i(x-y).\xi} \mathbf{q}(z,x,\xi) \psi(y)\, dy\, d\xi \ + \ \mathsf{R}\psi(x),$$

where $\mathsf{R}$ is a smooth family of smoothing operators, meaning that it is defined by a smooth vertical kernel $K(\mathsf{R})(x,y)$.

In particular, if as before $\mathsf{Q} \in \mathcal{A}(B, \Psi^{<-n}(\mathcal{E}))$, meaning that $\nu_i(z) < -n$ for $i = 0, 1, \ldots, \dim B + 1$, then for the corresponding vertical volume form, (1.3) is pointwise the differential form in $\mathcal{A}(B)$

$$\mathrm{Str}(\mathsf{Q})(z) = \int_{M/B} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{Str}(\mathbf{q}(z,x,\xi))\, d\xi\, \mathrm{vol}_{M/B} \ .$$

Notice that writing $\mathbf{q}$ according to form degree corresponds to writing

$$\mathsf{Q} = \mathsf{Q}_{[0]} + \mathsf{Q}_{[1]} + \ldots + \mathsf{Q}_{[\dim B]}$$

where $\mathsf{Q}_{[k]} \in \mathcal{A}^k(B, \Psi(\mathcal{E}))$ raises form degree in $\mathcal{A}(B, \pi_*(\mathcal{E}))$ by $k$. With respect to a local (weak) trivialization of $\pi_*(\mathcal{E})$ over $U \subset B$ one has

$$\mathcal{A}(U, \pi_*(\mathcal{E})_{|U}) \cong \mathcal{A}(U) \otimes \Gamma(M_{z_0}, \mathcal{E}^{z_0})$$

for $z_0 \in U$ and so a general $\mathsf{Q}_{[k]}$ can be written locally as a sum of simple vertical $\psi$dos $\mathsf{Q}_{[k]|U} = \sum_{j=0}^{J} \omega_{k,j} \otimes Q_j$ , with $\omega_{k,j} \in \mathcal{A}^k(U)$ and $Q_j \in \Gamma(U, \Psi^{\nu_j(z_0)}(\mathcal{E}_{z_0}))$.

Composition defines a canonical algebra structure on $\mathcal{A}(B, \Psi(\mathcal{E}))$, coinciding with the usual pointwise structure on $\Psi(\mathcal{E}_z)$, such that

$$\mathcal{A}^i(B, \Psi^{\nu}(\mathcal{E})) \times \mathcal{A}^j(B, \Psi^{\mu}(\mathcal{E})) \longrightarrow \mathcal{A}^{i+j}(B, \Psi^{\nu+\mu}(\mathcal{E})) \ . \tag{1.7}$$

The multiplication (1.7) is defined locally at the symbol level; if $\mathsf{A} \in \mathcal{A}^i(B, \Psi^{\nu}(\mathcal{E}))$, $\mathsf{B} \in \mathcal{A}^j(B, \Psi^{\nu}(\mathcal{E}))$ with simple local symbols given over $U_M$ by

$$\mathbf{a} = \omega_{[i]} \otimes a \in \Gamma\left((U_M \times_\pi U_M) \times \mathbb{R}^n \backslash \{0\},\ \pi^*(\wedge^i T^* U_B) \otimes \mathbb{R}^N \times (\mathbb{R}^N)^*\right) \ ,$$

$$\mathbf{b} = \sigma_{[j]} \otimes b \in \Gamma\left((U_M \times_\pi U_M) \times \mathbb{R}^n \backslash \{0\},\ \pi^*(\wedge^j T^* U_B) \otimes \mathbb{R}^N \times (\mathbb{R}^N)^*\right)$$

then $\mathsf{AB} \in \mathcal{A}^{i+j}(B, \Psi^{\nu+\mu}(\mathcal{E}))$ is defined by the vertical polyhomogeneous symbol

$$\mathbf{a} \circ \mathbf{b} = \omega_{[i]} \wedge \sigma_{[j]} \otimes (a \circ b) \ , \tag{1.8}$$

where, as elsewhere, $\otimes$ means the graded tensor product, and $a \circ b \sim \sum_j (a \circ b)_j$ with

$$(a \circ b)_j = \sum_{|\alpha|+k+l=j} \frac{(-i)^{\alpha}}{a!} \partial_\xi^\alpha (a)_k \partial_x^\alpha (b)_l \ .$$

The crucial property of a family of $\psi$dos in $\mathcal{A}(B, \Psi(\mathcal{E}))$ is that ellipticity properties are determined by its form degree zero component. For a family of $\psi$dos $\mathsf{P} \in \Gamma(B, \Psi(\mathcal{E}) = \mathcal{A}^0(B, \Psi(\mathcal{E}))$, with differential form degree zero, the principal symbol, defined in any local trivialization to be the leading term $\mathbf{p}_0$ in the asymptotic expansion (1.6), has an invariant realization as a smooth section

$$\mathbf{p}_0 \in \Gamma(T(M/B), \varphi^*(\mathrm{End}(\mathcal{E}))) \ ,$$

where $\varphi : T(M/B) \longrightarrow M$ is the tangent bundle along the fibres.

**Definition 1.1.** *A smooth family of* $\psi$dos $\mathsf{Q} \in \mathcal{A}(B, \Psi(\mathcal{E}))$ *with differential form coefficients is said to be elliptic with principal angle* $\theta$ *if its form degree zero component* $\mathsf{P} := \mathsf{Q}_{[0]} \in \Gamma(B, \Psi(\mathcal{E}))$ *is elliptic with principal angle* $\theta$. *This means that*

$$\mathbf{p}_0 - \lambda \mathcal{I} \in \Gamma(T(M/B) \backslash \{0\}, p^*(\mathrm{End}(\mathcal{E})))$$

*is an invertible bundle map for* $\lambda \in R_\theta = \{re^{i\theta} \mid r > 0\}$, *where* $\mathcal{I}$ *is the identity bundle operator,*

**Proposition 1.2.** *If* $\mathsf{Q} \in \mathcal{A}(B, \Psi(\mathcal{E}))$ *is elliptic with principal angle* $\theta$, *then there is an open sector* $\Gamma_\theta \subset \mathbb{C} - \{0\}$ *containing* $R_\theta$ *such that on any compact codimension zero submanifold* $B_c$ *of* $B$ *for large* $\lambda \in \Gamma_\theta$ *there is a smooth family of vertical resolvent* $\psi$dos

$$(\mathsf{Q} - \lambda \mathsf{I})^{-1} \in \mathcal{A}(B_c, \Psi(\mathcal{E})) \ . \tag{1.9}$$

*Here $\mathsf{I}$ is the vertical identity, defined by the symbol $\mathbf{I} = 1 \otimes I$, coinciding pointwise with the identity $I_z$ on $\Gamma(M_z, \Psi(\mathcal{E}_z))$, while the form degree zero component of* (1.9) *coincides pointwise with the usual* $\psi$do *resolvent for $\mathsf{P} := \mathsf{Q}_{[0]}$. Precisely, let*

$$\mathsf{Q}_{[>0]} = \mathsf{Q} - \mathsf{P} \in \mathcal{A}^1(B, \Psi(\mathcal{E}))\,)$$

*be the component of $\mathsf{Q}$ with non-zero form degree. Then for large $\lambda \in \Gamma_\theta$ the following identity holds in $\mathcal{A}(B_c, \Psi(\mathcal{E}))$*

$$(\mathsf{Q} - \lambda \mathsf{I})^{-1} = (\mathsf{P} - \lambda \mathsf{I})^{-1} + \sum_{k=1}^{\dim B} (-1)^k (\mathsf{P} - \lambda \mathsf{I})^{-1} \left( \mathsf{Q}_{[>0]} (\mathsf{P} - \lambda \mathsf{I})^{-1} \right)^k \ . \tag{1.10}$$

*Proof.* The resolvent for $\mathsf{P} \in \mathcal{A}^0(B, \Psi^{\nu_0}(\mathcal{E}))$ can be constructed using a standard procedure [S, Sh]. Locally, with respect to trivializations, for

$$\mu \in \Gamma_{\theta,m} = \{ \upsilon \in \mathbb{C} \backslash \{0\} \mid \upsilon^m \in \Gamma_\theta \}$$

inductively define vertical symbols $\mathbf{b}_j[\mu] \in \Gamma\left( (U_M \times_\pi U_M) \times \mathbb{R}^n \backslash \{0\},\ \mathbb{R}^N \times (\mathbb{R}^N)^* \right)$ homogeneous in $(\mu, \xi)$ of degree $-\nu - j$ by

$$\mathbf{b}_0[\mu](z, x, \xi) = (\mathbf{p}_0(z, x, \xi) - \mu^m I)^{-1} \ ,$$

(1.11)

$$\mathbf{b}_j[\mu](z, x, \xi) = (\mathbf{p}_0(z, x, \xi) - \mu^m I)^{-1} \sum_{\substack{|\alpha| + k + l = j \\ k < j}} \frac{(-i)^\alpha}{\alpha!} \partial_\xi^\alpha \mathbf{p}_k(z, x, \xi) \partial_x^\alpha \mathbf{b}_l[\mu](z, x, \xi) \ ,$$

so that

$$\left( \sum \mathbf{p}_k(z, x, \xi) - \mu^m I \right) \circ \left( \sum \mathbf{b}_j[\mu] \right) \sim \mathbf{I} \ . \tag{1.12}$$

It follows that for $\lambda = \mu^m$ there exists a vertical polyhomogeneous symbol $\mathbf{b}[\lambda] \sim \sum \mathbf{b}_j[\mu]$. By a standard partition of unity construction, and using the local trivializations, we can patch together to define $\mathsf{B} = \mathrm{OP}(\mathbf{b}) \in \Gamma(B, \Psi^{-\nu_0}(\mathcal{E}))$ and from (1.12) $(\mathsf{P} - \lambda \mathsf{I})\mathsf{B} = I - \mathsf{R}$ with $\mathsf{R} \in \Gamma(B, \Psi^{-\infty}(\mathcal{E}))$. The $L^2$ operator norm of $\mathsf{R}(z)$ is $O(|\lambda|^{-1})$ uniformly on $B_c$ and so $I - \mathsf{R}$ is invertible in $\Gamma(B_c, \Psi(\mathcal{E}))$ for sufficiently large $|\lambda|$ with

$$(\mathsf{P} - \lambda \mathsf{I})^{-1} = \mathsf{B} + \mathsf{B} \sum_{j \geq 1} \mathsf{R}^j \ . \tag{1.13}$$

The extension to (1.10) is a consequence of the nilpotence of the differential form coefficients of $\mathsf{Q}_{[>0]} = \sum_{k=1}^{\dim B} \mathsf{Q}_{[k]}$. From (1.7) and (1.8), if

$$\mathsf{A} \in \mathcal{A}^{>0}(B, \Psi(\mathcal{E})) := \sum_{i=1}^{\dim B} \mathcal{A}^i(B, \Psi(\mathcal{E})) \ ,$$

so $\mathsf{A}$ (strictly) raises form degree in $\mathcal{A}(B, \pi_*(\mathcal{E}))$, we have $\mathsf{A}^k = 0$ for $k > \dim B$. Hence the Neumann expansion exists, consisting of only finitely many terms giving the identity in $\mathcal{A}(B, \Psi(\mathcal{E}))$

$$(I - \mathsf{A})^{-1} = I + \mathsf{A} + \ldots + \mathsf{A}^{\dim B} \ .$$

Since $(\mathsf{Q}-\lambda\mathsf{I})^{-1} = (\mathsf{P}-\lambda\mathsf{I})^{-1}\left(\mathsf{I}+\mathsf{Q}_{[>0]}(\mathsf{P}-\lambda\mathsf{I})^{-1}\right)$ and $\mathsf{Q}_{[>0]}(\mathsf{P}-\lambda\mathsf{I})^{-1} \in \mathcal{A}^{>0}(B,\Psi(\mathcal{E}))$ we reach the conclusion. □

**Remarks.**

[1] Evidently, the proof can also be carried out directly by constructing a parametrix for the full local symbol $\mathbf{q} \in \Gamma\left((U_M \times_\pi U_M) \times \mathbb{R}^n\backslash\{0\},\ \pi^*(\wedge T^*U_B) \otimes \mathbb{R}^N \times (\mathbb{R}^N)^*\right)$ from the vertical symbol parametrix $\mathbf{b}$ for $\mathbf{p}(z,x,\xi) - \mu^m I$, as above, and then using the differential form nilpotence of $(\mathbf{q}-\lambda I) - (\mathbf{p}-\lambda I)$ and the corresponding Neumann expansion, defined for the symbol product (1.8), to construct a local symbol parametrix for $(\mathbf{q}-\lambda I)$ of the form (1.10).

[2] In the following we shall for brevity not distinguish between $B_c$ and $B$. In particular, this distinction in Theorem 0.1 and Theorem 0.4 is not pertinent.

**Theorem 1.3.** *Let $\mathsf{F} = \sum_{k=0}^{\dim B} \mathsf{F}_{[k]} \in \mathcal{A}(B,\Psi(\mathcal{E}))$ be a smooth simple family of elliptic $\psi$dos of constant order $(v_0, v_2, \ldots, v_{\dim B})$, so that $\mathsf{P} = \mathsf{F}_{[0]}$ is an elliptic family with parameter $\lambda \in \Gamma_\theta$ of constant order $r = v_0 > 0$, and $v_j = \mathrm{ord}(\mathsf{F}_{[j]}) \in \mathbb{R}^1$. Then with $w = \sum_{k=1}^{\dim B+1} v_j$, if $\lambda \in \Gamma_\theta$ and $m > \frac{w+n}{r}$ the resolvent derivative*

$$\partial_\lambda^{m-1}(\mathsf{F}-\lambda\mathsf{I})^{-1} \in \mathcal{A}(B,\Psi^{<-n}(\mathcal{E})) \tag{1.14}$$

*is a smooth family of $\psi$dos with continuous kernel $\mathsf{K}_m(x,y,\lambda)$ with asymptotic expansion on the diagonal $M \subset M \times_\pi M$, summing over differential form degree $p$,*

$$\begin{aligned}\mathsf{K}_m(x,x,\lambda) \;\sim\; \sum_{p=0}^{\dim M}\Bigg(&\sum_{j\geq 0,\,[p,k]} A_{j,[p,k]}(z,x)(-\lambda)^{\frac{w_k+n-j}{r}-(m+k)} \\ &+ \sum_{l\geq 0,\,[q,k]} (A'_{l,[p,q]}(z,x)\log\lambda + A''_{l,[p,q]}(z,x))(-\lambda)^{-l-(m+q)}\Bigg)\end{aligned} \tag{1.15}$$

*where $k, q \in \{0,1,\ldots,\dim M\}$ and $[p,k] = (p_{i_1},\ldots,p_{i_k})$ an ordered multi-index of $k$ non-negative integers $p_{i_1} < \ldots < p_{i_k}$ with*

$$|[p,k]| := p_{i_1} + \ldots + p_{i_k} = p \qquad \text{and} \qquad w_k = v_{i_1} + \ldots v_{i_k}\ ,$$

*and where the coefficients $A_{j,[p,k]}, A'_{l,[p,k]}$ are locally determined and $A''_{l,[p,k]}$ globally determined sections of the bundle*

$$(\pi^*(\wedge T^*B) \otimes |\wedge_\pi|) \otimes \mathrm{End}(\mathcal{E}) \subset \wedge^p T^*M \otimes \mathrm{End}(\mathcal{E})$$

*over $M$, depending smoothly on $z = \pi(x)$.*

*Consequently, taking the supertrace one has an asymptotic expansion as $\lambda \to \infty$ in $\Gamma_\theta$, summing over form degree $d = p - n \geq 0$,*

$$\begin{aligned}\mathrm{Str}(\partial_\lambda^{m-1}(\mathsf{F}-\lambda\mathsf{I})^{-1})(z) \;\sim\; \sum_{d=0}^{\dim B}\Bigg(&\sum_{j\geq 0,\,[d,k]} \alpha_{j,[d,k]}(z)(-\lambda)^{\frac{w_k+n-j}{r}-(m+k)} \\ &+ \sum_{l\geq 0,\,[d,q]} (\alpha'_{l,[d,q]}(z)\log\lambda + \alpha''_{l,[d,q]}(z))(-\lambda)^{-l-(m+q)}\Bigg)\ ,\end{aligned} \tag{1.16}$$

*where the coefficients* $\alpha_{j,[d,k]} = \int_{M/B} \mathrm{Str}(A_{j,[d+n,k]})$*, and similarly* $\alpha^{'}_{l,[d,q]}, \alpha^{''}_{l,[d,q]}$*, are elements of* $\mathcal{A}^d(B)$*.*

*If* $v_i \in \mathbb{N}$*, then* (1.16) *can be written more economically as*

(1.17)

$$\mathrm{Str}(\partial_\lambda^{m-1}(\mathsf{F}-\lambda\mathsf{I})^{-1})(z) \sim \sum_{d=0}^{\dim B}\left(\sum_{j\geq 0}\beta_{j,d}(z)(-\lambda)^{\frac{w+n-j}{r}-m} + \sum_{l\geq 0}(\beta^{'}_{l,d}(z)\log\lambda + \beta^{''}_{l,d}(z))(-\lambda)^{-l-m}\right)$$

$\beta_{j,d}, \beta^{'}_{l,[d,q]}, \beta^{''}_{l,[d,q]} \in \mathcal{A}^d(B)$.

*Proof.* Let $\mathsf{W} = \mathsf{F} - \mathsf{P} \in \mathcal{A}^{>0}(B, \Psi(\mathcal{E}))$. Then from (1.10) we have

$$(\mathsf{F} - \lambda\mathsf{I})^{-1} = \sum_{k=0}^{\dim B}(-1)^k(\mathsf{P} - \lambda\mathsf{I})^{-1}\left(\mathsf{W}(\mathsf{P} - \lambda\mathsf{I})^{-1}\right)^k \tag{1.18}$$

$$= \sum_{k=0}^{\dim B}(-1)^k(\mathsf{P} - \lambda\mathsf{I})^{-1}\left(\sum_{i=0}^{\dim B}\mathsf{W}_{[i]}(\mathsf{P} - \lambda\mathsf{I})^{-1}\right)^k . \tag{1.19}$$

Hence

$$(\mathsf{F} - \lambda\mathsf{I})^{-1}_{[d]} = (-1)^d \sum_{p_1+\ldots+p_k=d}(\mathsf{P} - \lambda\mathsf{I})^{-1}\mathsf{W}_{[p_1]}(\mathsf{P} - \lambda\mathsf{I})^{-1}\ldots\mathsf{W}_{[p_k]}(\mathsf{P} - \lambda\mathsf{I})^{-1} .$$

And consequently,

$$\frac{(-1)^d}{(m_0-1)!\ldots(m_k-1)!}\,\partial_\lambda^{m-1}(\mathsf{F} - \lambda\mathsf{I})^{-1}_{[d]} \tag{1.20}$$

$$= \sum_{p_1+\ldots+p_k=d}\left(\sum_{m_0+\ldots+m_k=m+k}(\mathsf{P} - \lambda\mathsf{I})^{-m_0}\,\mathsf{W}_{[p_1]}(\mathsf{P} - \lambda\mathsf{I})^{-m_1}\ldots\mathsf{W}_{[p_k]}(\mathsf{P} - \lambda\mathsf{I})^{-m_k}\right) .$$

For clarity we have assumed that with respect to any local trivialization

$$\mathsf{W}_{[p_j]} = w_{[p_j]} \otimes Q_j \tag{1.21}$$

with $w_{[p_j]}$ a $p_j$-form; for the general case take finite sums of the following expansions.

The task at hand, then, is to compute the asymptotic supertrace of the operator

$$\mathsf{R}(\lambda, m) = (\mathsf{P} - \lambda\mathsf{I})^{-m_0}\,\mathsf{W}_{[p_1]}(\mathsf{P} - \lambda\mathsf{I})^{-m_1}\ldots\mathsf{W}_{[p_k]}(\mathsf{P} - \lambda\mathsf{I})^{-m_k} .$$

First, $\mathsf{R}(\lambda, m)$ is a smooth family of $\psi$dos of order

$$v_1 + \ldots + v_k - (m_0+1)r - \ldots - (m_k+1)r = w_k - (m+k)r . \tag{1.22}$$

To satisfy (1.14) we hence need

$$w_k - (m+k)r < -n, \qquad k = 0, \ldots, \dim B ,$$

and so taking $m > (w+n)/r$ will do.

For simplicity we will treat only the case where $r$ is a positive integer, the general case is obtained by a generalization explained in [GH]. The proof follows broadly Theorem(2.7) of [GS1], and we use without comment the notation introduced there

$S^{\gamma,l}$ (or $S_z^{\gamma,l} = S^{\gamma,l}(M_z)$) for the symbol spaces (on $M_z$) [1]. Let $\mathsf{B} = \mathrm{OP}(\mathbf{b})$ be the parametrix for $\mathsf{P}-\lambda\mathsf{I}$ constructed from (1.11). Then $(\mathsf{P}-\lambda\mathsf{I})\mathsf{B} = I-\mathsf{R}$ with pointwise $\mathsf{R}(z) \in \mathrm{OP}(S_z^{-\infty,-r})$, while for large $\lambda \in \Gamma_\theta$ (1.13) implies

$$(\mathsf{P}-\lambda\mathsf{I})^{-m_i} = \mathsf{B}^{m_i} + \mathsf{B}\sum_{j\geq 1}\mathsf{R}_{(j)}$$

in $\Gamma(B,\Psi(\mathcal{E}))$ with $\mathsf{R}_{(j)}(z) \in \mathrm{OP}(S_z^{-\infty,-(j+1)r})$. Consequently

$$(\mathsf{P}-\lambda\mathsf{I})^{-m_0}\,\mathsf{W}_{[p_1]}(\mathsf{P}-\lambda\mathsf{I})^{-m_1}\ldots\mathsf{W}_{[p_k]}(\mathsf{P}-\lambda\mathsf{I})^{-m_k} \tag{1.23}$$

$$= \mathsf{B}^{m_0}\mathsf{W}_{[p_1]}\mathsf{B}^{m_1}\mathsf{W}_{[p_2]}\mathsf{B}^{m_2}\ldots\mathsf{W}_{[p_k]}\mathsf{B}^{m_k} + \sum_{j\geq 1} T(\mathsf{B},\mathsf{W}_{[p_1]},\ldots,\mathsf{W}_{[p_k]},\mathsf{R}_{(j)})$$

is a smooth family of weakly polyhomogeneous $\psi$dos with differential form degree $d$. The vertical operator coefficient of $T(\mathsf{B},\mathsf{W}_{[p_1]},\ldots,\mathsf{W}_{[p_k]},\mathsf{R}_{(j)})(z)$ with respect to (??) is in $\mathrm{OP}(S_z^{-\infty,-(j+1)r})$, while by Theorem (1.12) of [GS1] the expansion of the terms of local vertical symbol of $T(\mathsf{B},\mathsf{W}_{[p_1]},\ldots,\mathsf{W}_{[p_k]},\mathsf{R}_{(j)})$ are expansions in integer powers $\lambda = \mu^r$. By Theorem (2.1) of [GS1] the kernel

$$K_{T(\mathsf{B},\mathsf{W}_{[p_1]},\ldots,\mathsf{W}_{[p_k]},\mathsf{R}_{(j)})} \in \Gamma(M\times_\pi M, \pi^*(\wedge T^*B)\otimes(\mathcal{E}\otimes|\Lambda_\pi|^{1/2})\otimes(\mathcal{E}^*\otimes|\Lambda_\pi|^{1/2}))$$

has an expansion as $\lambda\to\infty$ in $\Gamma_\theta$

$$K_{T(\mathsf{B},\mathsf{W}_{[p_1]},\ldots,\mathsf{W}_{[p_k]},\mathsf{R}_{(j)})}(z,x,x,\lambda) \sim \sum_{\sigma\geq 0} C_{j,\sigma}(z,x)(-\lambda)^{-j-(m+k)-\sigma}\ , \tag{1.24}$$

with $C_{j,\sigma} \in \Gamma(M,(\pi^*(\wedge T^*B)\otimes|\wedge_\pi|)\otimes\mathrm{End}(\mathcal{E}))$ and highest power $(-\lambda)^{-1-m-k}$.

On the other hand, with the assumption (1.21), in a local trivialization let $\sigma(\mathsf{A})$ denote the local vertical symbol of the $\psi$do family in $\Gamma(B,\Psi(\mathcal{E}))$ coefficient to $\mathsf{A} \in \mathcal{A}^d(B,\Psi(\mathcal{E}))$, we have

$$\sigma((\mathsf{P}-\mu^r)^{-m_j})_z \in S_z^{-rm_j,0}\cap S_z^{0,-rm_j}\ .$$

Hence with $q = \sigma((\mathsf{P}-\lambda\mathsf{I})^{-m_0}\,\mathsf{W}_{[p_1]}(\mathsf{P}-\lambda\mathsf{I})^{-m_1}\ldots\mathsf{W}_{[p_k]}(\mathsf{P}-\lambda\mathsf{I})^{-m_k})$, (1.22) and the symbol calculus of [GS1] imply that

$$q_z \in S_z^{w_k-r(m+k),0}\cap S_z^{w_k,-r(m+k)}$$

with an expansion

$$q(z,x,\xi,\mu) \sim \sum_{j\geq 0} q_{w_k-r(m+k)-j}(z,x,\xi,\mu)$$

where $q_{w_k-r(m+k)-j} \in S_z^{w_k-j,(m+k)r}$. From [GS1] Thm (1.12), a symbol $p \in S^{k,d}$ has a Taylor expansion as $\mu\to\infty$

$$p(x,\xi,\mu) = \sum_{j=0}^{J} p^{(j)}(x,\xi)\mu^{d-j} + O((1+|\xi|^2)^{(k+J)/2}\,\mu^{d-J})$$

[1] Optimally, the proof here can be given for the vertical analogues of those symbol spaces on a fibration of manifolds $\pi : M\to B$, but since a full presentation of that generalized calculus is quite long we content ourselves here with a pointwise argument.

with $p^{(j)} \in S^{k+j}$. Consequently, locally since the kernel of $\mathsf{B}^{m_0}\mathsf{W}_{[p_1]}\mathsf{B}^{m_1}\mathsf{W}_{[p_2]}\mathsf{B}^{m_2}\ldots\mathsf{W}_{[p_k]}\mathsf{B}^{m_k}$ restricted to the diagonal is

$$K_{\mathrm{OP}(q)}(z,x,x,\mu) = \frac{i}{(2\pi)^n}\int_{\mathbb{R}^n} q(z,x,\xi,\mu)d\xi \otimes \omega_{[d]}(z)\otimes \upsilon_z$$

where $\upsilon_z$ is a local volume form on $U_z \subset M_z$ and $\omega_{[d]}$ the local coefficient $d$-form on the base $B$, and $\mu = \lambda^{1/r}$ relative to $R_\theta$, then by splitting the integral into three summands for $|\xi| \geq |\mu|, |\xi| \leq 1$ and $1 \leq |\xi| \leq |\mu|$ we obtain by the proof of [GS1] Theorem (2.1) a kernel expansion

$$K_{\mathsf{B}^{m_0}\mathsf{W}_{[p_1]}\mathsf{B}^{m_1}\ldots\mathsf{W}_{[p_k]}\mathsf{B}^{m_k}}(z,x,x,\mu) \tag{1.25}$$

$$\sim \sum_{j\geq 0} B_j(z,x)(-\lambda)^{\frac{w_k+n-j}{r}-(m+k)} + \sum_{l\geq 0}(B_l^{'}(z,x)\log\lambda + B_l^{''}(z,x))(-\lambda)^{-l-(m+q)} \ ,$$

where $B_j, B_l^{'}, B_l^{''} \in \Gamma(M, (\pi^*(\wedge T^*B)\otimes |\wedge_\pi|)\otimes \mathrm{End}(\mathcal{E}))$.

From (1.20), (1.23), (1.24) and (1.25) we obtain the expansion (1.15), from which the remaining statements are immediate consequences. □

**Remark.**

[1] With obvious modifications to the powers of $\lambda$, for an auxiliary $\mathsf{A} \in \mathcal{A}(B,\Psi(\mathcal{E}))$ the resolvent supertrace expansion (1.16) extends to

$$\mathrm{Str}(\partial_\lambda^{m-1}(\mathsf{A}(\mathsf{F}-\lambda\mathsf{I})^{-1}) \ . \tag{1.26}$$

Further, generalizations along the lines of [Gr1, L] yield expansions for log-polyhomogeneous operators with higher powers of $\log\lambda$.

It is important to see how the asymptotic expansions transform with respect to the rescaling by $t > 0$ of [B, BGV]

$$\delta_t : \mathcal{A}(B,\pi_*(\mathcal{E})) \to \mathcal{A}(B,\pi_*(\mathcal{E})), \qquad \delta_t\omega_{[i]} = t^{-i/2}\omega_{[i]} \ .$$

$\delta_t$ induces an automorphism of $\mathcal{A}(B,\Psi(\mathcal{E}))$ given by $\delta_t(\mathsf{A}) = \delta_t\cdot\mathsf{A}\cdot\delta_t^{-1}$. Let $\mathsf{F} \in \mathcal{A}(B,\Psi(\mathcal{E}))$ satisfying the assumptions of Theorem 1.3, and define

$$\mathsf{F}_t = t\delta_t(\mathsf{F}) \ . \tag{1.27}$$

Then

$$\mathrm{Str}(\partial_\lambda^m(\mathsf{F}_t-\lambda\mathsf{I})^{-1}) = \delta_t(\mathrm{Str}(\partial_\lambda^m(t\mathsf{F}-\lambda\mathsf{I})^{-1})) = t^{-m-1}\delta_t(\mathrm{Str}(\partial_\lambda^m(\mathsf{F}-\lambda t^{-1}\mathsf{I})^{-1})) \ ,$$

and from (1.16), since the coefficients are in $\mathcal{A}^d(B)$, we obtain

$$\begin{aligned}\mathrm{Str}(\partial_\lambda^{m-1}(\mathsf{F}_t-\lambda\mathsf{I})^{-1}) \ \sim \ & \sum_{d=0}^{\dim B}\left(\sum_{j\geq 0,\,[d,k]}\alpha_{j,[d,k]}(-\lambda)^{\frac{w_k+n-j}{r}-(m+k)}\ t^{\frac{j-w_k-n}{r}+k-1-\frac{d}{2}})\right.\\ & \left. + \sum_{l\geq 0,\,[d,q]}(\alpha_{l,[d,q]}^{'}\log(\lambda t^{-1}) + \alpha_{l,[d,q]}^{''})(-\lambda)^{-l-(m+q)}\ t^{l+q-1-\frac{d}{2}}\right) \ ,\end{aligned} \tag{1.28}$$

while in the case $v_i \in \mathbb{N}$, then (1.17) rescales to

$$\begin{aligned}\mathrm{Str}(\partial_\lambda^{m-1}(\mathsf{F}_t-\lambda\mathsf{I})^{-1}\,) \;\sim\; & \sum_{d=0}^{\dim B}\Bigg(\sum_{j\geq 0}\beta_{j,d}(-\lambda)^{\frac{w+n-j}{r}-m}\; t^{\frac{j-w-n}{r}-1-\frac{d}{2}} \\ & + \sum_{l\geq 0}(\beta^{'}_{l,d}\log\lambda(t^{-1})+\beta^{''}_{l,d})(-\lambda)^{-l-m}\; t^{l-1-\frac{d}{2}}\Bigg)\;. \end{aligned} \tag{1.29}$$

## 2. Zeta Forms and Zeta Determinant Forms

Let $\mathsf{F}\in\mathcal{A}(B,\Psi(\mathcal{E}))$ be a smooth family of elliptic $\psi$dos of constant order $(r>0, v_1, v_2,\ldots,v_{\dim B})$ with parameter $\lambda\in\Gamma_\theta$. Then, with $\mathsf{P}=\mathsf{F}_{[0]}$, for $\lambda\in R_\theta$ sufficiently large $\mathsf{P}-\lambda\mathsf{I}$ is a form degree zero family of invertible $\psi$dos of positive order $r$. If $\mathsf{P}-\lambda\mathsf{I}$, and hence $\mathsf{F}-\lambda\mathsf{I}$, is invertible for all $\lambda$ in the spectral cut $R_\theta=R_\theta\backslash\{0\}$ then the angle $\theta$ is called an *Agmon angle* for $\mathsf{F}$

We assume for the moment that

$$v_k=\mathrm{ord}(\mathsf{F}_{[k]})\leq r,\qquad k=1,\ldots,\dim B\;. \tag{2.1}$$

With (2.1) and using the expansion (1.19) we obtain an operator norm estimate in $\mathcal{A}(B)$ as $\lambda\to\infty$ in $\Gamma_\theta$

$$\|(\mathsf{F}-\lambda\mathsf{I})^{-1}\|^{(l)}_{M/Z}=O(|\lambda|^{-1})\;, \tag{2.2}$$

where for $l\in\mathbb{R}$

$$\|\;.\;\|^{(l)}_{M/B}:\mathcal{A}(M,\Psi^0(\mathcal{E}))\longrightarrow\mathcal{A}(B) \tag{2.3}$$

is the vertical operator Sobolev norm associated to the vertical metric

$$|\;.\;|_{M/B}:\mathcal{A}(B,\pi_*(\mathcal{E}))\longrightarrow\mathcal{A}(B)\;,\qquad |\pi^*(\alpha)\otimes\Psi\otimes\upsilon|_{M/B}=\alpha\int_{M/B}|\psi|^2\,\upsilon^2\;, \tag{2.4}$$

defined independently of the representation of a section as tensor product

$$\psi\otimes\pi^*(\alpha)\otimes\upsilon\in\Gamma(M,\pi^*(\wedge T^*B)\otimes\mathcal{E}\otimes|\wedge|_\pi)\;.$$

Pointwise for $z\in B$ the metric (2.4) reduces on the fibre $\Gamma(M_z,\mathcal{E}\otimes|\wedge_{M_z}|)$ to the canonical metric $|\psi_z|^2=\int_{M_z}|\psi_z(x)|^2$.

On the right side of (2.2), for $a:\mathbb{C}\to\mathcal{A}(B)$ we write $a(\lambda)=O(f(\lambda))$ if for each $l\in\mathbb{N}$ and relatively compact subset $U$ of $B$, there is a constant $C(l,U)$ such that $\|a(\lambda)\|_l\leq C(l,U)f(\lambda)$, where $\|\;.\;\|_l$ is the $C^l$ norm. The proof of (2.2) follows that of the classical result for a single operator [S, Sh] using the vertical parametrix $\mathsf{B}$ in (1.13). As for Seeley's analysis of the single operator case [S], it follows if $\mathsf{F}$ has Agmon angle $\theta$, that for $\mathrm{Re}(s)>0$ the complex powers can be defined by

$$\mathsf{F}_\theta^{-s}=\frac{i}{2\pi}\int_C\lambda_\theta^{-s}\,(\mathsf{F}-\lambda\mathsf{I})^{-1}\,d\lambda\;, \tag{2.5}$$

where $\lambda_\theta$ is the branch of $\lambda^{-s}$ defined by $\lambda_\theta^{-s}=|\lambda|^{-s}e^{-is\arg(\lambda)}$ with $\theta-2\pi\leq\arg(\lambda)<\theta$, and where $C$ is the negatively oriented contour which is the boundary of a sector

$$\Lambda_{\theta,\delta}=\{z\in\mathbb{C}\;|\;|\arg(z)-\theta|\leq\delta\;\text{ or }\;|z|\leq\rho\}$$

with $\delta$ so chosen that $\Lambda_{\theta,\delta}$ contains no eigenvalues of the operators $\mathsf{P}_z$ and $\rho$ such that $(\mathsf{F}-\lambda\mathsf{I})^{-1}$ is defined and holomorphic for $0<|\lambda|<\rho+\varepsilon$ for some $\varepsilon>0$.

The estimate (2.2) shows that $\mathsf{F}^{-s}$ converges in each vertical Sobolev norm and hence defines an operator from $\mathcal{A}(B,\pi_*(\mathcal{E}))$ into $\mathcal{A}(B,\pi_*(\mathcal{E}))$. From (1.19) we have for $\mathrm{Re}(s)>0$

(2.6)

$$\mathsf{F}_\theta^{-s}=\sum_{\substack{k\\ p_1+\ldots+p_m=k}}\frac{i}{2\pi}\int_C\lambda_\theta^{-s}\,(-1)^k(\mathsf{P}-\lambda\mathsf{I})^{-1}\mathsf{W}_{[p_1]}(\mathsf{P}-\lambda\mathsf{I})^{-1}\ldots\mathsf{W}_{[p_m]}(\mathsf{P}-\lambda\mathsf{I})^{-1}\,d\lambda.$$

Each of the summands in (2.6) is smooth family of $\psi$dos, of differential form degree $k$, represented locally by a sum of vertical polyhomogeneous symbols

$$(2.7)\quad \frac{i}{2\pi}\int_C\lambda_\theta^{-s}\,\mathbf{b}_{m_1}[\lambda](z,x,\xi)\circ\mathbf{w}_{p_l,i_1}(z,x,\xi)\ldots\mathbf{w}_{p_m,i_m}(z,x,\xi)\circ\mathbf{b}_{m_j}[\lambda](z,x,\xi)\,d\lambda$$

with $\mathsf{B}=\mathrm{OP}(\sum\mathbf{b}_i[\lambda])$ the parametrix for $\mathsf{P}-\lambda\mathsf{I}$ and $\mathsf{W}_{p_i}=\mathrm{OP}(\sum_\sigma\mathbf{w}_{p_i,\sigma})$ where $\mathbf{w}_{p_i,\sigma}(z,x,\xi)$ is homogeneous in $\xi$ of order $\nu_i-\sigma$ and $\mathbf{b}_i[\lambda])(z,x,\xi)$ homogeneous in $(\xi,\lambda^{1/r})$ of degree $r-i$, i.e. $\mathbf{b}_i[t^r\lambda])(z,x,t\xi)=t^{r-i}\mathbf{b}_i[\lambda])(z,x,\xi)$ for $t>0$, $\lambda,t^r\lambda\in\Lambda_\theta$. The degree of homogeneity of (2.7) is computed by replacing $\xi$ by $t\xi$ and $\lambda$ by $t^r\mu$ in the integrand of symbol products. In particular, setting $m_1=\ldots=m_j=0$, $\sigma=0$, we have that the principal symbol has degree $\nu_{i_1}+\ldots+\nu_{i_m}-(s+k)r$.

Proceeding in this way, applying the standard methods of [S, Sh] and the remark following Proposition 1.2, we obtain the following fact.

**Lemma 2.1.** *The vertical complex power defined for* $\mathrm{Re}(s)>0$ *by* (2.5) *(and generally without assumption* (2.1) *for* $\mathrm{Re}(s)>>0$*, see below,) is a smooth family of* $\psi$dos *of mixed differential form degree* $\mathsf{F}^{-s}\in\mathcal{A}(B,\Psi(\mathcal{E}))$ *such that if* $\mathbf{f}[\lambda]\sim\sum_j\mathbf{f}[\lambda]_j$ *is a local vertical polyhomogeneous symbol representing* $(\mathsf{F}-\lambda\mathsf{I})^{-1}$*, then* $\mathbf{f}_\theta^{-s}\sim\sum_j\mathbf{f}_{\theta,j}^{-s}$ *represents* $\mathsf{F}^{-s}$*, where*

$$\mathbf{f}_{\theta,j}^{-s}(z,x,\xi)=\frac{i}{2\pi}\int_C\lambda_\theta^{-s}\,\mathbf{f}[\lambda]_j(z,x,\xi)\,d\lambda\ .$$

The zeta form for $\mathsf{F}$ can now be constructed as follows. Since (2.2) implies for $\mathrm{Re}(s)>0$ the operator norm estimate in $\mathcal{A}(B)$

$$\|\lambda^{m-s}\,\partial_\lambda^{m-1}(\mathsf{F}-\lambda\mathsf{I})^{-1}\|_{M/Z}^{(l)}=O(|\lambda|^{-1})\ ,$$

as $\lambda\to\infty$ along $C$, we can integrate by parts in (2.5) to obtain

$$(2.8)\qquad \mathsf{F}^{-s}=\frac{1}{(s-1)\ldots(s-m)}\cdot\frac{i}{2\pi}\int_C\lambda^{m-s}\,\partial_\lambda^m(\mathsf{F}-\lambda\mathsf{I})^{-1}\,d\lambda\ .$$

Since

$$\partial_\lambda^m(\mathsf{F}-\lambda\mathsf{I})^{-1}=\sum_{\substack{k=0\\ m_0+\ldots+m_k=m}}^{\dim B}(-1)^k\partial_\lambda^{m_0}((\mathsf{P}-\lambda\mathsf{I})^{-1})\mathsf{W}\,\partial_\lambda^{m_2}((\mathsf{P}-\lambda\mathsf{I})^{-1})\ldots\mathsf{W}\,\partial_\lambda^{m_k}((\mathsf{P}-\lambda\mathsf{I})^{-1})\,d\lambda$$

and $\partial_\lambda^{m_i}(\mathsf{P}-\lambda\mathsf{I})^{-1} \in \mathcal{A}(B, \Psi^{-rm_i-r}(\mathcal{E}))$, then by taking $m \geq N$ for sufficiently large $N$ we may ensure an estimate $\|\partial_\lambda^m(\mathsf{F}-\lambda\mathsf{I})^{-1}\|^{(l)}_{M/Z} = O(|\lambda|^{-1})$ without assuming (2.1). For the general case, we hence define $\mathsf{F}^{-s}$ for $\mathrm{Re}(s) > m \geq N$ by (2.8).

Moreover, (2.8) and Theorem 1.3, equation (1.14), show for $\mathrm{Re}(s) > \frac{w+n}{r}$ that $\mathsf{F}^{-s} \in \mathcal{A}(B, \Psi^{<-n}(\mathcal{E}))$ is a smooth family of trace class $\psi$dos with kernel $K(\mathsf{F}^{-s})$ continuous over the diagonal $M \subset M \times_\pi M$. In that half-plane $\mathsf{F}$ therefore has a super-zeta form

$$\zeta_\theta(\mathsf{F}, s) := \mathrm{Str}(\mathsf{F}_\theta^{-s}) = \int_{M/B} \mathrm{Str}(K(\mathsf{F}_\theta^{-s})(x,x)) \in \mathcal{A}(B) \ . \tag{2.9}$$

From (2.8) for $\mathrm{Re}(s) > m > (w+n)/r$ we have

$$\zeta_\theta(\mathsf{F}, s) = \frac{1}{(s-1)\ldots(s-m)} . \frac{i}{2\pi} \int_C \lambda^{m-s}\, \mathrm{Str}\left(\partial_\lambda^m (\mathsf{F}-\lambda\mathsf{I})^{-1}\right) \, d\lambda \ . \tag{2.10}$$

We can use (2.10) to write down the singularity structure of the meromorphic continuation of the zeta form to all $s \in \mathbb{C}$. To do so requires the assumption that $\mathsf{P} = \mathsf{F}_{[0]}$ is smooth family of $\psi$dos such that $\mathrm{Ker}(P_z)$ has constant kernel dimension. Consequently the meromorphically continued zeta form $\zeta_\theta(\mathsf{F}, s)|^{\mathrm{mer}}$ is only defined for families of $\psi$dos with Agmon angle in $\mathcal{A}(B, \Psi(\mathcal{E}))$ modulo the regularizing subalgebra $\mathcal{A}(B, \Psi^{-\infty}(\mathcal{E}))$. This is essentially equivalent to $\zeta_\theta(\mathsf{F}, s)|^{\mathrm{mer}}$ being a characteristic class map on K-theory, taking values in $H^*(B)$.

Thus we assume that the family of $\psi$do projections onto the kernels define a smooth family of smoothing operators $\Pi \in \mathcal{A}(B, \Psi^{-\infty}(\mathcal{E}))$, defining a smooth finite-rank superbundle $\mathrm{Ker}(\mathsf{P})$ over $B$. With this assumption, (1.18) implies that at $\lambda = 0$ the resolvent trace form is meromorphic with Laurent expansion

$$\mathrm{Str}\left(\partial_\lambda^m(\mathsf{F}-\lambda\mathsf{I})^{-1}\right) = \sum_{k=0}^{\dim B} (-1)^k \frac{(m+k)!}{k!} (-\lambda)^{-k-1-m} \mathrm{Str}\left((\Pi\cdot\mathsf{W}\cdot\Pi)^k\right) + O(|\lambda|^m) \ . \tag{2.11}$$

The asymptotic expansion (1.16) as $\lambda \to \infty$ in $\Gamma_\theta$ along with the expansion (2.11) at $\lambda = 0$ now imply by a standard transition argument, for example [GS2] Proposition (2.9), that $\zeta_\theta(\mathsf{F}, s)$ extends meromorphically to $\mathbb{C}$ with the singularity structure

$$\frac{\pi}{\sin(\pi s)} \zeta_\theta(\mathsf{F}, s)|^{\mathrm{mer}} \sim \tag{2.12}$$

$$- \sum_{j=-\dim B-1}^{-1} \frac{\mathrm{Str}\left((\Pi\cdot\mathsf{W}\cdot\Pi)^{-j-1}\right)}{(s-j-1)} \ +$$

$$\sum_{d=0}^{\dim B} \left( \sum_{j\geq 0,\, [d,k]} \frac{a_{j,[d,k]}}{\left(s+k+\frac{j-n-w_k}{r}-1\right)} + \sum_{l\geq 0,\, [d,q]} \frac{a'_{l,[d,q]}}{(s+l+q-1)^2} + \frac{a''_{l,[d,q]}}{(s+l+q-1)} \right) \ ,$$

with coefficients $a_{j,[d,k]}, a^{'}_{l,[d,q]}, a^{''}_{l,[d,q]} \in \mathcal{A}^d(B)$ related to the coefficients of (1.16) by universal multiplicative constants. Specifically,

$$a_{j,[d,k]} = \Gamma(\frac{j-n-w_k}{r} + k)\Gamma(\frac{j-n-w_k}{r} + k + m)^{-1}\, \alpha_{j,[d,k]}(m) \tag{2.13}$$

independently of m, with $\Gamma(s)$ the Gamma function. In the more general case where we allow non-constant order $\nu_k \in \Gamma(B, \mathbb{R})$ then the factors are replaced by the corresponding universal functions.

If $v_i \in \mathbb{N}$, then (2.14) takes the simpler form

$$\begin{aligned} \frac{\pi}{\sin(\pi s)}\zeta_\theta(\mathsf{F}, s)|^{\mathrm{mer}} \sim \\ -\sum_{j=-\dim B-1}^{-1} \frac{\mathrm{Str}\,((\Pi \cdot \mathsf{W} \cdot \Pi)^{-j-1})}{(s-j-1)^{k+1}} \; + \\ \sum_{d=0}^{\dim B} \left( \sum_{j\geq 0} \frac{b_{j,d}}{(s + \frac{j-n-w}{r} - 1)} + \sum_{l \geq 0} \frac{b^{'}_{l,d}}{(s+l-1)^2} + \frac{b^{''}_{l,d}}{(s+l-1)} \right) \;, \end{aligned} \tag{2.14}$$

with coefficients $b_{j,d}, b^{'}_{l,d}, b^{''}_{l,d} \in \mathcal{A}^d(B)$ related to the $\beta_{j,d}, \beta^{'}_{l,d}, \beta^{''}_{l,d}$ by constants, and

$$b_{j,d} = \Gamma(\frac{j-n-w}{r})\Gamma(\frac{j-n-w}{r} + m)^{-1}\, \beta_{j,d}(m) \tag{2.15}$$

The pole structure (2.12) can also be computed directly from the meromorphically continued symbol representation of $\mathsf{F}^{-s}$ in Lemma 2.1.

**Definition 2.2.** *A family of $\psi$dos $\mathsf{F} \in \mathcal{A}(B, \Psi(\mathcal{E}))$ admitting an Agmon angle $\theta$ is said to be $\zeta$-admissible if when $l+q-1 \in \{0, 1, \ldots, \dim B\}$ then $a^{'}_{l,[d,q]} = 0$ for $d \geq 1$ in* (2.12) *(for $d = 0$ this is guaranteed by the ellipticity of $P = \mathsf{F}_{[0]}$). Similarly, for* (2.14) *this requires $b^{'}_{l,d} = 0$ for $l - 1 \in \{0, 1, \ldots, \dim B\}$.*

This ensures that $\zeta_\theta(\mathsf{F}, s)|^{\mathrm{mer}}$ is holomorphic for $s$ around $0, 1, \ldots, \dim B$. This property is needed for the differential $\zeta$ form, but when $\mathsf{F}$ is the curvature of a superconnection is irrelevant at the cohomological level, since in that case the forms are all exact.

The complex powers $\mathsf{F}^{-s}$ defined by (2.5) for $\mathrm{Re}(s) > 0$ if (2.1) holds, and in general by (2.8) for $\mathrm{Re}(s) > m$ if it does not, are extended by Seeley's method [S] to all $s \in \mathbb{C}$ by choosing any positive integer $N$ with $\mathrm{Re}(s) + N > m$ and defining

$$\mathsf{F}^{-s} = \mathsf{F}^{-s-N}\mathsf{F}^N \in \mathcal{A}(B, \Psi(\mathcal{E})) \;. \tag{2.16}$$

More precisely, the map $s \longmapsto K(\mathsf{F}^{-s})$ assigning to $\mathsf{F}^{-s}$ its (distributional) kernel is a holomorphic map of $\{s \mid \mathrm{Re}(s) > (w+n)/r\}$ into (in a local trivialization) matrices of continuous functions. Restricted to any compact subset $V$ of $M \times M \backslash \Delta(M)$ the map $s \in \mathbb{C} \longmapsto K(\mathsf{F}^{-s})_{|V}$ is holomorphic map from $\mathbb{C}$ to smooth matrices, while along the diagonal $s \longmapsto K(\mathsf{F}^{-s})(x, x)$ is a meromorphic function on all of $\mathbb{C}$ with discrete poles at the points indicated in (2.12).

We then define the logarithm of $\mathsf{F}$ to be the smooth vertical family of *log-polyhomogeneous* $\psi$dos

$$\log_\theta \mathsf{F} := -\partial_s|_{s=0}\mathsf{F}_\theta^{-s} \in \mathcal{A}(B, \Psi_{\log}(\mathcal{E}))\ .$$

Thus, omitting the $\theta$ subscript, $\partial_s \mathsf{F}^{-s} = -\log \mathsf{F}\,\mathsf{F}^{-s}$, where for $\mathrm{Re}(s) > 0$ if (2.1) holds

$$\log \mathsf{F}\,\mathsf{F}^{-s} = \frac{i}{2\pi}\int_C \log\lambda\,\lambda^{-s}\,(\mathsf{F}-\lambda\mathsf{I})^{-1}\,d\lambda\ ,$$

and similarly using (2.8) for the general case.

Here $\mathcal{A}(B, \Psi_{\log}(\mathcal{E}))$ is the extension of $\mathcal{A}(B, \Psi(\mathcal{E}))$ to operators represented by vertical log-polyhomogeneous symbols. This means that with respect to local coordinates on $\mathcal{E}$, an operator $\mathsf{T} \in \mathcal{A}(B, \Psi_{\log}(\mathcal{E}))$ is represented by a vertical symbol $\mathbf{t} \in \Gamma\left( (U_M \times_\pi U_M) \times \mathbb{R}^n\backslash\{0\},\ \pi^*(\wedge T^*U_B) \otimes \mathbb{R}^N \times (\mathbb{R}^N)^*\right)$ of the form

$$\mathbf{t}(z,x,\xi) \sim \sum_{j\geq 0}\sum_{p=0}^{1} \mathbf{t}_{-j,p}(z,x,\xi)\log^p|\xi|\ , \tag{2.17}$$

with $\mathbf{t}_{-j,p}(z,x,\xi)$ a vertical homogeneous symbol in the sense of (**??**).

It is readily verified that $\log_\theta \mathsf{F}$ is log-polyhomogeneous locally represented by the vertical log-polyhomogeneous symbol $\log\mathbf{f} \sim \sum_{j\geq 0}\log_j \mathbf{f}$ with

$$\log_j \mathbf{f}(x,\xi) = \frac{i}{2\pi}\int_C \log\lambda\,\mathbf{b}[\lambda]_j(z,x,\xi)\,d\lambda$$

and $\partial_s \mathbf{f}^{-s} = (\log\mathbf{f})\circ\mathbf{f}^{-s}$ and furthermore $(\log\mathsf{F})_{[0]} = \log\mathsf{P}$.

**Definition 2.3.** *For $\zeta$-admissible $\mathsf{F} \in \mathcal{A}(B, \Psi(\mathcal{E}))$ with Agmon angle $\theta$, the zeta-determinant form* $\det_{\zeta,\theta}\mathsf{F} \in \mathcal{A}(B)$ *is defined by*

$$\log\det{}_{\zeta,\theta}\mathsf{F} \in \mathcal{A}(B) = -\partial_s|_{s=0}^{\mathrm{mer}}\ \mathrm{Str}(\mathsf{F}^{-s}) = \mathrm{Str}(\log \mathsf{F}\,.\,\mathsf{F}^{-s})|_{s=0}^{\mathrm{mer}}\ . \tag{2.18}$$

Lemma 0.3 is an immediate corollary of the following.

**Lemma 2.4.** *One has $\zeta(\mathsf{F},s)_{[0]} = \zeta(\mathsf{P},s)$.*

To see this, notice from (2.8) and (1.18) that for $\mathrm{Re}(s) >> 0$

$$\mathsf{F}^{-s} = \mathsf{P}^{-s} + \sum_{k=0}^{\dim B}\frac{1}{(s-1)\ldots(s-m)}.\frac{i}{2\pi}\int_C \lambda^{m-s}\,\partial_\lambda^m\left((\mathsf{P}-\lambda\mathsf{I})^{-1}(\mathsf{W}(\mathsf{P}-\lambda\mathsf{I})^{-1})^k\right)d\lambda\,,$$

and hence that for $m > (w+n)/r$

$$\zeta(\mathsf{F},s)|^{\mathrm{mer}} = \zeta(\mathsf{P},s)|^{\mathrm{mer}} +$$

$$\sum_{k=0}^{\dim B}\frac{1}{(s-1)\ldots(s-m)}.\frac{i}{2\pi}\int_C \lambda^{m-s}\,\mathrm{Str}\left(\partial_\lambda^m((\mathsf{P}-\lambda\mathsf{I})^{-1}(\mathsf{W}(\mathsf{P}-\lambda\mathsf{I})^{-1})^k)\right)d\lambda\,|^{mer}\ .$$

The second term on the right is meromorphic on $\mathbb{C}$, and holomorphic at zero, and of non-zero form degree, since $\mathsf{W} \in \mathcal{A}^{>0}(B, \Psi(\mathcal{E}))$, with the pole structure of (2.12) but with $d \geq 1$.

Replacing $\mathsf{F}$ by $\mathsf{F}_t = t\delta_t(\mathsf{F})$ in (2.12), we can use (1.28) to write down the $t$-rescaled singularity structure. This means that the rescaled left side of (2.12) minus the rescaled sums on the right-side for $j \leq N$ terms is $O(t^{\frac{N-w_k-n}{r}+k-\frac{d}{2}})$. Taking the $s$ derivative and evaluating at $s=0$, this implies the following.

**Proposition 2.5.** *There is an asymptotic expansion as $t \longrightarrow 0+$ in $\mathcal{A}(B)$*
(2.19)

$$\log\det_\zeta \mathsf{F}_t \sim \sum_{d=0}^{\dim B} \sum_{j\geq 0,\,[d,k]} c_{j,[d,k]}\, t^{\frac{j-w_k-n}{r}+k-1-\frac{d}{2}} + \sum_{l\geq 0,\,[d,q]} (c'_{l,[d,q]} \log t + c''_{l,[d,q]})\, t^{l-1-\frac{d}{2}} \ ,$$

*where the degree $d$ differential forms $c_{j,[d,k]}, c'_{l,[d,q]}$ are determined locally, while $c''_{l,[d,q]}$ are globally determined.*

In the case where eigenvalues of the principal symbol $\mathbf{p}_0 \in \Gamma(T(M/B)\backslash\{0\}, p^*(\mathrm{End}(\mathcal{E})))$ of $\mathsf{P}$ lie pointwise for $(x,\xi) \in T(M/B)\backslash\{0\}$ in a subsector of the right-half plane, then, the zeta form and zeta determinant form can be equivalently formulated by a vertical heat trace form

$$\mathrm{Str}(e^{-\mathsf{F}}) = \frac{i}{2\pi}\int_{\mathcal{C}} e^{-\lambda}\mathrm{Str}(\partial_\lambda^m(\mathsf{F}-\lambda\mathsf{I})^{-1})\, d\lambda \ . \tag{2.20}$$

where $m > (w+n)/r$ and $\mathcal{C}$ is a contour coming in on a ray with argument in $(0,\pi/2)$, encircling the origin, and leaving on a ray with argument in $(-\pi/2,0)$. We hence obtain an asymptotic expansion, which for brevity we state only for the case $v_i \in \mathbb{N}$, that as $t \longrightarrow 0+$

$$\mathrm{Str}(e^{-\mathsf{F}_t}) \sim \sum_{d=0}^{\dim B}\left(\sum_{j\geq 0} \tilde{b}_{j,d}\, t^{\frac{j-w-n}{r}-1-\frac{d}{2}} + \sum_{l\geq 0}(\tilde{b}'_{l,d}\log t + \tilde{b}''_{l,d})\, t^{l-1-\frac{d}{2}}\right) \tag{2.21}$$

with $\tilde{b}_{j,d}, \tilde{b}'_{l,q}, \tilde{b}''_{l,d} \in \mathcal{A}^d(B)$ related to those in the expansion (2.14) by

$$\tilde{b}_{j,d} = \Gamma(\frac{j-n-w_k}{r})^{-1} b_{j,d} \ , \quad \tilde{b}'_{l,d} = \Gamma(l)^{-1} b'_{l,d} \ , \quad \tilde{b}''_{l,d} = \Gamma(l)^{-1} b''_{l,d} \ . \tag{2.22}$$

## 3. Homotopy Properties

A superconnection [Q, B, BGV] on $\pi_*(\mathcal{E})$ adapted to a smooth family of formally self-adjoint elliptic $\psi$dos $\mathsf{P} = \begin{bmatrix} 0 & \mathsf{P}^- \\ \mathsf{P}^+ & 0 \end{bmatrix} \in \mathcal{A}^0(B, \Psi^r(\mathcal{E}))$ of order $r>0$ is a classical $\psi$do $\mathbb{A}$ on $\mathcal{A}(B,\pi_*(\mathcal{E})) = \Gamma(M, \pi^*(\wedge T^*B)\otimes\mathcal{E}\otimes|\wedge_\pi|^{1/2})$ of odd-parity with respect to the $\mathbb{Z}_2$-grading, such that

$$\mathbb{A}(\omega\,\psi) = d\omega\,\psi + (-1)^{|\omega|}\omega\,\mathbb{A}(\psi) \ , \tag{3.1}$$

for $\omega \in \mathcal{A}(B)$ and $\psi \in \mathcal{A}(B,\pi_*(\mathcal{E}))$, and such that $\mathbb{A}_{[0]} = \mathsf{P}$ , where $\mathbb{A} = \sum_{i=0}^{\dim B}\mathbb{A}_{[i]}$ and $\mathbb{A}_{[i]} : \mathcal{A}^d(B,\pi_*(\mathcal{E})) \to \mathcal{A}^{d+i}(B,\pi_*(\mathcal{E}))$ is the component which raises form degree by $i$. It follows from (3.1) that $\mathbb{A}_{[1]}$ is a connection in the classical (ungraded) sense, while each remaining term is a smooth family of $\psi$dos $\mathbb{A}_{[i]} \in \mathcal{A}^i(B,\Psi(\mathcal{E}))$ if $i \neq 1$. In a local weak trivialization $\mathcal{A}(U,\pi_*(\mathcal{E})_{|U}) \cong \mathcal{A}(U)\otimes\Gamma(M_{z_0},\mathcal{E}^{z_0})$ for $z_0 \in U$, $\mathbb{A}$ takes the local coordinate form $\mathbb{A}_{|U} = d_U + \sum_I P_I dz_I$, where $dz_I = dz_{i_1}\ldots dz_{i_m}$ and $P_I^z$ a classical $\psi$do on $\Gamma(M_{z_0},\mathcal{E}^{z_0})$.

The curvature of $\mathbb{A}$ is the smooth family of $\psi$dos with differential form coefficients $\mathbb{A}^2 \in \mathcal{A}(B, \pi_*(\mathcal{E}))$. Since $\mathbb{A}^2_{[0]} = \mathsf{P}^2 = \mathsf{P}^-\mathsf{P}^+ \oplus \mathsf{P}^+\mathsf{P}^- \in \mathcal{A}^0(B, \pi_*(\mathcal{E}))$ we have that $\mathbb{A}^2$ is elliptic with Agmon angle $\pi$.

**Proposition 3.1.** *Let $m \in \mathbb{N}$ with $m > \frac{w+n}{r}$ with $w = \sum_i \mathrm{ord}((\mathbb{A}^2)_{[i]})$. Then for $\lambda \in \Gamma_\pi$ the resolvent trace form $\mathrm{Str}(\partial_\lambda^{m-1}(\mathbb{A}^2 - \lambda I)^{-1}) \in \mathcal{A}(B)$ is a closed differential form and a homotopy invariant of the superconnection $\mathbb{A}$.*

*Proof.* Let $\mathbb{A}_\sigma$ be a 1-parameter family of superconnections on $\pi_*(\mathcal{E})$ adapted to $\mathsf{P}$. Then using the identity

$$(\mathbb{A}_\sigma^2 - \lambda \mathsf{I})^{-1}\mathbb{A}_\sigma = \mathbb{A}_\sigma(\mathbb{A}_\sigma^2 - \lambda \mathsf{I})^{-1} \tag{3.2}$$

we have formally

$$\begin{aligned}
\partial_\sigma \mathrm{Str}(\partial_\lambda^{m-1}(\mathbb{A}_\sigma^2 - \lambda \mathsf{I})^{-1}) &= \mathrm{Str}(\partial_\lambda^{m-1}\partial_\sigma(\mathbb{A}_\sigma^2 - \lambda \mathsf{I})^{-1}) \\
&= -\mathrm{Str}\left(\partial_\lambda^{m-1}\left((\mathbb{A}_\sigma^2 - \lambda \mathsf{I})^{-1}(\dot{\mathbb{A}}_\sigma\mathbb{A}_\sigma + \mathbb{A}_\sigma\dot{\mathbb{A}}_\sigma)^{-1}(\mathbb{A}_\sigma^2 - \lambda \mathsf{I})^{-1}\right)\right) \\
&= -\mathrm{Str}\left(\left[\mathbb{A}_\sigma, \partial_\lambda^{m-1}((\mathbb{A}_\sigma^2 - \lambda \mathsf{I})^{-1}\dot{\mathbb{A}}_\sigma(\mathbb{A}_\sigma^2 - \lambda \mathsf{I})^{-1}\right]\right) \\
&= -d\ \mathrm{Str}\left(\partial_\lambda^{m-1}\left((\mathbb{A}_\sigma^2 - \lambda \mathsf{I})^{-1}\dot{\mathbb{A}}_\sigma(\mathbb{A}_\sigma^2 - \lambda \mathsf{I})^{-1}\right)\right) \qquad (3.3) \\
&= -d\ \mathrm{Str}\left(\dot{\mathbb{A}}_\sigma\, \partial_\lambda^{m}\left((\mathbb{A}_\sigma^2 - \lambda \mathsf{I})^{-1}\right)\right)\ , \qquad (3.4)
\end{aligned}$$

where each step is easily justified rigourously using the kernel $K(\partial_\lambda^{m-1}(\mathbb{A}_\sigma^2 - \lambda \mathsf{I})^{-1})$, which depends smoothly on $\sigma$. The equality (3.3) follows by essentially the same argument as that in [BGV] Lemma (9.15), using a parametrix for $\mathsf{P} = \mathbb{A}_{[0]}$ to see the supertrace vanishes on the supercommutators of vertical $\psi$dos that arise in (3.3). (Generally, if $\mathsf{R} \in \mathcal{A}(B, \Psi(\mathcal{E}))$ has sufficiently negative $\psi$do order, then one has $d\,\mathrm{Str}(\mathsf{R}) = \mathrm{Str}([\mathbb{A}, \mathsf{R}])$, but in (3.3) no such subtleties enter.) Since the variation of the resolvent trace form is exact, this proves the homotopy invariance.

Similarly, we obtain

$$d\ \mathrm{Str}(\partial_\lambda^{m-1}(\mathbb{A}^2 - \lambda \mathsf{I})^{-1}) = \mathrm{Str}([\mathbb{A}, \partial_\lambda^{m-1}(\mathbb{A}^2 - \lambda \mathsf{I})^{-1})] \tag{3.5}$$

which, since the resolvent form has even parity, vanishes by (3.2), proving closure. □

From here on we restrict our attention to the scaled superconnection $\mathbb{A}_t := t^{1/2}\delta_t(\mathbb{A})$ with curvature $\mathsf{F}_t := \mathbb{A}_t^2 = t\delta_t(\mathbb{A}^2) \in \mathcal{A}(B, \Psi(\mathcal{E}))$.

The small time asymptotics (1.28) of the resolvent trace form for the superconnection curvature and equation (3.5) now yield the next result, given in terms of the corresponding coefficient forms in the zeta-form singularity structure (2.12).

**Proposition 3.2.** *For*

$$j \neq r + \frac{rd}{2} + w_k - k + n$$

*the $C^\infty$ differential forms $a_{j,[d,k]}$ are exact, for*

$$l \neq 1 + \frac{d}{2} - q$$

*the forms* $a'_{l,[d,q]}, a''_{l,[d,q]}$ *are exact. The forms*

$$a_{r+\frac{rd}{2}+w_k-k+n,[d,k]}\ , \quad a'_{1+\frac{d}{2}-q,[d,q]}\ , \quad a''_{1+\frac{d}{2}-q,[d,q]}\ , \tag{3.6}$$

*are closed in* $\mathcal{A}(B)$.

*Similarly, if* $\nu_k \in \mathbb{N}$, *if*

$$j \neq r + \frac{rd}{2} + w + n$$

*the* $C^\infty$ *differential forms* $b_{j,d}$ *are exact, for*

$$l \neq 1 + \frac{d}{2}$$

*the forms* $b'_{l,d}, b''_{l,d}$ *are exact. The forms*

$$b_{r+\frac{rd}{2}+w+n,d}\ , \quad b'_{1+\frac{d}{2},d}\ , \quad b''_{1+\frac{d}{2},d}\ , \tag{3.7}$$

*are closed in* $\mathcal{A}(B)$.

For the large time asymptotics we assume that $\mathsf{P} = \mathsf{F}_{[0]}$ satisfies the constant kernel dimension condition, so that we have the finite-rank superbundle $\mathrm{Ker}(\mathsf{P})$ over $B$ with the induced connection $\nabla_0 = \Pi_0 \cdot \mathbb{A}_{[1]} \cdot \Pi_0$ (in the usual sense), as in Theorem 0.1, and so we have the corresponding classical resolvent trace form

$$\mathrm{Str}\left((\nabla_0^2 - \lambda I)^{-1}\right) \in \mathcal{A}(B)\ .$$

**Proposition 3.3.** *The following limit holds*

$$\lim_{t\to\infty} \mathrm{Str}\left(\partial_\lambda^m (\mathsf{F}_t - \lambda \mathsf{I})^{-1}\right) = \partial_\lambda^m \mathrm{Str}\left((\nabla_0^2 - \lambda I)^{-1}\right) \tag{3.8}$$

*in each* $C^l$ *norm on compact subsets of* $B$. *For* $t > 0$ *one has in* $\mathcal{A}(B)$

$$\partial_\lambda^m \mathrm{Str}\left((\nabla_0^2 - \lambda I)^{-1}\right) = \mathrm{Str}\left(\partial_\lambda^m (\mathsf{F}_t - \lambda \mathsf{I})^{-1}\right) - d \int_t^\infty \mathrm{Str}\left(\dot{\mathbb{A}}_s\, \partial_\lambda^m \left((\mathbb{A}_s^2 - \lambda \mathsf{I})^{-1}\right)\right) ds\ . \tag{3.9}$$

*Proof.* We follow the method of [BGV], Corollary 9.32. to see that

$$(\mathsf{F}_t - \lambda \mathsf{I})^{-1} \ = \ \begin{bmatrix} (\nabla_0^2 - \lambda I)^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} O(t^{-1/2}) & O(t^{-1/2}) \\ O(t^{-1/2}) & O(t^{-1}) \end{bmatrix} \tag{3.10}$$

as $t \to \infty$, and hence that for $m > (w+n)/r$ in each $C^l$ norm the kernel estimate

$$\|K(\partial_\lambda^m (\mathsf{F}_t - \lambda \mathsf{I})^{-1}) - K(\partial_\lambda^m (\nabla_0^2 - \lambda I)^{-1})\|_l \leq C_l t^{-1/2}$$

holds uniformly on compact subsets of $M \times_\pi M$, from which (3.8) follows. By integrating (3.4) we have for $0 < t < T < \infty$

$$\mathrm{Str}(\partial_\lambda^{m-1} (\mathsf{F}_T - \lambda \mathsf{I})^{-1}) - \mathrm{Str}(\partial_\lambda^{m-1} (\mathsf{F}_t - \lambda \mathsf{I})^{-1}) = -d \int_t^T \mathrm{Str}\left(\dot{\mathbb{A}}_s\, \partial_\lambda^m \left((\mathbb{A}_s^2 - \lambda \mathsf{I})^{-1}\right)\right) ds\ . \tag{3.11}$$

On the other hand, the estimate (3.10) implies that

$$\|\mathrm{Str}(\dot{\mathbb{A}}_s\, \partial_\lambda^m ((\mathbb{A}_s^2 - \lambda \mathsf{I})^{-1}))\|_l = O(t^{-3/2}) \tag{3.12}$$

as $t \to \infty$ in each $C^l$ norm on compact subsets of $B$, and so with (3.8) the identity (3.9) follows. $\square$

**Corollary 3.4.** *The following cohomology identity holds. For $m > (w+n)/r$ in $H^*(B)$*

$$\partial_\lambda^m \mathrm{Str}\left((\nabla_0^2 - \lambda I)^{-1}\right) = \mathrm{LIM}_{t\to 0}\mathrm{Str}\left(\partial_\lambda^m (F_t - \lambda I)^{-1}\right) \tag{3.13}$$

$$= \sum_{d,k=0}^{\dim B} (\alpha_{w_k+n-rk+r+\frac{rd}{2},[d,k]} + \alpha'_{1-k+\frac{d}{2},[d,k]}\log(\lambda) + \alpha''_{1-k+\frac{d}{2},[d,k]})\,(-\lambda)^{-1-\frac{d}{2}-m}\ , \tag{3.14}$$

*where* (3.14) *follows from* (1.28), *and, at the level of differential forms, the coefficients are closed, differing from the forms in* (3.6) *by constants.*

Notice that (2.20) and (3.9) prove the Chern character transgression formula of [B, BGV]. Indeed, the above formulas are the governing transgression formulas for all characteristic class forms on $\pi_*(\mathcal{E})$.

## 4. Zeta Forms and the Family Index Theorem

This Section consists of the proof of Theorem 0.1. Throughout

$$\mathbb{A}_t := t^{1/2}\delta_t(\mathbb{A})\ , \qquad \mathsf{F}_t := \mathbb{A}_t^2 = t\delta_t(\mathbb{A}^2)\ . \tag{4.1}$$

Evidently $\mathbb{A} = \mathbb{A}_1, \mathsf{F} = \mathsf{F}_1$.

**Proposition 4.1.** *The zeta form $\zeta(\mathbb{A}^2, s)|^{\mathrm{mer}}$ is canonically exact. One has in $\mathcal{A}(B)$*

$$\zeta(\mathbb{A}^2, s)|^{\mathrm{mer}} = d\int_1^\infty \zeta(\dot{\mathbb{A}}_\sigma.F_\sigma, s)|^{\mathrm{mer}}\, d\sigma\ . \tag{4.2}$$

*Proof.* For simplicity we assume (2.1) holds, the modifications for the general case are obvious.

For $\mathrm{Re}(s) > 0$ we then have

$$\mathsf{F}_\theta^{-s} = \frac{i}{2\pi}\int_C \lambda_\theta^{-s}\,(\mathsf{F} - \lambda\mathsf{I})^{-1}\, d\lambda\ .$$

It follows that on $\mathrm{Ker}(\mathsf{P})$

$$\mathsf{F}^{-s}_{|\mathrm{Ker}(\mathsf{P})} \equiv 0\ .$$

For, from (1.18)

$$(\mathsf{F} - \lambda\mathsf{I})^{-1}_{|\mathrm{Ker}(\mathsf{P})} = -\sum_{i=0}^{\dim B} \lambda^{-i-1}\,(\Pi_0\cdot\mathsf{W}\cdot\Pi_0)^i\ , \tag{4.3}$$

and $\frac{i}{2\pi}\int_C \lambda_\theta^{-s-i-1}\, d\lambda = 0$ for $i \geq 0$ and $\mathrm{Re}(s) > 0$. Hence we have

$$\mathrm{Str}(\mathsf{F}^{-s}_{|\mathrm{Ker}(\mathsf{P})}) = 0\ , \qquad \mathrm{Re}(s) > 0\ . \tag{4.4}$$

On the other hand, (3.10) gives

$$\mathrm{Str}\left((\mathsf{F}_T - \lambda\mathsf{I})^{-1}_{|\mathrm{Ker}(\mathsf{P})^\perp}\right) = O(T^{-1}) \quad \text{as } T \longrightarrow \infty\ .$$

Consequently from (3.11)

$$\mathrm{Str}(\partial_\lambda^{m-1}(\mathsf{F} - \lambda\mathsf{I})^{-1}_{|\mathrm{Ker}(\mathsf{P})^\perp}) = d\int_1^\infty \mathrm{Str}\left(\dot{\mathbb{A}}_\sigma\,\partial_\lambda^m\left((\mathbb{A}_\sigma^2 - \lambda\mathsf{I})^{-1}\right)\right)\, d\sigma\ . \tag{4.5}$$

From (4.4), (4.5) and (2.10) we find for $\mathrm{Re}(s) > m > (w+n)/r$

$$\zeta_\theta(\mathsf{F}, s)$$

$$= d\ \left(\int_1^\infty \frac{1}{(s-1)\dots(s-m)}.\frac{i}{2\pi}\int_C \lambda^{m-s}\,\mathrm{Str}\left(\dot{\mathbb{A}}_\sigma\,\partial_\lambda^m(\mathbb{A}_\sigma^2-\lambda\mathsf{I})^{-1}\right)\,d\lambda\,d\sigma\right)$$

$$= d\ \int_1^\infty \zeta(\dot{\mathbb{A}}_\sigma.\mathsf{F}_\sigma, s)\,d\sigma\ .$$

Hence in $\mathcal{A}(B)$

$$\zeta_\theta(\mathsf{F}, s) - d\ \int_1^\infty \zeta(\dot{\mathbb{A}}_\sigma.\mathsf{F}_\sigma, s)\,d\sigma = 0\ , \qquad \mathrm{Re}(s) > (w+n)/r\ . \tag{4.6}$$

Elsewhere in $\mathbb{C}$, from (4.3) and [GS1] Proposition (2.9), we see that $\Gamma(s)\mathrm{Str}(\mathsf{F}^{-s}_{|\mathrm{Ker}(\mathsf{P})})$ has no poles. (Less strong, but more general, and sufficient, the zeta form for any family of finite rank operators extends without poles.) Hence $\mathrm{Str}(\mathsf{F}^{-s}_{|\mathrm{Ker}(\mathsf{P})})|^{\mathrm{mer}}$ is a holomorphic extension of zero to all of $\mathbb{C}$, and consequently

$$\mathrm{Str}(\mathsf{F}^{-s}_{|\mathrm{Ker}(\mathsf{P})})|^{\mathrm{mer}} = 0\ . \tag{4.7}$$

Likewise

$$(\,d\ \int_1^\infty \zeta(\dot{\mathbb{A}}_\sigma.\mathsf{F}_\sigma, s)\,d\sigma\ )|^{\mathrm{mer}} = d\ (\int_1^\infty \zeta(\dot{\mathbb{A}}_\sigma.\mathsf{F}_\sigma, s)|^{\mathrm{mer}}\,d\sigma\ )\ , \tag{4.8}$$

since from (3.4) and (2.12) both sides of (4.8) have the same pole structure. Hence we find that $\zeta_\theta(\mathsf{F}, s)|^{\mathrm{mer}} - d\ \int_1^\infty \zeta(\dot{\mathbb{A}}_\sigma.\mathsf{F}_\sigma, s)|^{\mathrm{mer}}\,d\sigma$ is holomorphic on $\mathbb{C}$ and so from (4.6) it is identically zero. □

Evidently, then, (0.10) now follows with

$$\sum_{k=0}^{\dim B}\frac{1}{k!}\,\zeta_\pi(\mathsf{F}, -k)|^{\mathrm{mer}} = d\left(\sum_{k=0}^{\dim B}\frac{1}{k!}\ \int_1^\infty \zeta(\dot{\mathbb{A}}_\sigma.\mathsf{F}_\sigma, -k)|^{\mathrm{mer}}\,d\sigma\right)\ . \tag{4.9}$$

For clarity, and as it applies to the case of the Bismut connection which is the superconnection of primary geometric interest, we restrict our formulas from here on to the case

$$\nu_i = \mathrm{ord}(\mathsf{F}_{[i]}) \in \mathbb{N}\ . \tag{4.10}$$

The case for any real $\nu_i$ is the same but with more indices to track.

From (1.17) the $t$-rescaled singularity structure equation (2.14) is

$$\frac{\pi}{\sin(\pi s)}\zeta_\theta(\mathsf{F}_t, s)|^{\mathrm{mer}} \sim \tag{4.11}$$

$$-\sum_{j=-\dim B-1}^{-1}\frac{\mathrm{Str}\left((\Pi\cdot\mathsf{W}_t\cdot\Pi)^{-j-1}\right)}{(s-j-1)^{k+1}}\ +$$

$$\sum_{d=0}^{\dim B}\left(\sum_{j\geq 0}\frac{b_{j,d}}{\left(s+\frac{j-n-w}{r}-1\right)}t^{\frac{j-n-w}{r}-1-\frac{d}{2}} + \sum_{l\geq 0}\frac{b''_{l,d}}{(s+l-1)}t^{l-1-\frac{d}{2}}\right)\ ,$$

where

$$\mathsf{W}_t = t\delta_t(\mathsf{W}) = \mathsf{F}_t - t\mathsf{P}^2 \ . \tag{4.12}$$

For $s$ in small neighborhood of $-k$ there is a Laurent expansion

$$\frac{\sin(\pi s)}{\pi} = (-1)^k(s-k) + O((s-k)^3) \ ,$$

and hence from (4.11)
(4.13)

$$(-1)^k\zeta_\pi(\mathsf{F}_t, -k)|^{\rm mer} = -\mathrm{Str}\left((\Pi\cdot\mathsf{W}\cdot\Pi)^k\right) + \sum_{d=0}^{\dim B}\left(b_{n+w+r+rk,d} + b^{''}_{k+1,d}\right)t^{k-\frac{d}{2}} \ .$$

On the other hand, (3.7) of Proposition 3.2 says that the forms $b_{n+w+r+rk,d}$ are exact except possibly when

$$n+w+r+rk = r + \frac{rd}{2} + w + n \ .$$

That is, when $d=2k$. Like wise for the $b^{''}_{k+1,d}$, and so (4.13) can be written

$$\zeta_\pi(\mathsf{F}_t, -k)|^{\rm mer} = -(-1)^k\mathrm{Str}\left((\Pi\cdot\mathsf{W}_t\cdot\Pi)^k\right) + b_{n+w+r+rk,2k} + b^{''}_{k+1,2k} + d\gamma^t_{n,w,r,k} \ ,$$

where $\gamma^t_{n,w,r,k}\in\mathcal{A}(B)$ such that $d\gamma^t_{n,w,r,k}$ is the sum of exact forms in (4.13) minus the two closed forms above. Hence

$$\begin{aligned}\sum_{k=0}^{\dim B}\frac{1}{k!}\,\zeta_\pi(\mathsf{F}_t, -k)|^{\rm mer} = & -\sum_{k=0}^{\dim B}\frac{(-1)^k}{k!}\mathrm{Str}\left((\Pi\cdot\mathsf{W}_t\cdot\Pi)^k\right)\\ & +\sum_{k=0}^{\dim B}\frac{1}{k!}\,(b_{n+w+r+rk,2k} + b^{''}_{k+1,2k}) + d\sum_{k=0}^{\dim B}\frac{1}{k!}\,\gamma^t_{n,w,r,k}\end{aligned}$$

— which since $\mathrm{Str}\left((\Pi\cdot\mathsf{W}_t\cdot\Pi)^k\right)\in\mathcal{A}^{>0}(B)$ for $k>0$ —

$$\begin{aligned} = & -\ \mathrm{Str}\left(e^{-(\Pi\cdot\mathsf{W}_t\cdot\Pi)^k}\right)\\ & +\sum_{k=0}^{\dim B}\frac{1}{k!}\,(b_{n+w+r+rk,2k} + b^{''}_{k+1,2k}) + d\sum_{k=0}^{\dim B}\frac{1}{k!}\,\gamma^t_{n,w,r,k} \ .\end{aligned} \tag{4.14}$$

Now from (4.13) we have $\mathrm{LIM}_{t\to 0}\ d\gamma^t_{n,w,r,k} = 0$. On the other hand, the $t$-independent component of $\mathsf{W}_t$ is the 2-form piece which is the curvature form $\mathbb{A}^2_{[1]}$. Hence we find $\mathrm{LIM}_{t\to 0}\ \mathrm{Str}\left((\Pi\cdot\mathsf{W}_t\cdot\Pi)^k\right) = \mathrm{Str}\left(\nabla_0^{2k}\right)$, with $\nabla_0 = \Pi\cdot\mathbb{A}_{[1]}\cdot\Pi$ the induced connection on the superbundle $\mathrm{Ker}(\mathsf{P})$. Consequently, since (2.21) and (2.22) give

$$\mathrm{LIM}_{t\to 0}\ \mathrm{Str}(e^{-\mathsf{F}_t}) = \sum_{k=0}^{\dim B}\frac{1}{k!}\,(b_{n+w+r+rk,2k} + b^{''}_{k+1,2k}) \ ,$$

taking the regularized limit as $t \to 0+$ of (4.14) we find

$$\text{(4.15)} \qquad \mathrm{LIM}_{t\to 0} \sum_{k=0}^{\dim B} \frac{1}{k!}\, \zeta_\pi(\mathsf{F}_t, -k)|^{\mathrm{mer}} = -\mathrm{ch}(\mathrm{Ker}(\mathsf{P}), \nabla_0) + \mathrm{LIM}_{t\to 0}\ \mathrm{ch}(\mathbb{A}_t)\ .$$

Since, from (4.9), the left-side of (4.15) is equal to

$$\text{(4.16)} \qquad d\ \mathrm{LIM}_{t\to 0} \int_t^\infty \mathrm{Str}(\ \dot{\mathbb{A}}_\sigma \sum_{k=0}^{\dim B} \frac{(-1)^k}{k!}\ \mathsf{F}_\sigma^{-s}|^{\mathrm{mer}}_{s=-k}\ )\ \ d\sigma$$

$$= d\ \mathrm{LIM}_{t\to 0} \int_t^\infty \mathrm{Str}(\ \dot{\mathbb{A}}_\sigma e^{-\mathsf{F}_\sigma})\ \ d\sigma$$

this completes the proof of Theorem 0.1.

## 5. Zeta-Chern Forms

In this Section we prove Theorem 0.4.

**Definition 5.1.** *Let $\mathbb{A}$ be a superconnection on $\pi_*(\mathcal{E})$ adapted to a family of formally self-adjoint $\zeta$-admissible $\psi$dos $P \in \Gamma(B, \Psi^{r>0}(\mathcal{E}))$ of odd-parity. The zeta Chern form of $\mathbb{A}$ is the differential form of mixed order on $B$ defined by the super zeta determinant form*

$$\text{(5.1)} \qquad \mathsf{c}_\zeta(\mathbb{A}) = \mathrm{sdet}_{\zeta,\pi}(I + \mathbb{A}^2) \in \mathcal{A}(B)\ .$$

*Here $I \in \mathcal{A}^0(B, \Psi^0(\mathcal{E}))$ is the vertical identity operator and $\mathbb{A}^2 \in \mathcal{A}(B, \Psi(\mathcal{E}))$ the superconnection curvature.*

Thus, with $\mathsf{F} = \mathbb{A}^2$ we have

$$\text{(5.2)} \qquad -\log \mathsf{c}_\zeta(\mathbb{A}) = \partial_s \mathrm{Str}\left((\mathsf{I} + \mathsf{F}^2)^{-s}\right)|^{\mathrm{mer}}_{s=0} = \mathrm{Str}\left(\log(\mathsf{I}+\mathsf{F})\,(\mathsf{I}+\mathsf{F})^{-s}\right)|^{\mathrm{mer}}_{s=0}\ .$$

Notice that since

$$\text{(5.3)} \qquad (\mathsf{I} + \mathsf{F})_{[0]} = \mathsf{I} + \mathsf{P}^2$$

then $\mathsf{I} + \mathsf{F} \in \mathcal{A}(B, \Psi^{r>0}(\mathcal{E}))$ is elliptic and invertible with Agmon angle $\theta = \pi$. Further, since

$$\mathrm{sdet}_{\zeta,\pi}(\mathsf{I} + \mathsf{P}^2) = 1\ ,$$

then by Lemma 0.3 we have $\log \mathsf{c}_\zeta(\mathbb{A})_{[0]} = 0$ and hence

$$\text{(5.4)} \qquad \mathsf{c}_\zeta(\mathbb{A})_{[0]} = 1\ .$$

To write down the singularity structure of the zeta-form $\zeta_\pi(\mathsf{I}+\mathsf{F}, s)$ it is convenient to use the function introduced in [GS2] defined for $\mathrm{Re}(-t) < \mathrm{Re}(s) < 0$ by

$$\text{(5.5)} \qquad F_t(s) = \frac{i}{2\pi} \int_C \mu^{-s-1}(1-\mu)^{-t}\, d\mu$$

with $C$ a contour around $R_\pi$. $F_t(s)$ extends meromorphically to all of $\mathbb{C}$ and satisfies

$$\text{(5.6)} \qquad F_t(s) = \frac{\Gamma(s+t)}{\Gamma(t)\Gamma(s+1)}.$$

Since

$$\mathrm{Str}\left(\partial_\lambda^{m-1}((\mathsf{I}+\mathsf{F})-\lambda\mathsf{I})^{-1}\right)=\mathrm{Str}\left(\partial_\lambda^{m-1}(\mathsf{F}+(1-\lambda)\mathsf{I})^{-1}\right)$$

we have from (1.17) an asymptotic expansion of the resolvent trace as $\lambda\to\infty$ in $\Lambda_\pi$

$$\begin{aligned}\mathrm{Str}(\partial_\lambda^{m-1}(\mathsf{F}-(1-\lambda)\ \mathsf{I})^{-1}\ ) \;&=\; \sum_{d=0}^{\dim B}\left(\sum_{j=0}^{N-1}(-1)^{\frac{w+n-j}{r}}\beta_{j,d}(1-\lambda)^{\frac{w+n-j}{r}-m}\right.\\ &\quad\left.+\sum_{l=0}^{N-1}(-1)^{-l}(\beta^{'}_{l,d}\log(1-\lambda)+\beta^{''}_{l,d})(1-\lambda)^{-l-m}\right)\\ &\quad+O(|1-\lambda|^{\frac{w+n-N}{r}-m})\ . \qquad (5.7)\end{aligned}$$

Hence in $\mathcal{A}(B)$

$$\begin{aligned}\mathrm{Str}((\mathsf{I}+\mathsf{F})^{-s}\ )|^{\mathrm{mer}} \;&=\; \sum_{d=0}^{\dim B}\left(\sum_{j=0}^{N-1}(-1)^{\frac{w+n-j}{r}}b_{j,d}\ F_{\frac{j-w-n}{r}}(s-1)|^{\mathrm{mer}}\right.\\ &\quad\left.+\sum_{l=0}^{N-1}(-1)^{-l}b^{'}_{l,d}\ \partial_s F_l(s-1)|^{\mathrm{mer}}+\sum_{l=0}^{N-1}b^{''}_{l,d}\ F_l(s-1)|^{\mathrm{mer}}\right)\\ &\quad+h_N(s)\ , \qquad (5.8)\end{aligned}$$

where $h_N(s)\in\mathcal{A}(B)$ is holomorphic for

$$1-\left(\frac{N-n-w}{r}\right)<\mathrm{Re}(s)<N+1\ , \qquad (5.9)$$

and $b_{j,d},b^{'}_{l,[d,q]},b^{''}_{l,[d,q]}\in\mathcal{A}^d(B)$.

**Proposition 5.2.** *The differential form $\mathsf{c}_\zeta(\mathbb{A})$ is closed.*

*Proof.* For $\mathrm{Re}(s)>>0$ we have

$$\mathrm{Str}\left(\log(\mathsf{I}+\mathsf{F})\,(\mathsf{I}+\mathsf{F})^{-s}\right)=\frac{1}{(s-m)\dots(s-1)}\frac{i}{2\pi}\int_C\log\lambda\,\lambda^{m-s}\,\mathrm{Str}(\partial_\lambda^{m-1}(\mathsf{F}{+}(1{-}\lambda)\mathsf{I})^{-1})\ d\lambda$$

and hence from Proposition 3.1

$$d\ \mathrm{Str}\left(\log(\mathsf{I}+\mathsf{F})\,(\mathsf{I}+\mathsf{F})^{-s}\right)=0\ ,\qquad \mathrm{Re}(s)>>0\ . \qquad (5.10)$$

Elsewhere from (5.8)

$$\begin{aligned}\mathrm{Str}\left(\log(\mathsf{I}+\mathsf{F})\,(\mathsf{I}+\mathsf{F})^{-s}\right)|^{\mathrm{mer}} \;&=\; \sum_{d=0}^{\dim B}\left(\sum_{j=0}^{N-1}(-1)^{\frac{w+n-j}{r}}b_{j,d}\ \partial_s F_{\frac{j-w-n}{r}}(s-1)|^{\mathrm{mer}}\right.\\ &\quad\left.+\sum_{l=0}^{N-1}(-1)^{-l}b^{'}_{l,d}\ \partial_s^2 F_l(s-1)|^{\mathrm{mer}}+\sum_{l=0}^{N-1}b^{''}_{l,d}\ \ \partial_s F_l(s-1)|^{\mathrm{mer}}\right)\\ &\quad+\partial_s h_N(s)\ , \qquad (5.11)\end{aligned}$$

But from Proposition 3.2 the forms $b_{j,d},b^{'}_{l,[d,q]},b^{''}_{l,[d,q]}$ are all closed. Hence

$$d\ \mathrm{Str}\left(\log(\mathsf{I}+\mathsf{F})\,(\mathsf{I}+\mathsf{F})^{-s}\right)|^{\mathrm{mer}}=d\ \partial_s h_N(s)\ . \qquad (5.12)$$

The right-side of (5.12) is independent of $N$ and holomorphic. By (5.10) we obtain that $d\ \mathrm{Str}\,(\log(\mathsf{I}+\mathsf{F})\,(\mathsf{I}+\mathsf{F})^{-s})\,|^{\mathrm{mer}}$ is a holomorphic extension of zero and hence vanishes identically on all of $\mathbb{C}$, proving the assertion. □

The zeta-Chern form $\mathsf{c}_\zeta(\mathbb{A})$ hence defines a mixed degree cohomology class in $H^*(B)$. To see this is the Chern class of the index bundle we have the following transgression results.

**Proposition 5.3.** *If $\mathbb{A}_\sigma$ is a 1-parameter family of superconnections adapted to $P \in \mathcal{A}^0(B, \Psi(\mathcal{E}))$ with curvature $F_\sigma = \mathbb{A}_\sigma^2$, then*

$$\partial_\sigma \log \mathsf{c}_\zeta(\mathbb{A}_\sigma) = -d\ \zeta((F_\sigma + I)^{-1}\dot{\mathbb{A}}_\sigma, F_\sigma + I, 0)|^{\mathrm{mer}} \ . \tag{5.13}$$

*Here,* $\zeta((F_\sigma + I)^{-1}\dot{\mathbb{A}}_\sigma, F_\sigma + I, s)|^{\mathrm{mer}} = \mathrm{Str}((F_\sigma + I)^{-s-1}\dot{\mathbb{A}}_\sigma)|^{\mathrm{mer}}$.

*Proof.* From (3.4) we have for sufficiently large $m$

$$\partial_\sigma\ \mathrm{Str}(\partial_\lambda^{m-1}(\mathsf{F}_\sigma + (1-\lambda)\mathsf{I})^{-1}) = -d\ \mathrm{Str}(\partial_\lambda^{m}(\mathsf{F}_\sigma + (1-\lambda)\mathsf{I})^{-1}\dot{\mathbb{A}}_\sigma) \ . \tag{5.14}$$

Hence for $\mathrm{Re}(s) >> 0$ we have integrating by parts

$$\partial_\sigma\ \mathrm{Str}\left(\log(\mathsf{I}+\mathsf{F}_\sigma)\,(\mathsf{I}+\mathsf{F}_\sigma)^{-s}\right) \tag{5.15}$$

$$\begin{aligned}
&= -d\left(\frac{1}{(s-m)\ldots(s-1)}\frac{i}{2\pi}\int_C \log\lambda\,\lambda^{m-s}\,\mathrm{Str}(\partial_\lambda^m(\mathsf{F}_\sigma + (1-\lambda)\mathsf{I})^{-1}\dot{\mathbb{A}}_\sigma)\ d\lambda\right)\\
&= -d\left(\frac{1}{(s-m)\ldots(s-1)}\frac{i}{2\pi}\int_C \lambda^{m-s-1}\,\mathrm{Str}(\partial_\lambda^{m-1}(\mathsf{F}_\sigma + (1-\lambda)\mathsf{I})^{-1}\dot{\mathbb{A}}_\sigma)\ d\lambda\right)\\
&\quad + s\,d\left(\frac{1}{(s-m)\ldots(s-1)}\frac{i}{2\pi}\int_C \log\lambda\,\lambda^{m-s-1}\,\mathrm{Str}(\partial_\lambda^{m-1}(\mathsf{F}_\sigma + (1-\lambda)\mathsf{I})^{-1}\dot{\mathbb{A}}_\sigma)\ d\lambda\right)\\
&= -d\ \zeta((\mathsf{F}_\sigma+\mathsf{I})^{-1}\dot{\mathbb{A}}_\sigma, \mathsf{F}_\sigma+\mathsf{I}, s) + s\,d\ \partial_s\zeta((\mathsf{F}_\sigma+\mathsf{I})^{-1}\dot{\mathbb{A}}_\sigma, \mathsf{F}_\sigma)+\mathsf{I}, s) \ .
\end{aligned}$$

Since $\zeta((\mathsf{F}_\sigma+\mathsf{I})^{-1}\dot{\mathbb{A}}_\sigma, \mathsf{F}_\sigma)+\mathsf{I}, s)$ is holomorphic around $s=0$, as we consider $\zeta$-admissible vertical operators, then evaluating at zero (5.15) gives

$$\partial_\sigma\ \mathrm{Str}\left(\log(\mathsf{I}+\mathsf{F}_\sigma)\,(\mathsf{I}+\mathsf{F}_\sigma)^{-s}\right)|_{s=0}^{\mathrm{mer}} = -\left(d\ \zeta((\mathsf{F}_\sigma+\mathsf{I})^{-1}\dot{\mathbb{A}}_\sigma, \mathsf{F}_\sigma+\mathsf{I}, s)\right)|_{s=0}^{\mathrm{mer}} \ .$$

Since the singularity expansion (5.11) shows that the derivatives can be commuted with $|^{\mathrm{mer}}$, this completes the proof. □

We now restrict attention to the rescaled superconnection and curvature (4.1).

**Proposition 5.4.** *Let $\tau_{t,T}(\mathbb{A}) = -d\ \int_t^T \zeta((F_\varepsilon + I)^{-1}\dot{\mathbb{A}}_\varepsilon, F_\varepsilon + I, 0)|^{\mathrm{mer}} d\varepsilon$ with $0 < t < T < +\infty$. Then in $\mathcal{A}(B)$ one has*

$$\frac{\mathsf{c}_\zeta(\mathbb{A}_T)}{\mathsf{c}_\zeta(\mathbb{A}_t)} = e^{\tau_{t,T}(\mathbb{A})} \ . \tag{5.16}$$

*Equivalently,*

$$\mathsf{c}_\zeta(\mathbb{A}_T) = \mathsf{c}_\zeta(\mathbb{A}_t) + d\,\omega_{t,T}(\mathbb{A}) \ , \tag{5.17}$$

*where*

$$\omega_{t,T}(\mathbb{A}) = \mathsf{c}_{\zeta}(\mathbb{A}_t) \wedge \sum_{k\geq 1} \frac{1}{k!} \tau_{t,T}(\mathbb{A}) \wedge (d\,\tau_{t,T}(\mathbb{A}))^{k-1} \ . \tag{5.18}$$

*If $\mathsf{P} \in \mathcal{A}^0(B, \Psi(\mathcal{E}))$ has constant kernel dimension, then the limit $\lim_{T\to\infty} \omega_{t,T}(\mathbb{A})$, denoted $\omega_{t,\infty}(\mathbb{A})$ exists in all $C^l$-norms on compact subsets of $B$, and one has*

$$\mathsf{c}(\mathrm{Ker}(\mathsf{P}), \nabla_0) = \mathsf{c}_{\zeta}(\mathbb{A}_t) + d\ \omega_{t,\infty}(\mathbb{A}) \ . \tag{5.19}$$

*with notation as in Theorem 0.4.*

*Proof.* From (5.4) the quotient on the left-side of (5.16) is well-defined in $\mathcal{A}(B)$. The identity is immediate from integrating (5.13) and exponentiating both sides. Since $\mathsf{c}_{\zeta}(\mathbb{A}_t)$ is closed, (5.17) is immediate.

By Proposition 3.3 and (5.11) we find

$$\begin{aligned}\lim_{T\to\infty} \mathsf{c}_{\zeta}(\mathbb{A}_T) &= \frac{1}{(s-m)\dots(s-1)} \frac{i}{2\pi} \int_C \log\lambda\, \lambda^{m-s} \lim_{T\to\infty} \mathrm{Str}(\partial_{\lambda}^{m-1}(\mathsf{F}_T + (1-\lambda)\mathsf{I})^{-1}\ d\lambda|^{\mathrm{mer}}_{s=0} \\ &= \frac{i}{2\pi} \int_C \log\lambda\, \mathrm{Str}((\nabla_0^2 + (1-\lambda)\mathsf{I})^{-1}\ d\lambda = \log\ \mathrm{sdet}(I + \nabla_0^2)) \ .\end{aligned}$$

Since (3.12)implies the $\zeta$-form $C^l$ estimate $\|\zeta((\mathsf{F}_{\varepsilon}+\mathsf{I})^{-1}\dot{\mathbb{A}}_{\varepsilon}, \mathsf{F}_{\varepsilon}+\mathsf{I}, 0)|^{\mathrm{mer}}\|_l \leq c(l)t^{-3/2}$ we obtain the existence of the limit $\lim_{T\to\infty} \tau_{t,T}(\mathbb{A})$ and hence the limit $\lim_{T\to\infty} \omega_{t,T}(\mathbb{A})$. □

We turn next to the proof of the local index density formula (0.19).

We suppose that the fibre bundle $\pi : M \longrightarrow B$ has even-dimensional fibre and that it is endowed with a connection

$$TM = \pi^*(TB) \oplus T(M/B) \ , \tag{5.20}$$

defined by a choice of bundle projection $P : TM \to T(M/B)$. Suppose also that $TM$ has a spin structure and that there are Riemannian metrics $g_{M/B}, g_B$ on $T(M/B)$ and $TB$. The vertical bundle $\mathcal{E}$ is assumed to be a bundle of Clifford modules equipped with a connection which restricts to a Clifford connection on $\mathcal{E}_{|M_z}$. Let $\mathsf{D}$ be the associated family of compatible Dirac operators, let $\nabla^{\pi_*(\mathcal{E})}$ be the canonical Hermitian connection induced on $\pi_*(\mathcal{E})$ [B], [BGV] Proposition(9.13), and let $\mathsf{c}(T) \in \mathcal{A}^2(B, \mathrm{End}(\pi_*(\mathcal{E})))$ denote Clifford multiplication by the torsion tensor of the fibration associated to the connection (5.20). For $t > 0$ the scaled Bismut superconnection on $\pi_*(\mathcal{E})$ is then defined by [B, BGV]

$$\mathbb{A}_t = t^{1/2}\mathsf{D} + \nabla^{\pi_*(\mathcal{E})} + \frac{1}{4t^{1/2}}\,\mathsf{c}(T) \ . \tag{5.21}$$

Its crucial property [BGV] Proposition (10.28) is that the curvature 4-form

$$\mathsf{F}_t = \mathbb{A}_t^2 \in \mathcal{A}^4(B, \Psi(\mathcal{E}))$$

in a small enough neighborhood $U$ of $x \in M$ with local geodesic coordinates $(x_1, \ldots, x_n)$ along the fibres takes the form as $t \to 0+$

$$\mathsf{F}_t = -\sum_i \left( \partial_i - \frac{1}{4} \sum_j (R^{M/B} \partial_i, \partial_j) x_j \right)^2 + R^{\mathcal{E}/S} + O(t^{1/2}) \ , \tag{5.22}$$

where the $O(t^{1/2})$ term is of non-zero form degree. Let

$$\widehat{A}(M/B) = \det{}^{1/2} \left( \frac{R^{M/B}/2}{\sinh(R^{M/B}/2)} \right)$$

be the vertical $\widehat{A}$-genus form for the connection $\nabla^{M/B} = P \cdot \nabla^M \cdot P$, with $\nabla^M$ the Levi-Civita connection defined by the metric $g_{M/B} \oplus \pi^*(g_B)$, and let $\mathrm{ch}'(\mathcal{E}) = \mathrm{Str}_{\mathcal{E}/S}(e^{-R^{\mathcal{E}/S}})$ be the relative Chern character of $\mathcal{E}$ and the spin bundle $S$ on $M$.

**Proposition 5.5.** *With $\mathsf{F}_t$ the Bismut superconnection curvature, the differential form has a limit as $t \to 0$ given by the formula*

$$\lim_{t \longrightarrow 0} \mathrm{Str}\left( \partial_\lambda^{m-1} (\mathsf{F}_t - \lambda I)^{-1} \right) \tag{5.23}$$

$$= (m-1)! \sum_{k=0}^{[\dim B/2]} (-\lambda)^{-1-k-m} \, k! \left( (2\pi)^{-\frac{n}{2}} \int_{M/B} \widehat{A}(M/B) \mathrm{ch}'(\mathcal{E}) \right)_{[2k]} .$$

*Proof.* Since the component terms $(\mathsf{F}_t)_{[i]} \in \mathcal{A}^i(B, \Psi(\mathcal{E}))$, $i = 0, \ldots, 4$, are smooth families of *differential* operators, with $(\mathsf{F}_t)_{[0]} = t\mathsf{D}^2$ so that $r = 2$, it follows from [GS1] that there are no log terms in the resolvent supertrace and that all the coefficients are local, determined by only finitely many terms of the vertical symbol.; that is, $\beta'_{l,d} = 0, \beta''_{l,d} = 0$ in (1.29), so that as $t \to 0+$ there is an asymptotic expansion

$$\mathrm{Str}(\partial_\lambda^{m-1} (\mathsf{F}_t - \lambda \mathsf{I})^{-1}) \sim \sum_{d=0}^{\dim B} \sum_{j \geq 0} \beta_{j,d} (-\lambda)^{\frac{w+n-j}{2} - m} \, t^{\frac{j-w-n}{2} - 1 - \frac{d}{2}} . \tag{5.24}$$

On the other hand, it follows from (5.22) that

$$\mathrm{Str}(\partial_\lambda^{m-1} (\mathsf{F}_t - \lambda \mathsf{I})^{-1}_{|U}) \tag{5.25}$$

$$= \sum_{k=0}^{\dim B} \partial_\lambda^{m-1} \left( (\sum_i \partial_i^2 - \lambda \mathsf{I})^{-1} \left( \omega(R^{M/B}, R^{\mathcal{E}/S}) \, (\sum_i \partial_i^2 - \lambda \mathsf{I})^{-1} \right)^k \right) + O(t^{1/2}) \ ,$$

with $\omega(R^{M/B}, R^{\mathcal{E}/S}) \in \mathcal{A}^{k \geq 2}(B)$ a form of mixed degree 2 or greater, and hence that the resolvent supertrace has a limit as $t \to 0$.

Consequently, the expansion (5.24) begins with the $t^0$ term, that is when

$$\frac{j - n - w}{2} = 1 + \frac{d}{2} \ , \tag{5.26}$$

and since from (5.25) the form degree is always even $d = 2k$, then as $t \to 0+$

$$\mathrm{Str}(\partial_\lambda^{m-1} (\mathsf{F}_t - \lambda \mathsf{I})^{-1}) = \sum_{k=0}^{[\dim B/2]} \beta_{n+w+2+2k, 2k}[m] (-\lambda)^{-1-k-m} + O(t^{1/2}) \ . \tag{5.27}$$

We hence obtain from (4.11)

$$\frac{\pi}{\sin(\pi s)}\zeta_\theta(\mathsf{F}_t,s)|^{\mathrm{mer}} = \tag{5.28}$$

$$-\sum_{l=0}^{\dim B}\frac{\mathrm{Str}\left((\Pi\cdot\mathsf{W}_t\cdot\Pi)^l\right)}{(s+l)} \;+\sum_{k=0}^{[\dim B/2]}\frac{b_{n+w+2+2k,2k}}{(s+k)}+G_{t^{1/2}}(s)\ ,$$

where $G_{t^{1/2}}(s)=O(t^{1/2})$ and is meromorphic on $\mathbb{C}$ with no poles at the negative integers, and according to (2.15) and (2.22)

$$\beta_{n+w+2+2k,2k}[m]=(m-1)!\ b_{n+w+2+2k,2k}=(m-1)!\ k!\ \tilde{b}_{n+w+2+2k,2k}\ . \tag{5.29}$$

Since $\tilde{b}_{n+w+2+2k,2k}=\mathrm{Str}(e^{-\mathsf{F}_t})_{[2k]}$ the result can now be deduced by direct appeal to the Bismut Local Family Index Theorem formula [BGV], Theorem(10.23), but let us rather outline how one can deduce this from the local resolvent symbols. This computation is part of joint work with Don Zagier [SZ].

For brevity we will consider the case where $\mathcal{E}$ is trivial with zero curvature; the general case follows easily from this one. Then from the local formula (5.22) it is sufficient to compute $\tilde{b}_{n+w+2+2k,2k}$ for the local operator

$$H_z=-\sum_i\left(\partial_i-\frac{1}{4}\,r_i\,x_i\right)^2 \tag{5.30}$$

since the vertical skew-adjoint matrix of 2-forms $(R^{M/B}\partial_i,\partial_j)$ can be written with respect to a particular vertical orthonormal basis as the direct sum of $2\times 2$ blocks of 2-forms $\begin{bmatrix}0 & -r_j\\ r_j & 0\end{bmatrix}$ along the fibres. By definition one then has

$$\tilde{b}_{n+w+2+2k,2k}(z)=\frac{1}{(2\pi)^n}\int_{M/B}\int_{\mathbb{R}^n}\frac{i}{2\pi}\int_{C_0}e^{-\lambda}\ \mathbf{q}_{2k}(z,x,\xi,\lambda)\ d\lambda\ d\xi \tag{5.31}$$

where $\mathbf{q}_j(z,x,\xi,\lambda)$ are $2j$-forms which are the $\xi$-homogeneous terms of the symbol of the resolvent operator $(H_z-\lambda I)^{-1}$ and $C_0$ is a contour coming in on a ray with argument in $(0,\pi/2)$, encircling the origin, and leaving on a ray with argument in $(-\pi/2,0)$. From (5.30) one easily sees using the product formula for symbols that the $\mathbf{q}_j$ are determined by the following recurrence relation: let $\Delta=\sum_{k=1}^n\partial^2_{x_k}$, and set $\mathbf{q}_{-1}=0$, $\mathbf{q}_0=(|\xi|^2-\lambda I)^{-1}$, then one has

$$\mathbf{q}_{j+1}=\mathbf{q}_0\,(\Delta-a_2)\,\mathbf{q}_{j-1}-\mathbf{q}_0\,a_1\,r_j \qquad (j\geq 0)\ . \tag{5.32}$$

where with $\xi=(\xi_1,\ldots,\xi_{2n})$

$$a_1(x,\xi)=i\sum_{j=1}^{n/2}\left(\frac{r_j}{2}\right)(x_{2j-1}\,\xi_{2j}-x_{2j}\,\xi_{2j-1}),\quad a_2(x,\xi)=-\frac{1}{4}\sum_{j=1}^{n/2}\left(\frac{r_j}{2}\right)^2(x_{2j-1}^2+x_{2j}^2)\,.$$

It is convenient to write $\mathbf{q}_j$ explicitly as a polynomial in $T=(|\xi|^2-\lambda I)^{-1}$

$$\mathbf{q}_j(T)=\sum_{\mu+2\nu=j}\mathbf{q}_{\mu,\nu}\,T^{\mu+\nu+1}\,, \tag{5.33}$$

where the polynomials $\mathbf{q}_{\mu,\nu}$ are now given recursively by

$$\mathbf{q}_{\mu,\nu} \cong \begin{cases} 1 & \text{if } \mu=\nu=0, \\ -a_1\,\mathbf{q}_{\mu-1,\nu} + (\Delta - a_2)\,\mathbf{q}_{\mu,\nu-1} & \text{otherwise,} \end{cases} \tag{5.34}$$

with the convention that $\mathbf{q}_{*,*}$ is to be interpreted as 0 if either index is negative. Then (5.34) implies that

$$\sum_{\mu,\,\nu\geq 0} \frac{\mathbf{q}_{\mu,\nu}(z,x,\xi)}{(\mu+\nu)!} = \left(\prod_{i=1}^{n} \frac{1}{\cosh \hat{r}_i}\right)^{1/2} \exp\left(|\xi|^2 - \sum_{i=1}^{n} \frac{\tanh \hat{r}_i}{\hat{r}_i}\,(\xi_i + \frac{1}{2}\,\hat{r}_i x_i)^2\right), \tag{5.35}$$

where $\hat{r}_{2j-1} = -\hat{r}_{2j} = i\,r_j$. Hence

$$\begin{aligned} &\sum_{j=0}^{n/2} \int_{\mathbb{R}^n} \frac{i}{2\pi} \int_{C_0} e^{-\lambda}\, \mathbf{q}_{2j}(x,\xi,\lambda)\; d\lambda\; d\xi \\ =& \int_{\mathbb{R}^n} \sum_{\mu,\nu\geq 0} \mathbf{q}_{\mu,\nu}(x,\xi,\lambda)\, \frac{i}{2\pi} \int_{C_0} e^{-\lambda}\, (|\xi|^2 - \lambda)^{-(\mu+\nu+1)}\; d\lambda\; d\xi \\ =& \int_{\mathbb{R}^n} \sum_{\mu,\nu\geq 0} \frac{\mathbf{q}_{\mu,\nu}(x,\xi,\lambda)}{(\mu+\nu)!}\, \frac{i}{2\pi} \int_{C_0} e^{-\lambda}\, (|\xi|^2 - \lambda)^{-1}\; d\lambda\; d\xi \\ =& \int_{\mathbb{R}^n} \sum_{\mu,\nu\geq 0} \frac{\mathbf{q}_{\mu,\nu}(x,\xi,\lambda)}{(\mu+\nu)!}\; e^{-|\xi|^2}\; d\xi \\ =& \int_{\mathbb{R}^n} \prod_{i=1}^{n} (\cosh \hat{r}_i)^{-1/2} \exp\left(-\sum_{i=1}^{n} \frac{\tanh \hat{r}_i}{\hat{r}_i}\,(\xi_i + \frac{1}{2}\,\hat{r}_i x_i)^2\right) d\xi \\ =& \; 2\pi^{n/2}\, \widehat{A}(M/B)\,, \end{aligned}$$

from which we have that

$$b_{n+w+2+2k,2k} = k!\left((2\pi)^{-\frac{n}{2}} \int_{M/B} \widehat{A}(M/B)\right)_{[2k]}. \tag{5.36}$$

Equations (5.27), (5.29), (5.36) thus combine to prove (5.23). □

**Corollary 5.6.** *With $\mathsf{F}_t$ the Bismut superconnection curvature, the resolvent trace differential form $\log \mathsf{c}_\zeta(\mathbb{A}_t)$ has a limit in $\mathcal{A}(B)$ as $t \to 0$ given by the formula*

$$\lim_{t\longrightarrow 0} \log \mathsf{c}_\zeta(\mathbb{A}_t) = \sum_{k=0}^{[\dim B/2]} (-1)^k (k-1)! \left((2\pi)^{-\frac{n}{2}} \int_{M/B} \widehat{A}(M/B)\mathrm{ch}'(\mathcal{E})\right)_{[2k]}. \tag{5.37}$$

*Proof.* By the previous proof, we have

$$\begin{aligned} \mathrm{Str}(\partial_\lambda^{m-1}((\mathsf{I}+\mathsf{F}_t) - \lambda \mathsf{I})^{-1}) \;\sim\; & \sum_{k=0}^{[\dim B/2]} (-1)^{-1-k} \beta_{n+w+2+2k,2k} (1-\lambda)^{-1-k-m} \\ & + \sum_{k=0}^{[\dim B/2]} \sum_{j\geq 1} \beta_{j,2k} (1-\lambda)^{-\frac{j}{2}-1-k-m}\, t^{\frac{j}{2}}\,, \end{aligned} \tag{5.38}$$

while (5.23) becomes

$$\lim_{t\longrightarrow 0} \mathrm{Str}(\partial_\lambda^{m-1}((\mathsf{I}+\mathsf{F}_t)-\lambda\mathsf{I})^{-1}) \tag{5.39}$$

$$= (m-1)! \sum_{k=0}^{[\dim B/2]} (-1)^{-1-k}(1-\lambda)^{-1-k-m}\, k! \left((2\pi)^{-\frac{n}{2}} \int_{M/B} \widehat{A}(M/B)\mathrm{ch}'(\mathcal{E})\right)_{[2k]} ,$$

and (5.11) becomes

$$\mathrm{Str}\left(\log(\mathsf{I}+\mathsf{F}_t)\,(\mathsf{I}+\mathsf{F}_t)^{-s}\right)|^{\mathrm{mer}}$$

$$= \sum_{k=0}^{[\dim B/2]} (-1)^k k! \left((2\pi)^{-\frac{n}{2}} \int_{M/B} \widehat{A}(M/B)\mathrm{ch}'(\mathcal{E})\right)_{[2k]} \partial_s F_{1+k}(s-1)|^{\mathrm{mer}}$$

$$+\ \partial_s h_{2k+3+n+w,t}(s)\ , \tag{5.40}$$

where $h_{N,t}(s)$ is the remainder term $h_N(s)$ for the $t$-rescaled Bismut superconnection with $N = 2k+3+n+w$. The coefficients in the above formulas are precisely related as explained in the previous proof. The powers in the expansion (5.38) have the crucial consequence for the globally determined remainder term in (5.40) that

$$\partial_s h_{2k+3+n+w,t}(s) = O(t^{1/2}) \tag{5.41}$$

as $t \to 0+$. It remains then to compute the sum in (5.40). We have for $-1-\mathrm{Re}(\alpha) < \mathrm{Re}(s) < 0$

$$\begin{aligned}
\partial_s F_{1+\alpha}(s-1) &= \partial_s \frac{i}{2\pi}\int_C \mu^{-s}(1-\mu)^{-\alpha-1}\, d\mu \\
&= -\frac{1}{\alpha}\frac{i}{2\pi}\int_C \log\mu\, \mu^{-s}\partial_\mu(1-\mu)^{-\alpha}\, d\mu \\
&= \frac{1}{\alpha}\frac{i}{2\pi}\int_C \mu^{-s-1}(1-\mu)^{-\alpha}\, d\mu - \frac{s}{\alpha}\frac{i}{2\pi}\int_C \log\mu\, \mu^{-s-1}(1-\mu)^{-\alpha}\, d\mu \\
&= \frac{1}{\alpha}F_\alpha(s) - \frac{s}{\alpha}\,\partial_s F_\alpha(s)\ . \qquad (5.42)
\end{aligned}$$

Since $\Gamma(z)|^{\mathrm{mer}}$ is holomorphic for $z \neq 0,-1,-2,\ldots$, then from (5.6) we have that $F_\alpha(s)|^{\mathrm{mer}}$ is holomorphic near $s=0$ and hence that

$$\left(\frac{s}{\alpha}\,\partial_s F_\alpha(s)\right)|^{\mathrm{mer}}_{s=0} = 0\ .$$

The meromorphic extension of the Gamma function is obtained by the identity $\Gamma(s+1) = s\Gamma(s)$ ,and hence from (5.6) we find that

$$F_k(s)|^{\mathrm{mer}}_{s=0} = 1\ .$$

Hence

$$\partial_s F_{1+k}(s-1)|^{\mathrm{mer}}_{s=0} = \frac{1}{k}\ .$$

From (5.40) and (5.41) we obtain the asserted identity □

Using Proposition 3.3, one has for the Bismut superconnection that the limit

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon}\zeta_\pi((\mathsf{I}+\mathsf{F}_\varepsilon)^{-1}\dot{\mathbb{A}}_\varepsilon,\mathsf{I}+\mathsf{F}_\varepsilon,0)|^{\mathrm{mer}}\,d\varepsilon$$

exists uniformly in all $C^l$ norms on compact subsets of $B$, and hence that

$$\tau_{0,\infty}(\mathbb{A}):=\lim_{\varepsilon\to 0}\tau_{\varepsilon,\varepsilon^{-1}}(\mathbb{A})$$

exists. With Proposition 5.4, this completes the proof of the local family index formula for the zeta-Chern class:

$$\begin{aligned}\log\mathsf{c}(\mathrm{Ker}(\mathsf{D}),\nabla^0) &= \sum_{k=0}^{[\dim B/2]}(-1)^k(k-1)!\left((2\pi)^{-\frac{n}{2}}\int_{M/B}\widehat{A}(M/B)\mathrm{ch}'(\mathcal{E})\right)_{[2k]}\\ &\quad + d\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon}\zeta_\pi((\mathsf{I}+\mathsf{F}_\varepsilon)^{-1}\dot{\mathbb{A}}_\varepsilon,\mathsf{I}+\mathsf{F}_\varepsilon,0)|^{\mathrm{mer}}\,d\varepsilon\ ,\end{aligned}$$

or, exponentiating,

$$\mathsf{c}(\mathrm{Ker}(\mathsf{D}),\nabla^0)=\prod_{k=0}^{[\dim B/2]}e^{(-1)^k(k-1)!\left((2\pi)^{-\frac{n}{2}}\int_{M/B}\widehat{A}(M/B)\mathrm{ch}'(\mathcal{E})\right)_{[2k]}}+d\omega_{0,\infty}\ .$$

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