A SYMBOL PROOF OF THE LOCAL INDEX THEOREM

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Abstract. The local index theorem is proved by computing the generating function for the symbol of the resolvent of the Dirac Laplacian. This is accomplished without use of Mehler's formula.

Let M be a compact Riemannian manifold of even dimension n and let

$$\mathsf{D} = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix} : \Gamma(M,\mathcal{E}) \longrightarrow \Gamma(M,\mathcal{E})$$

be a first-order differential operator of Dirac-type acting on the space of sections of a bundle of graded Clifford modules $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. It is a fact that has been well-known since the early days of index theory that the index of D can be computed as the supertrace of the heat operator $\mathrm{e}^{-t\mathsf{D}^2}$, or, equivalently, via the meromorphically continued value at $s = 0$ of the supertrace of the complex power D^{-2s}. It is, though, a fact which is surprising. It is surprising because the supertrace of a trace-class pseudodifferential operator (ΨDO) acting on $\Gamma(M,\mathcal{E})$ is prima facia a global spectral invariant, while the index of D is a local invariant. It follows that a (potentially complicated) cancellation mechanism of the global part, and non-contributing local parts, of the heat kernel must take place inside the supertrace of $\mathrm{k}(\mathrm{e}^{-t\mathsf{D}^2})(x)$, or, equivalently, inside the supertrace of the power kernel $\mathrm{k}(\mathsf{D}^{-2s})(x)|^{\mathrm{mer}}$. An insightful analysis of that process led to Getzler's proof [Ge] of the local index theorem

$$\lim_{t\to 0} \mathrm{str}(\mathrm{k}(\mathrm{e}^{-t\mathsf{D}^2})(x)) = (2\pi)^{-n/2}\widehat{A}(R)\,\mathrm{ch}(\mathcal{E}/S)_{[n]}\ , \tag{0.1}$$

utilizing the property that the heat kernel for a general harmonic oscillator can be written in closed form (Mehler's formula).

The subscript on the right-side of (1.6) indicates the n-form component of the differential form written there, composed of the $\widehat{A}$-form, built from the Riemannian curvature form R, and the relative Chern character form $\mathrm{ch}(\mathcal{E}/S)$. A formula identifying all form components of the index density $\widehat{A}(R)\,\mathrm{ch}(\mathcal{E}/S)$ with corresponding local contributions of the heat kernel was subsequently given in the monograph [BeGeVe]; it is that formula which we shall be concerned with here.

Getzler's proof and the subsequent 'short' proofs by Bismut [Bi], Atiyah [At], Witten [Wi] were driven by ideas of QFT and supersymmetry from physics. However, the first analytic proof of a local index theorem had been given 15 years earlier by Patodi [Pa] for the Riemann-Roch theorem on Kahler manifold by constructing an explicit parametrix for $\mathrm{k}(\mathrm{e}^{-t\mathsf{D}^2})(x)$. In Atiyah-Bott-Patodi [AtBoPa] a (rather indirect) heat kernel proof for general Dirac-type operators was subsequently obtained by using Gilkey's [Gi2] theory of $O(n)$ invariants to check enough special cases to conclude that the heat kernel had to converge to the $\widehat{A}$-genus form.

The approach we have taken in this paper may perhaps be seen to be rather closer to that of Patodi than to the later physics inspired proofs. The method we adopt is to explicitly compute from a generating function the relevant terms in the symbol of the resolvent of the Dirac Laplacian. In this way, an explicit identity is established between the classical generating function for the characteristic numbers (the $\widehat{A}$ and Todd polynomials) and the generating function determining the polynomial functions of ΨDO symbols which arise when computing the operator index.

1. Statement of the Theorem

In this Section we formulate the local index theorem at the level of ΨDO symbols; since it appears not to be available in the literature, we do so initially for the complex power operators, before recalling the equivalence with the usual heat kernel statements.

For $\mathrm{Re}(s) > n/2$ the complex power D^{-2s} is a classical ΨDO of order $-2s$ with absolutely integrable kernel $K(\mathsf{D}^{-2s}, x, y)$ continuous in $(x, y) \in M \times M$ (C^∞ away from the diagonal). The restriction of $K(\mathsf{D}^{-2s}, x, y)$ to the diagonal considered as a defined on M defines a smooth global section

$$\mathrm{k}[\mathsf{D}^{-2s}](x) := K(\mathsf{D}^{-2s}, x, x)\, j_g(x)^{-1} \in \Gamma(M, \mathrm{End}(\mathcal{E})) \qquad \mathrm{Re}(s) > n/2 \ , \tag{1.1}$$

where $j_g(x) = \sqrt{\det(g(x))}$, g the Riemannian metric. From [Se], the 'kernel' (1.1) extends meromorphically $k[\mathsf{D}^{-2s}](x)|^{\mathrm{mer}}$ to all of $\mathbb{C}$ with pole structure

$$\Gamma(s)\ \mathrm{k}[\mathsf{D}^{-2s}](x)|^{\mathrm{mer}} \sim \sum_{j\geq 0} \frac{k_{2j}(x)}{s + j - \frac{n}{2}} - \frac{\Pi_0(\mathsf{D})(x)}{s} \ , \tag{1.2}$$

where $\Gamma(s)$ is the Gamma function, $k_{2j}(x) \in \Gamma(M, \mathrm{End}(\mathcal{E}))$, and $\Pi_0(\mathsf{D})(x) = \mathrm{Ker}(x, x)$ for $\mathrm{Ker}(x, y)$ the smooth kernel of the orthogonal projection onto the finite-dimensional kernel $\mathrm{Ker}(\mathsf{D}) = \mathrm{Ker}(\mathsf{D}^2)$. The coefficients $k_0(x), k_2(x), \ldots, k_{n-2}(x)$ correspond to simple poles of $k[\mathsf{D}^{-2s}](x)|^{\mathrm{mer}}$ located at $s = \frac{n}{2}, \frac{n}{2} - 1, \ldots, 1$. These coefficients along with the coefficient $k_n(x)$, which does not correspond to a pole but is equal to the regular value $\mathrm{k}[\mathsf{D}^{-2s}](x)|^{\mathrm{mer}}_{s=0} + \Pi_0(\mathsf{D})(x)$, will determine the index density $\widehat{A}(R)\,\mathrm{ch}(\mathcal{E}/S)$. The above coefficients can be expressed in the following uniform manner using the ΨDO symbol $\mathbf{r}(x, \xi, \lambda)$ of the resolvent $(\mathsf{D}^2 - \lambda\mathsf{I})^{-1}$. The latter is a classical elliptic ΨDO of order -2 with parameter $\lambda \in \mathbb{C}\backslash\mathbb{R}_+$ and local asymptotic expansion of $\mathbf{r}$ over open $U \in M$

$$\mathbf{r}(x,\xi,\lambda) \sim \sum_{j\geq 0} \mathbf{r}_j(x,\xi,\lambda)\ , \tag{1.3}$$

meaning that $\mathbf{r}(x,\xi,\lambda) - \sum_{j=0}^{J-1} \mathbf{r}_j(x,\xi,\lambda)$ is an element of the space $S^{-2-J}(U)$ of ΨDO symbols of order $-2-J$, and $\mathbf{r}_j(x,t\xi,t^m\lambda) = t^{m-j}\,\mathbf{r}_j(x,\xi,\lambda)$ for $t \geq 1, |\xi| \geq 1$.

Proposition 1.1. *For $j = 0,1,\ldots,\frac{n}{2}$ one has*

$$k_{2j}(x) = -\frac{1}{2}\int_{|\xi|=1}\int_{C_0} \log(\lambda)\ \partial_\lambda^{\frac{n}{2}-j}\,\mathbf{r}_{2j}(x,\xi,\lambda)\ \hat{d}\lambda\ \hat{d}S(\xi)\ , \tag{1.4}$$

where $\hat{d}\lambda = \frac{i}{2\pi}\,d\lambda$, and $\hat{d}S(\xi) = (2\pi)^{-n}\,dS(\xi)$ with $dS(\xi)$ the surface measure on the $n-1$ sphere S^{n-1} (indicated by $|\xi| = 1$ in the integral).

Proof. For $j = 0,1,\ldots,\frac{n}{2}-1$ this is immediate by integration by parts of [Se] eqn(), [**?**] eqn(), using $\partial_\lambda^k \log\lambda = (-1)^k(k-1)!\ \lambda^{-k}$. For $j = \frac{n}{2}$ equation (1.4) follows from a micro-local identification of the kernel $\mathrm{k}[\mathsf{D}^{-2s}](x)|_{s=0}^{\mathrm{mer}}$ with the Wodzicki residue density associated to the logarithmic ΨDO $\log\mathsf{D}$. This is carried out in [Sc]. ☐

Using the isomorphism $\mathrm{End}(\mathcal{E}) \cong C(M)\otimes \mathrm{End}_{C(M)}(\mathcal{E})$, where $C(M)$ is the bundle of Clifford algebras associated to T^*M, and the isomorphism $\sigma : C(M) \longrightarrow \wedge T^*M$ (the 'symbol' map), there is an identification by σ of $\Gamma(M,\mathrm{End}(\mathcal{E}))$ with the space $\mathcal{A}(M,\mathrm{End}_{C(M)}(\mathcal{E}))$ of forms with values in $\mathrm{End}_{C(M)}(\mathcal{E})$. The filtration $C(M) = \cup_i C^i(M)$ with $C^i(M) = \sum_{k=0}^{i}\sigma^{-1}(\wedge^k T^*M)$ leads to the symbol map of degree i $\sigma_i : C^i(M) \longrightarrow \wedge^i T^*M$, $\sigma_i(a) = \sigma(a)_{[i]}$. Hence, because

$$k_{2j} \in \Gamma(M, C^{2j}(M)\otimes \mathrm{End}_{C(M)}(\mathcal{E}))\ , \tag{1.5}$$

it follows that $\sigma_{2j}(k_{2j}) \in \mathcal{A}^{2j}(M,\mathrm{End}_{C(M)}(\mathcal{E}))$

The local Atiyah-Singer index theorem is the following.

Theorem 1.2. *As elements of $\mathcal{A}(M,\mathrm{End}_{C(M)}(\mathcal{E}))$*

$$\sum_{j=0}^{n/2}\sigma_{2j}(k_{2j})\ =\ (4\pi)^{-n/2}\,\widehat{A}(R)\exp(-F^{\mathcal{E}/\mathcal{S}})\ , \tag{1.6}$$

where $R \in \mathcal{A}^2(M,\mathrm{End}(\mathcal{E}))$ is the Riemannian curvature endomorphism-valued 2-form and $\widehat{A}(R) = \det^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)$ is the $\widehat{A}$-genus characteristic class.

From Proposition 1.1, (1.6) is the assertion that

$$\sum_{j=0}^{n/2}\int_{|\xi|=1}\int_{C_0}\log(\lambda)\ \partial_\lambda^{\frac{n}{2}-j}\sigma_{2j}(\,\mathbf{r}_{2j}(x,\xi,\lambda))\ \hat{d}\lambda\ dS(\xi) = -2\pi^{n/2}\,\widehat{A}(R)\exp(-F^{\mathcal{E}/\mathcal{S}}). \tag{1.7}$$

The proof we shall give here of the local index formula proceeds by evaluating the left-side of (1.7) through by computing the symbols $\mathbf{r}_{2j}(x,\xi,\lambda)$ via a generating function.

The relative supertrace $\mathrm{str}_{\mathcal{E}/\mathcal{S}} : \mathcal{A}(M, \mathrm{End}_{C(M)}(\mathcal{E})) \longrightarrow \mathcal{A}(M)$ is the Berezin map picking out the top form component. On the other hand, there is a canonical identification of $\Gamma(M, \mathrm{End}(\mathcal{E}))$ with $\Gamma(M, \mathrm{End}(\mathcal{E}) \otimes |\wedge_M|) \subset \mathcal{A}^n(M, \mathrm{End}(\mathcal{E}))$ via multiplication by the Riemannian measure $j_g(x)dx$ and as elements of $\mathcal{A}^n(M)$

$$\mathrm{str}_{\mathcal{E}}(k(x)) = (-2i)^{n/2}\mathrm{str}_{\mathcal{E}/\mathcal{S}}(\sigma(k)) \ . \tag{1.8}$$

Since $\mathrm{str}_{E/S}$ vanishes on $C^i(T^*_xM)$ for $i < n$ the supertrace of the residues (1.4) for $j < \frac{n}{2}$ vanish. Thus, though the kernel $k[\mathsf{D}^{-2s}](x)|^{\mathrm{mer}}$ has poles at the points indicated in (1.2), the pointwise supertrace $\mathrm{str}_E(k[\mathsf{D}^{-2s}]|^{\mathrm{mer}})(x)$ is *holomorphic* on all of $\mathbb{C}$, the remaining poles in (1.2) due solely to the Gamma function.

By integrating $\mathrm{str}_E(k[\mathsf{D}^{-2s}]|^{\mathrm{mer}})(x)$ over M with respect to the Riemannian measure we obtain the (super) zeta-function $\zeta(\mathsf{D}^2, s)|^{\mathrm{mer}} := \int_M \mathrm{str}_{\mathcal{E}}(K(\mathsf{D}^{-2s}, x, x)|^{\mathrm{mer}})dx$. Since $\mathsf{D}^+\mathsf{D}^-$ and $\mathsf{D}^-\mathsf{D}^+$ have the same non-zero eigenvalues, while D^{-2s} vanishes on $\mathrm{Ker}(\mathsf{D})$, then $\zeta(\mathsf{D}^2, s)|^{\mathrm{mer}}$ vanishes identically for all s. Evaluating at $s = 0$ we obtain the (Atiyah-Bott-Seeley) index formula $\zeta(\mathsf{D}^2, 0)|^{\mathrm{mer}} = 0$ which by (1.2), and $\mathrm{ind}\,(\mathsf{D}) = \int_M \mathrm{str}_{\mathcal{E}}(\Pi_0(\mathsf{D})(x))dx$, is the formula $\int_M \mathrm{str}(k_n(x))dx_g = \mathrm{ind}\,(\mathsf{D})$, while from the vanishing of $\zeta(\mathsf{D}^2, s)|^{\mathrm{mer}}$ elsewhere $\int_M \mathrm{str}(k_{2j}(x))dx_g = 0$ for $j \neq n/2$.

From (1.8) the volume form $\mathrm{str}_E(k_n(x))\, dx_g$ is equal to the n-form $(-2i)^{n/2}\mathrm{str}_{E/S}(\sigma_n(\mathrm{k}_n))$ and so (1.6) implies the topological Atiyah-Singer Index Theorem

$$\mathrm{ind}\,(\mathsf{D}) \ = \ (2\pi i)^{-n/2} \int_M \widehat{A}(R)\ \mathrm{ch}(\mathcal{E}/\mathcal{S}) \ . \tag{1.9}$$

On the other hand, the Mellin transform

$$\Gamma(s)\,\mathrm{k}(\mathsf{D}^{-2s})(x) \ = \ \int_0^\infty t^{s-1}\mathrm{k}(\mathrm{e}^{-t\mathsf{D}^2})(x)\ dt \ , \qquad \mathrm{Re}(s) > n/2 \ , \tag{1.10}$$

shows this to be equivalent to the heat equation formulation of the local index theorem [BeGeVe]. Here, $\mathrm{k}(\mathrm{e}^{-t\mathsf{D}^2})(x) := K(\mathrm{e}^{-t\mathsf{D}^2}, x, x)\, j_g(x)^{-1}$, where $K(\mathrm{e}^{-t\mathsf{D}^2}, x, y)$ is the (smoothing) heat kernel. By a standard transition (see [?] for example) the pole structure (1.2) is equivalent to the existence of an asymptotic expansion as $t \to 0+$

$$\mathrm{k}(\mathrm{e}^{-t\mathsf{D}^2})(x) \ \sim \ \sum_{j\geq 0} t^{j-\frac{n}{2}}\ k_{2j}(x) \ . \tag{1.11}$$

Proposition 1.3. *One has*

$$k_{2j}(x) = \int_{\mathbb{R}^n}\int_{C_0} e^{-\lambda}\ \mathbf{r}_{2j}(x,\xi,\lambda)\ \hat{d}\lambda\ \hat{d}\xi \ . \tag{1.12}$$

The coefficients in (1.11) *are the same as those in* (1.2)*; in particular, for* $j = 0, 1, \ldots, \frac{n}{2}$ *the expressions on the right side of* (1.4) *and on the right-side of* (1.12) *are equal.*

Proof. The existence of the expansion (1.11) and the formula (1.12) are well known, see [**?**]. The equality of the expressions (1.4) and (1.12), follows from the general transition formulae already referenced, but this will be shown explicitly below on the way to proving Theorem 1.2. $\square$

From (1.7) it follows that the theorem to be proved can be restated as

$$\sum_{j=0}^{n/2}\int_{\mathbb{R}^n}\int_{C_0} e^{-\lambda}\,\sigma_{2j}(\,\mathbf{r}_{2j}(x,\xi,\lambda))\;\hat{d}\lambda\;\hat{d}\xi = -2\pi^{n/2}\,\widehat{A}(R)\exp(-F^{\mathcal{E}/\mathcal{S}}). \tag{1.13}$$

Note that the holomorphicity of the pointwise supertrace $\mathrm{str}_E(k[\mathsf{D}^{-2s}]|^{\mathrm{mer}})(x)$ on $\mathbb{C}$ is equivalent to the statement that the pointwise supertrace $\mathrm{str}_E(\mathrm{k}(\mathrm{e}^{-t\mathsf{D}^2}(x))$ has a limit as $t\to 0+$.

2. A RECURSION FORMULA FOR THE $\mathbf{r}_j(x,\xi,\lambda)$

In a local trivialization of $\mathcal{E}$, D^2 has the form $\sum_{0\le|\alpha|\le 2} a_\alpha(x)D_x^\alpha$, where the sum is over multi-indices $\alpha=(\alpha_1,\dots,\alpha_n)$, with $D_x^\alpha=(-i)^{|\alpha|}\partial_x^{|\alpha|}$, and so, replacing D_x^α by ξ^α, it has a matrix valued polynomial local symbol $\mathbf{a}(x,\xi)$ in $\xi^\alpha=\xi_1^{\alpha_1},\dots,\xi_n^{\alpha_n}$

$$\mathbf{a}(x,\xi) = \mathbf{a}_0(x,\xi) + \mathbf{a}_1(x,\xi) + \mathbf{a}_2(x,\xi)\ , \tag{2.1}$$

with $\mathbf{a}_j(x,\xi)=\sum_{|\alpha|=2-j} a_\alpha(x)\xi^\alpha$ homogeneous of degree $2-j$. As D^2 is of Laplace type

$$\mathbf{a}_0(x,\xi) = \sum_{i,j} g^{ij}(x)\xi_i\xi_j\,I = |\xi|^2\,I \tag{2.2}$$

where $g^{ij}(x) =< dx_k, dx_l >_x$ is the metric in the cotangent bundle. Hence the local symbol $\mathbf{r}(x,\xi,\lambda)\sim\sum_{j\ge 0}\mathbf{r}_j(x,\xi,\lambda)$ of $Q_\lambda=(\mathsf{D}^2-\lambda\mathsf{I})^{-1}$ (the object of interest in (1.7) or (1.13)) is defined for $\lambda\in\mathbb{C}\backslash\mathbb{R}^+$ and is constructed by solving

$$(\,\mathbf{a}(x,\xi) - \lambda\,\mathbf{I})\circ\,\mathbf{r}(x,\xi,\lambda) = \mathbf{I}\ , \tag{2.3}$$

with $\mathbf{I}=(I,0,0,\dots)$ the identity symbol, and where $\circ$ is the usual symbol product. Solving (2.3) recursively yields [1]

$$\mathbf{r}_0(x,\xi,\lambda) = (\,\mathbf{a}_0(x,\xi) - \lambda\,\mathbf{I})^{-1}\ , \tag{2.4}$$

$$\mathbf{r}_j(x,\xi,\lambda) = -\,\mathbf{r}_0(x,\xi,\lambda)\sum_{\substack{|\mu|+k+l=j\\ l<j}}\frac{1}{\mu!}\,\partial_\xi^\mu\,\mathbf{a}_k(x,\xi,\lambda)\,D_x^\mu\,\mathbf{r}_l(x\xi,\lambda)\ , \tag{2.5}$$

for $\lambda\notin\mathbb{R}_+$. The $\mathbf{r}_j(x,\xi,\lambda)$ are C^∞ at $\xi=0$ provided $\lambda\neq 0$, and can be made C^∞ also at $\lambda=0$ by multiplication by an excision function $\chi((|\xi|^2+|\lambda|^2)^{1/2})$, where $\chi(t)\in C^\infty(\mathbb{R}^1)$, $\chi(t)=1$ for $t\ge 1$, $\chi(t)=0$ for $t\le 1/2$.

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[1] The sum is over $\mu=(\mu_1,\dots,\mu_n), k, l$ with $\mu!=\mu_1!\dots\mu_n!, |\mu|=\mu_1+\dots+\mu_n$.

Since the principal symbol $\mathbf{a}_0(x,\xi)$ is scalar valued the factors $\mathbf{r}_0(x,\xi,\lambda)$ can be grouped together in (2.5) to see that it has the form

$$\mathbf{r}_j(x,\xi,\lambda) = \sum_{\frac{j}{2}\le k\le 2j} (|\xi|^2-\lambda)^{-k-1}\, p_{j,k}(x,\xi) \ , \tag{2.6}$$

where $p_{j,k}(x,\xi)$ is a finite product of derivatives of $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, and therefore polynomial in ξ and independent of λ.

The first formula for the $\mathbf{r}_j$ we shall give is deduced without further precision in the $\mathbf{a}_k$. By observing that in the sum (**??**) k can take values $0,1$ or 2, while $\partial^{\mu}_{\xi}\,\mathbf{a}_k(x,\xi)=0$ if $|\mu|>2$, and $\mathbf{a}_2$ is independent of ξ_j so that $\partial^{\mu}_{\xi}\,\mathbf{a}_2(x,\xi)=0$ if $|\mu|>0$, and $\partial^{|\mu|}_{\xi_k}\,\mathbf{a}_1=0$ for $|\mu|>1$, and also using (2.2), the following is easily established.

Lemma 2.1. *With the convention that* $\mathbf{r}_k := 0$ *for* $k<0$, *one has for* $j>0$

$$\begin{aligned}\mathbf{r}_0^{-1}\,\mathbf{r}_j \;&=\; \left(\sum_{k,l=1}^{n}\frac{1}{2}g^{kl}(x)\partial_{x_k}\partial_{x_l} - \mathbf{a}_2\right)\mathbf{r}_{j-2} \;+\; \mathbf{a}_1\,\mathbf{r}_{j-1}\\ &\quad -i\left(\sum_{k=1}^{n}\partial_{\xi_k}\,\mathbf{a}_1\,\partial_{x_k}\,\mathbf{r}_{j-2} + \sum_{k,l=1}^{n} 2\,g^{kl}(x)\xi_l\,\partial_{x_k}\,\mathbf{r}_{j-1}\right) \ .\end{aligned}$$

For the Dirac Laplacian, the $\mathbf{a}_j$, and hence the $\mathbf{r}_j$, can be computed using the Lichnerowicz formula which states that in local geodesic coordinates

$$\mathsf{D}^2 = \sum_{i,j} g^{ij}\left(\nabla_{\partial_i}\nabla_{\partial_j} + \sum_k \Gamma^k_{ij}\nabla_{\partial_k}\right) + \sum_{i<j} F^{\mathcal{E}/S}(e_i,e_j)c^ic^j + \frac{1}{4}r_M \ . \tag{2.7}$$

Here, the first sum on the right is the metric Laplacian, with Γ^k_{ij} the Levi-Civita coefficients, r_M the scalar curvature, and c^i the Clifford action of the orthonormal frame e^i. In geodesic coordinates one has

$$\nabla_{\partial_i} = \partial_i + \frac{1}{4}\sum_{j;k<l} R_{ijkl}x^jc^kc^l + \sum_{k<l} f_{ijkl}(x)c^kc^l + g_i(x) \ , \tag{2.8}$$

where $R_{ijkl} = (R(\partial_i,\partial_j)e_k,e_l)$ is the Riemannian curvature at x_0 in M and $f_{ijkl}(x) = O(|x|^2)$, $g_i(x) = F_{i,j}x^j + O(|x|^2)$.

Getzler rescaling [Ge, BeGeVe] reduces these local expressions reduce the recursion to the following form. Let

$$\Delta = \sum_{k=1}^{n}\partial^2_{x_k} \ , \qquad |\xi|^2 = \sum_i \xi_i^2 I \ .$$

Proposition 2.2. *At* $x_0\in M$, $\mathbf{r}_j(x_0,\xi,\lambda)$ *is computed by setting* $x=0$ *in* $r_j(x,\xi,\lambda)$ *defined for* $j\ge 2$ *by the recursion*

$$r_{-1} = 0 \ , \qquad r_0 = (|\xi|^2-\lambda I)^{-1} \ , \tag{2.9}$$

$$r_{j+1} = r_0\,(\Delta - a_2)\,r_{j-1} - r_0\,a_1\,r_j \qquad (j \geq 0)\ . \tag{2.10}$$

where

$$a_0(x,\xi) = |\xi|^2\ , \qquad a_1(x,\xi) = i\sum_{j=1}^{n/2}\left(\frac{\theta_j}{2}\right)(x_{2j-1}\,\xi_{2j} - x_{2j}\,\xi_{2j-1})\ , \tag{2.11}$$

$$a_2(x,\xi) = -\,\frac{1}{4}\sum_{j=1}^{n/2}\left(\frac{\theta_j}{2}\right)^2 (x_{2j-1}^2 + x_{2j}^2)\ . \tag{2.12}$$

3. THE POLYNOMIALS r_j

We begin by restating the problem in a more convenient form, simplifying and streamlining the notations a little, and looking at the first few polynomials r_j,

We consider three multi-variables $\alpha = (\alpha_1, \ldots, \alpha_n)$, $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$; here n is to be thought of as large, and the final results will be stable in n in a suitable sense. We consider a supplementary variable T and define polynomials $r_j = r_j(\alpha, x, \xi, T)$ recursively by the equations

$$r_{-1} = 0\,, \qquad r_0 \;=\; T\,, \qquad r_{j+1} \;=\; -a_1\,T\,r_j \;+\; T\,(\Delta - a_2)\,r_{j-1} \qquad (j \geq 0)\,, \tag{3.1}$$

where $a_1 = a_1(\alpha, x, \xi)$ and $a_2 = a_2(\alpha, x)$ are the polynomials

$$a_1 \;=\; \sum_{i=1}^{n} \alpha_i\,x_i\,\xi_i\,, \qquad a_2 \;=\; \frac{1}{4}\sum_{i=1}^{n}\alpha_i^2\,x_i^2\ . \tag{3.2}$$

In the notations of the index problem, $T = (|\xi|^2 - \lambda)^{-1}$ and the α_j are given by $\alpha_{2j-1} = -\alpha_{2j} = i\theta_j$ for $1 \leq j \leq n/2$, with n always assumed to be even, but there is no need to make these restrictions on the form of the variables, and dropping them simplifies both the formulation and the proof of the desired result.

First, it is convenient to write $r_j(T)$ explicitly as a polynomial in T. Clearly (3.1) implies

$$r_j(T) \;=\; \sum_{\mu+2\nu=j} r_{\mu,\nu}\,T^{\mu+\nu+1}\,, \tag{3.3}$$

where the polynomials $r_{\mu,\nu} = r_{\mu,\nu}(\alpha, x, \xi)$ $(\mu, \nu \geq 0)$ are now given recursively by

$$r_{\mu,\nu} \;=\; \begin{cases} 1 & \text{if } \mu = \nu = 0, \\ -a_1\,r_{\mu-1,\nu} \;+\; (\Delta - a_2)\,r_{\mu,\nu-1} & \text{otherwise,} \end{cases} \tag{3.4}$$

with the convention that $r_{*,*}$ is to be interpreted as 0 if either index is negative.

The first key remark is that all the $r_{\mu,\nu}$ are polynomials (with coefficients independent of n) in the variables
(3.5)

$$A_p \;=\; \sum_{i=1}^{n} \alpha_i^{2p}, \quad B_p \;=\; \sum_{i=1}^{n} \alpha_i^{2p} x_i^2, \quad C_p \;=\; \sum_{i=1}^{n} \alpha_i^{2p-1} x_i \xi_i, \quad D_p \;=\; \sum_{i=1}^{n} \alpha_i^{2p} \xi_i^2 \quad (p \geq 1)\,.$$

Indeed, in view of the recursive formula for the $r_{\mu,\nu}$, to prove this it is sufficient to prove that the ring $\mathcal{R} = \mathbb{Q}[A_1, A_2, \ldots, B_1, B_2, \ldots, C_1, C_2, \ldots, D_1, D_2, \ldots]$ contains a_1 and a_2 and is closed under Δ. The first assertion is obvious since $a_1 = C_1$ and $a_2 = \frac{1}{4}B_1$, and the latter follows from the easily checked formula

$$\Delta(\Phi) \;=\; 2\sum_{p=1}^{\infty} A_p \,\frac{\partial \Phi}{\partial B_p} \;+\; \sum_{p,\,q=1}^{\infty} \biggl(4B_{p+q} \,\frac{\partial^2 \Phi}{\partial B_p \partial B_q} \;+\; 4C_{p+q} \,\frac{\partial^2 \Phi}{\partial B_p \partial C_q} \;+\; D_{p+q-1} \,\frac{\partial^2 \Phi}{\partial C_p \partial C_q} \biggr)$$

for $\Phi \in \mathcal{R}$. Using this formula and the recursion (3.4), we easily compute the first values:

	$\nu = 0$	1	2	3
$\mu = 0$	T	$-\frac{1}{4}B_1$	$-\frac{1}{2}A_1 + \frac{1}{16}B_1^2$	$\frac{3}{8}A_1B_1 - \frac{1}{64}B_1^3 + \frac{1}{2}B_2$
1	$-C_1$	$\frac{1}{2}B_1C_1$	$\frac{3}{2}A_1C_1 - \frac{3}{16}B_1^2C_1 + 2C_2$	
2	C_1^2	$-\frac{3}{4}B_1C_1^2 + 2D_1$	$-3A_1C_1^2 + \frac{3}{8}B_1^2C_1^2 - 2B_1D_1 - 8C_1C_2$	
3	$-C_1^3$	$B_1C_1^3 - 8C_1D_1$		
4	C_1^4	$-\frac{5}{4}B_1C_1^4 + 20C_1^2D_1$		
5	$-C_1^5$			
6	C_1^6			

where the values are given corresponding to $j \leq 6$. Giving the full formulas for further polynomials r_j would take up too much space, but the specializations to $x = 0$ (corresponding to setting all B's and C's equal to 0), which are all we need for the local index theorem, are relatively simple and we give them here up to r_{12}:

$$\begin{aligned}
r_0 \;&=\; T\,, \qquad\qquad r_4 \;=\; -\tfrac{1}{2}A_1\,T^3 \;+\; 2D_1\,T^4\,,\\
r_8 \;&=\; (\tfrac{3}{4}A_1^2 + A_2)\,T^5 \;-\; (10A_1D_1 \;+\; 16D_2)\,T^6 \;+\; 40D_1^2\,T^7\,,\\
r_{12} \;&=\; -(\tfrac{15}{8}A_1^3 + \tfrac{15}{2}A_1A_2 + 8A_3)\,T^7 \;+\; (\tfrac{105}{2}A_1^2D_1 + 168A_1D_2 + 70A_2D_1 + 272D_3)\,T^8\\
&\quad\; -(560A_1D_1^2 + 1792D_1D_2)\,T^9 \;+\; 2240D_1^3\,T^{10}\,.
\end{aligned}$$

It is amusing to check (1.7) for the first few values using these explicit formulas for the r_j and the identities

$$(3.6) \qquad -\int_{C_0} \log(\lambda)\; \partial_\lambda^k T^{h+1}\; \hat{d}\lambda = \frac{(k+h-1)!}{h!}\,, \quad (h,\,k \geq 0,\; h+k \geq 1)\,,$$

$$\int_{S^{4m-1}} dS(\xi) \;=\; \frac{2\,\pi^{2m}}{(2m-1)!}\,, \qquad \int_{S^{4m-1}} \xi_i^2\; dS(\xi) \;=\; \frac{\pi^{2m}}{(2m)!}\,. \tag{3.7}$$

When $j = 0$, since $\widehat{A}(R)_{[0]} = 1$ the right-side of (1.7) is $-2\pi^{2m}\,\widehat{A}(R)_{[0]} = -2\pi^{2m}$, while the left-side is then

$$\int_{S^{4m-1}} \int_{C_0} \log(\lambda)\;\partial_\lambda^{2m} T\;\hat{d}\lambda\; dS(\xi) = \int_{S^{4m-1}} dS(\xi)\;\int_{C_0} \log(\lambda)\;\partial_\lambda^{2m} T\;\hat{d}\lambda = -2\pi^{2m}\,,$$

When $j = 1$, the right-side of (1.7) is $-2\pi^{2m}\,\widehat{A}(R)_{[4]} = -\left(\frac{\pi^{2m}}{12}\right)\;\sum_{i=1}^{2m} \theta_i^2$, while we computed above that

$$r_4(0,\xi,\lambda) \;=\; T^3 \sum_{k=1}^{2m} \left(\frac{\theta_k}{2}\right)^2 \;-\; 2T^4 \sum_{k=1}^{2m} \left(\frac{\theta_k}{2}\right)^2\;(\xi_{2k-1}^2 + \xi_{2k}^2)$$

so that the left-side is then

(3.8)

$$\int_{S^{4m-1}} \int_{C_0} \log(\lambda)\;\partial_\lambda^{2m-2} r_4\;\hat{d}\lambda\; dS(\xi) = \int_{S^{4m-1}} dS(\xi)\;\left(\int_{C_0} \log(\lambda)\;\partial_\lambda^{2m-2} T^3\;\hat{d}\lambda\right)\;\sum_{k=1}^{2m} \left(\frac{\theta_k}{2}\right)^2$$

$$-\,2\left(\int_{C_0} \log(\lambda)\;\partial_\lambda^{2m-2} T^4\;\;\hat{d}\lambda\right)\;\left(\int_{S^{4m-1}} \xi_{2k-1}^2\; dS(\xi) + \int_{S^{4m-1}} \xi_{2k}^2\; dS(\xi)\right)\;\sum_{k=1}^{2m} \left(\frac{\theta_k}{2}\right)^2$$

which using (3.6) and (3.7) is easily seen to be equal to $-\left(\frac{\pi^{2m}}{12}\right)\;\sum_{i=1}^{2m} \theta_i^2$.

The computability of the local index theorem makes the theorem widely useful.

4. THE AVERAGING OPERATOR A_n AND THE RESIDUE OPERATOR R_k

For the general case we proceed as follows. We introduce an *averaging operator* A_n on polynomials in ξ, defined by

$$\mathsf{A}_n\big[F\big] \;=\; \frac{1}{2\pi^{n/2}} \int_{|\xi|=1} F(\xi)\,dS(\xi) \;=\; \frac{1}{\Gamma(n/2)}\,\big(\text{average of } F(\xi) \text{ on } S^{n-1} \subset \mathbb{R}^n\big)\,, \tag{4.1}$$

and a *residue operator* R_k on polynomials in T without constant term, defined by

$$\mathsf{R}_k\big[P(T)\big] \;=\; -\int_{C_0} \log(\lambda)\;\partial_\lambda^k P\big(\frac{1}{1-\lambda}\big)\;\hat{d}\lambda \;\;(k \geq 0)\,. \tag{4.2}$$

Since R_k can be described explicitly by its values on monomials given in equation (3.6) we may simply take that as our definition of the operator R_k.

The identity which has to be proved can then be formulated as

$$\mathsf{A}_n \mathsf{R}_{\frac{1}{2}(n-j)}\big[r_j(\alpha,0,\xi,T)\big] \;=\; \widehat{A}(\alpha)_{[j/2]}\,, \tag{4.3}$$

where $\widehat{A}(\alpha) = \prod_{i=1}^n \left(\frac{\alpha_i}{\sinh\alpha_i}\right)^{1/2}$ and $\widehat{A}(\alpha)_{[j/2]}$ denotes the component of $\widehat{A}(\alpha)$ of degree $j/2$. For the application to the index theorem, only the case when $n \geq j \geq 0$, $n \equiv j \equiv 0$

(mod 4), and $\alpha_{2i-1} = -\alpha_{2i}$ is needed, but we will prove (4.3) without any of these restrictions. (We can even allow odd n if we interpret $(h+k-1)!$ in (3.6) as $\Gamma(h+k)$. Of course, j must always be 0 mod 4 since otherwise both sides of (4.3) are 0.)

The basic observation is that each polynomial $r_{\mu,\nu}$ is homogeneous in the ξ variables (of degree μ), and that for homogeneous polynomials $P(\xi)$ there is an easy way to compute the average $\mathsf{A}_n\big[P\big]$. Namely, for any polynomial $P(\xi)$ in $\mathbb{R}^n$ we define the "radial Laplace transform" $\mathsf{L}_n[P](z)$ by

$$\mathsf{L}_n\big[P(\xi)\big](z) \;=\; \int_{\mathbb{R}^n} P(\xi)\, e^{-|\xi|^2 z}\, d\xi\,.$$

This is a priori much easier to compute than A_n since the domain of integral is Euclidean space, which is a product, rather than the unit sphere, which is not. Thus for monomials with even exponents (monomials having at least one odd exponent clearly give 0) we find immediately

$$\mathsf{L}_n\big[\xi_1^{2s_1}\cdots\xi_n^{2s_n}\big](z) \;=\; \prod_{i=1}^{n} \mathsf{L}_1\big[\xi^{2s_i}\big](z) \;=\; \frac{\Gamma(s_1+\frac12)\cdots\Gamma(s_n+\frac12)}{z^{s_1+\cdots+s_n+n/2}}\,. \tag{4.4}$$

On the other hand, if P is homogeneous of degree μ, then its integral over the sphere of radius r in $\mathbb{R}^n$ equals $2\pi^{n/2}\mathsf{A}_n\big[P\big]\, r^{n+\mu-1}$, so we have

$$\mathsf{L}_n\big[P(\xi)\big](z) \;=\; 2\pi^{n/2}\mathsf{A}_n\big[P\big]\int_0^\infty r^{n+\mu-1}\, e^{-zr^2}\, dr \;=\; \pi^{n/2}\,\Gamma\Big(\frac{n+\mu}{2}\Big)\,\mathsf{A}_n\big[P\big]\, z^{-(n+\mu)/2}\,. \tag{4.5}$$

Note that by comparing formulas (4.4) and (4.5) we immediately obtain the evaluation

$$\mathsf{A}_n\big[\xi_1^{2s_1}\cdots\xi_n^{2s_n}\big] = \frac{\Gamma(s_1+\frac12)\cdots\Gamma(s_n+\frac12)}{\pi^{n/2}\,\Gamma(s_1+\cdots+s_n+n/2)} \tag{4.6}$$

of the averaging operator on arbitrary monomials (and hence on arbitrary polynomials), going well beyond the special evaluations (3.7). However, we will only use the relation (4.5) , since we are going to end up applying the operators A_n and L_n to Gaussian (heat) generating functions rather than to functions which are expressed explicitly as linear combinations of monomials.

If we apply the general formula (4.5) to the polynomial $r_{\mu,\nu}(\alpha,x,\xi)$, which is homogeneous of degree μ in ξ (as one sees immediately from (3.4) by induction, since Δ and a_2 do not involve ξ), and combine the result with the definition (3.6) of the operator R_k, we find

$$\mathsf{A}_n\mathsf{R}_{\frac12(n-j)}\big[r_{\mu,\nu}(\alpha,x,\xi)\,T^{\mu+\nu+1}\big] \;=\; \frac{\pi^{-n/2}}{(\mu+\nu)!}\, z^{(n+\mu)/2}\,\mathsf{L}_n\big[r_{\mu,\nu}(\alpha,x,\xi)\big](z)\,, \tag{4.7}$$

for $\mu,\,\nu \ge 0$ with $\mu+2\nu = j$ and for any z. The main point here, and the reason why the whole proof will work, is that the gamma-factor $\Gamma\big(\frac{n+\nu}{2}\big)$ in (4.5) has cancelled the factor $(h+k-1)!$ from (3.6), leaving only a factorial independent of n in (4.7).

In particular, (4.7) yields the identification of the right side of (1.4) and (1.12) asserted in Proposition 1.3.

5. The Generating Function

Comparing the left-hand side of the desired identity (4.3) with the equation (4.7) just obtained, we see that we need to apply L_n to expressions of the form $\sum_{\mu+2\nu=j} r_{\mu,\nu}(\alpha,x,\xi)z^\nu/(\mu+\nu)!$ for each j. This suggests that we should introduce the generating function

$$R(\alpha,x,\xi;X,Y) \;=\; \sum_{\mu,\,\nu=0}^{\infty} r_{\mu,\nu}(\alpha,x,\xi)\,\frac{X^\mu Y^\nu}{(\mu+\nu)!}\,. \tag{5.1}$$

The first few terms of this power series are given by

$$R(\alpha,x,\xi;X,Y) \;=\; 1 \,-\, C_1X \,-\, \frac{B_1}{4}Y \,+\, \frac{C_1^2}{2}X^2 \,+\, \frac{B_1C_1}{4}XY \,+\, \big(\frac{B_1^2}{32}-\frac{A_2}{4}\big)Y^2 \,+\,\cdots\,,$$

which looks very much like the exponential of something much simpler, and this impression is reinforced by the observation that the right-hand side of the identity (4.3) which we want to prove is a product over $i=1,\dots,n$, whereas the expressions (3.2) or (3.5) are sums over the same set. This suggests that we should look at the *logarithm* of the generating function R. When we do this for the first few terms, using the values calculated in §3, we discover that indeed there is a huge simplification: while R itself contains all monomials in the infinitely many variables (3.5), its logarithm is simply a *linear* combination of these variables, and what's more, of a very special form. Explicitly, the expansion of $\log R$ begins

$$\begin{aligned}\log\,R(\alpha,x,\xi;X,Y) \;&=\; -\big(\tfrac14B_1Y+C_1X+\tfrac14A_1Y^2\big)\\ &\quad+\tfrac13\left(\tfrac14B_2Y^3+C_2XY^2+D_1X^2Y+\tfrac18A_2Y^4\right)\\ &\quad-\tfrac{2}{15}\left(\tfrac14B_3Y^5+C_3XY^4+D_2X^2Y^3+\tfrac1{12}A_3Y^6\right)\\ &\quad+\tfrac{17}{315}\left(\tfrac14B_4Y^7+C_4XY^6+D_3X^2Y^5+\tfrac1{16}A_4Y^8\right)\,+\,\cdots\end{aligned}$$

The numerical coefficients $\beta_1=1$, $\beta_2=-\frac13$, $\beta_3=\frac2{15}$, $\beta_4=-\frac{17}{315}$, … are easily recognized as the coefficients of the generating functions $\sum_{p=1}^\infty \beta_p\,x^{2p-1} \;=\; \tanh x$ and $\sum_{p=1}^\infty \beta_p\,\frac{x^{2p}}{2p} \;=\; \log(\cosh x)$. Based on computer calculations we are therefore led to guess the following result.

Proposition 5.1. *The generating function* (5.1) *is given by*

$$R(\alpha,x,\xi;X,Y) \;=\prod_{i=1}^{n}\frac{1}{\sqrt{\cosh(\alpha_iY)}}\,\exp\left[-\frac{\tanh(\alpha_iY)}{\alpha_i}\,\left(\frac12\alpha_ix_i\,+\,\frac{X}{Y}\xi_i\right)^2\,+\,\frac{X^2}{Y}\,\xi_i^2\right]. \tag{5.2}$$

Proof. To prove equation (5.2), we observe that the initial conditions and recursion given in equation (3.4) are equivalent to the partial differential equation

$$X\,\frac{\partial R}{\partial X} \,+\, Y\,\frac{\partial R}{\partial Y} \;=\; -X\,a_1\,R \,+\, Y\,(\Delta - a_2)\,R \tag{5.3}$$

for the generating function (5.1) together with the boundary condition $R(\alpha, x, \xi; 0, 0) = 1$. If we write $R = e^F$, then (5.3) is transformed into a non-linear partial differential equation for the logarithm F:

$$X\,\frac{\partial F}{\partial X} \,+\, Y\,\frac{\partial F}{\partial Y} \;=\; -X\,a_1 \,-\, Y\,a_2 \,+\, Y\,\sum_{i=1}^{n}\biggl[\frac{\partial^2 F}{\partial x_i^2} \,+\, \Bigl(\frac{\partial F}{\partial x_i}\Bigr)^2\biggr]\,. \tag{5.4}$$

Now we try choosing for F a function of the form

$$F(\alpha, x, \xi; X, Y) \;=\; \sum_{i=1}^{n}\biggl(A(\alpha_i Y) + B(\alpha_i Y)\,\frac{x_i^2}{Y} + C(\alpha_i Y)\,\frac{x_i \xi_i X}{Y} + D(\alpha_i Y)\,\frac{\xi_i^2 X^2}{Y}\biggr) \tag{5.5}$$

(note that this is a function of the variables αY, $xY^{-1/2}$, and $\xi X Y^{-1/2}$, in accordance with the remark at the end of §5), where $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are power series in one variable satisfying

$$A(t) = \mathrm{O}(t)\,, \qquad B(t) = \mathrm{O}(t^2), \qquad C(t) = \mathrm{O}(t), \qquad D(t) = \mathrm{O}(t^2) \tag{5.6}$$

in order to ensure that $F = \mathrm{O}(X) + \mathrm{O}(Y)$ as X and Y go to 0. The differential equation (5.4) is satisfied if and only if the four functions A, B, C and D satisfy

$$\begin{aligned} tA'(t) \;&=\; 2B(t)\,, \qquad & tB'(t) \,-\, B(t) \;&=\; -\frac{1}{4}\,t^2 \,+\, 4B(t)^2\,, \\ tC'(t) \;&=\; -t \,+\, 4B(t)C(t)\,, \qquad & tD'(t) \,+\, D(t) \;&=\; C(t)^2\,. \end{aligned} \tag{5.7}$$

A solution of the system of equations (5.7), and in fact the unique power series solution subject to the growth conditions (5.6), is given by

$$A(t) = -\frac{1}{2}\log(\cosh t), \quad B(t) = -\frac{1}{4}\,t\,\tanh t, \quad C(t) = -\tanh t, \quad D(t) = 1 - \frac{\tanh t}{t}.$$

Substituting these formulas into (5.5) and exponentiating, we obtain precisely the formula given in equation (5.2). This completes the proof of Proposition 5.1. □

6. PROOF OF THE MAIN IDENTITY

In Proposition 5.1, the variables X and Y can be absorbed into the other variables and have only been included to keep track of the degrees. Specifically, one checks by induction from (3.4) that if $r_{\mu,\nu}$ contains a monomial of (total) degree a in the α's, b in the x's, and c

in the ξ's, then $\mu = c$ and $\nu = a-(b+c)/2$, so $R(\alpha, x, \xi; X, Y) = R(\alpha Y,\, xY^{-1/2},\, \xi X Y^{-1/2})$. Hence (5.2) can be stated equivalently in the simpler form

$$\sum_{\mu,\,\nu\geq 0} \frac{r_{\mu,\nu}(\alpha, x, \xi)}{(\mu+\nu)!} \;=\; \Bigg(\prod_{i=1}^{n} \frac{1}{\cosh \alpha_i}\Bigg)^{1/2} \exp\Bigg(|\xi|^2 - \sum_{i=1}^{n} \frac{\tanh \alpha_i}{\alpha_i}\,\big(\xi_i + \frac{1}{2}\alpha_i x_i\big)^2\Bigg)\,. \tag{6.1}$$

With the specializations to the index problem, $T = (|\xi|^2 - \lambda)^{-1}$, $\alpha_{2j-1} = -\alpha_{2j} = i\theta_j$ for $1 \leq j \leq n/2$, and using the formula $\int_{\mathbb{R}} e^{-\lambda\xi^2}\, d\xi = (\pi/\lambda)^{1/2}$, we use this to compute

$$\sum_{j=0}^{n/2} \int_{\mathbb{R}^n} \int_{C_0} e^{-\lambda}\; \sigma_{2j}(\,\mathbf{r}_{2j}(x, \xi, \lambda))\; \hat{d}\lambda\; \hat{d}\xi$$

$$\begin{aligned}
&= \int_{\mathbb{R}^n} \sum_{\mu,\nu\geq 0} r_{\mu,\nu}(x, \xi, \lambda) \int_{C_0} e^{-\lambda}\; (|\xi|^2 - \lambda)^{-(\mu+\nu+1)}\; \hat{d}\lambda\; \hat{d}\xi \\
&= \int_{\mathbb{R}^n} \sum_{\mu,\nu\geq 0} \frac{r_{\mu,\nu}(x, \xi, \lambda)}{(\mu+\nu)!} \int_{C_0} e^{-\lambda}\; (|\xi|^2 - \lambda)^{-1}\; \hat{d}\lambda\; \hat{d}\xi \\
&= \int_{\mathbb{R}^n} \sum_{\mu,\nu\geq 0} \frac{r_{\mu,\nu}(x, \xi, \lambda)}{(\mu+\nu)!}\; e^{-|\xi|^2}\; \hat{d}\xi \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{n} (\cosh \alpha_i)^{-1/2}\; \exp\Bigg(-\sum_{i=1}^{n} \frac{\tanh \alpha_i}{\alpha_i}\, \big(\xi_i + \frac{1}{2}\alpha_i x_i\big)^2\Bigg)\; \hat{d}\xi \\
&= -2\pi^{n/2}\, \widehat{A}(R)\,.
\end{aligned}$$

Interestingly, note that this formula holds independently of x.

7. GENERALIZATIONS

The above calculations can be performed without making any of the specializations to $X = Y = 1$, $x = 0$ or $z = 1$. Indeed, from (4.7) we find

$$\begin{aligned}
(z/\pi)^{n/2}\, \mathsf{L}_n\big[R(\alpha, x, \xi; X, Y)\big](z) \;&=\; (z/\pi)^{n/2} \sum_{j=0}^{\infty} \sum_{\mu+2\nu=j} \mathsf{L}_n\big[r_{\mu,\nu}(\alpha, x, \xi)\big](z)\, \frac{X^{\mu} Y^{\nu}}{(\mu+\nu)!} \\
&=\; \sum_{j=0}^{\infty} \sum_{\mu+2\nu=j} \mathsf{A}_n \mathsf{R}_{\frac{1}{2}(n-j)}\big[r_{\mu,\nu}(\alpha, x, \xi)\, T^{\mu+\nu+1}\big]\; X^{\mu}\, Y^{\nu}\, z^{-\mu/2} \\
&=\; \sum_{j=0}^{\infty} \mathsf{A}_n \mathsf{R}_{\frac{1}{2}(n-j)}\big[r_j(\alpha, x, \frac{X\xi}{\sqrt{Yz}}, T)\big]\; Y^{j/2} \\
&=\; \sum_{j=0}^{\infty} \mathsf{A}_n \mathsf{R}_{\frac{1}{2}(n-j)}\big[r_j(\alpha Y, \frac{x}{\sqrt{Y}}, \frac{X\xi}{\sqrt{Yz}}, T)\big]\,,
\end{aligned} \tag{7.1}$$

while (5.2) gives

$$(7.2)\qquad (z/\pi)^{n/2}\,\mathsf{L}_n\big[R(\alpha,x,\xi;X,Y)\big](z) \;=\; \prod_{i=1}^{n} A\Big(\alpha_i Y,\, \frac{\alpha_i x_i\sqrt{Y}}{2},\, \frac{X^2}{Yz}\Big)$$

with $A(\alpha,\beta,t)$ defined by

$$(7.3)\qquad A(\alpha,\beta,t) \;=\; \frac{1}{\sqrt{\pi t\cosh\alpha}}\int_{-\infty}^{\infty}\exp\Big(-\frac{u^2}{t}\,+\,u^2\,-\,\frac{\tanh\alpha}{\alpha}(u+\beta)^2\Big)\,du\,.$$

For (4.3) we needed only the double specialization of $A(\alpha,\beta,t)$ to $\beta=0$, $t=1$. Making only one of these specializations leads to equally simple formulae, namely

$$(7.4)\qquad A(\alpha,\beta,1) \;=\; \frac{1}{\sqrt{\pi\cosh\alpha}}\int_{-\infty}^{\infty} e^{-\frac{\tanh\alpha}{\alpha}(u+\beta)^2}\,du \;=\; \sqrt{\frac{\alpha}{\sinh\alpha}}$$

(independent of β!) and
(7.5)

$$A(\alpha,0,t) \;=\; \frac{1}{\sqrt{\pi t\cosh\alpha}}\int_{-\infty}^{\infty} e^{-(\frac{1}{t}-1+\frac{\tanh\alpha}{\alpha})u^2}\,du \;=\; \Big(\big(1-t\big)\cosh\alpha\,+\,t\,\frac{\sinh\alpha}{\alpha}\Big)^{-1/2},$$

but we can just as easily calculate the general case, using the standard identity $\int_{-\infty}^{\infty}e^{-At^2+Bt}\,dt = (\pi/A)^{1/2}e^{B^2/4A}$, obtaining

$$(7.6)\;\; A(\alpha,\beta,t) \;=\; \Big(\big(1-t\big)\cosh\alpha\,+\,t\,\frac{\sinh\alpha}{\alpha}\Big)^{-1/2}\,\exp\Big(-\frac{(1-t)\,\beta^2\,\sinh\alpha}{(1-t)\,\alpha\,\cosh\alpha\,+\,\tanh\alpha}\Big).$$

This in turn has two further nice special cases:

$$(7.7)\qquad A(\alpha,\beta,0) \;=\; \frac{1}{\sqrt{\cosh\alpha}}\,\exp\Big(-\beta^2\frac{\sinh\alpha}{\alpha}\Big)$$

and

$$(7.8)\qquad A(0,\beta,t) \;=\; \exp\big(-(1-t)\,\beta^2\big)\,.$$

By comparing (7.1) and (7.2) and using (7.6), we now obtain the following generalization of the original identity (4.3) .

Theorem 7.1. *For any (even) integer $j\ge 0$ and any h, the quantity $\mathsf{A}_n\mathsf{R}_{\frac{1}{2}(n-j)}\big[r_j(\alpha,x,h\xi,T)\big]$ is equal to the homogeneous part of degree j of $\prod_{i=1}^{n}A(\alpha_i,\frac{1}{2}\alpha_i x_i,h^2)$, where $A(\alpha,\beta,t)$ is given by* (7.6) *and where the variables α_i, x_i and h have degrees 2, -1 and 0, respectively.*

It would be interesting to check whether this result, or any of its special cases corresponding to equations (7.3), (7.5), (7.7) or (7.8), has an interpretation in the context of the Atiyah-Singer index theorem. In particular, (7.3) is a possible candidate for this since the corresponding version of the theorem is that $\mathsf{A}_n\mathsf{R}_{\frac{1}{2}(n-j)}\big[r_j(\alpha,x,\xi,T)\big]$ equals $\widehat{A}(\alpha)_{[j/2]}$ for any x, not just for $x=0$.

APPENDIX A. THE RECURRENCE RELATION

We have

$$-\mathbf{r}_0^{-1}\ \mathbf{r}_j = \mathbf{a}_2\,\mathbf{r}_{j-2} + \mathbf{a}_1\,\mathbf{r}_{j-1} - \sum_{k=1}^{n} \partial_{x_k}^2\,\mathbf{r}_{j-2} + \sum_{k=1}^{n} \partial_{\xi_k}\,\mathbf{a}_1\,D_{x_k}\,\mathbf{r}_{j-2} + \sum_{k=1}^{n} 2\xi_k\,D_{x_k}\,\mathbf{r}_{j-1}\ . \tag{A.1}$$

Let

$$T_1 = \sum_{k=1}^{n} \partial_{\xi_k}(\,\mathbf{a}_1)\,D_{x_k}\ , \qquad T_2 = \sum_{k=1}^{n} 2\xi_k\,D_{x_k}\ . \tag{A.2}$$

We have to prove that

$$T_1\ \mathbf{r}_{j-2}\ +\ T_2\,\mathbf{r}_{j-1}\ =\ 0\ , \qquad j \geq 2\ . \tag{A.3}$$

It is easily verified that

$$T_1\,\mathbf{a}_2\ =\ 0\ . \tag{A.4}$$

$$T_2\,\mathbf{a}_1\ =\ 0\ . \tag{A.5}$$

$$T_1\,\mathbf{a}_1 + T_2\,\mathbf{a}_2\ =\ 0\ . \tag{A.6}$$

The proof of (A.3) now proceeds by induction. From (**??**)

$$\mathbf{r}_1\ =\ -\,\mathbf{r}_0^2\ \mathbf{a}_1\ ,$$

and so, since $\mathbf{r}_0$ is independent of the x_i, by (A.5)

$$T_2\,\mathbf{r}_1\ =\ -\,\mathbf{r}_0^2\ T_2\,\mathbf{a}_1 = 0\ . \tag{A.7}$$

Since clearly $T_1\,\mathbf{r}_0 = 0$ then (A.3) holds for $j = 2$. Assume, then, (A.3) holds for $j \leq m$. Then, in particular, it holds for $j = m-1$ and $j = m$, so that from (A.1)

$$-\,\mathbf{r}_0^{-1}\ \mathbf{r}_{m-1} = \mathbf{a}_2\,\mathbf{r}_{m-3} + \mathbf{a}_1\,\mathbf{r}_{m-2} - \sum_{k=1}^{n} \partial_{x_k}^2\,\mathbf{r}_{m-3}\ , \tag{A.8}$$

$$-\,\mathbf{r}_0^{-1}\ \mathbf{r}_{m} = \mathbf{a}_2\,\mathbf{r}_{m-2} + \mathbf{a}_1\,\mathbf{r}_{m-1} - \sum_{k=1}^{n} \partial_{x_k}^2\,\mathbf{r}_{m-2}\ . \tag{A.9}$$

Hence

$$-\,\mathbf{r}_0^{-1}(T_1\,\mathbf{r}_{m-1} + T_2\,\mathbf{r}_m) = \tag{A.10}$$

$$T_1(\,\mathbf{a}_2\,\mathbf{r}_{m-3}) + T_1(\,\mathbf{a}_1\,\mathbf{r}_{m-2}) + T_2(\,\mathbf{a}_2\,\mathbf{r}_{m-2}) + T_2(\,\mathbf{a}_1\,\mathbf{r}_{m-1})$$

$$- \left(T_1 \sum_{k=1}^{n} \partial_{x_k}^2 \, \mathbf{r}_{m-3} \; + \; T_2 \sum_{k=1}^{n} \partial_{x_k}^2 \, \mathbf{r}_{m-2} \right) \; .$$

Since the coefficients of D_{x_k} in T_1, T_2 are at most linear in x_j then

$$[T_1 \; , \; \partial_{x_k}^2 \;] = 0 \; , \qquad [T_2 \; , \; \partial_{x_k}^2 \;] = 0 \; ,$$

and hence the final (bracketed) term on the right-side of (A.10) is equal to

$$-\sum_{k=1}^{n} \partial_{x_k}^2 \, (T_1 \, \mathbf{r}_{m-3} \; + \; T_2 \, \mathbf{r}_{m-2}) \tag{A.11}$$

and this is zero by the inductive hypothesis for $j = m-1$.

Since T_1, T_2 are are first-order linear differential operators

$$T_1(\, \mathbf{a}_2 \, \mathbf{r}_{m-3}) = T_1(\, \mathbf{a}_2) \; \mathbf{r}_{m-3} + \mathbf{a}_2 \, T_1(\, \mathbf{r}_{m-3}) \; , \tag{A.12}$$

$$T_2(\, \mathbf{a}_2 \, \mathbf{r}_{m-2}) = T_2(\, \mathbf{a}_2) \; \mathbf{r}_{m-2} + \mathbf{a}_2 \, T_2(\, \mathbf{r}_{m-2}) \; , \tag{A.13}$$

$$T_1(\, \mathbf{a}_1 \, \mathbf{r}_{m-2}) = T_1(\, \mathbf{a}_1) \; \mathbf{r}_{m-2} + \mathbf{a}_1 \, T_1(\, \mathbf{r}_{m-2}) \; , \tag{A.14}$$

$$T_2(\, \mathbf{a}_1 \, \mathbf{r}_{m-1}) = T_2(\, \mathbf{a}_1) \; \mathbf{r}_{m-1} + \mathbf{a}_1 \, T_2(\, \mathbf{r}_{m-1}) \; . \tag{A.15}$$

The sum of the second terms on the right-side of (A.12) and (A.13) is zero by the inductive hypothesis for $j = m-1$, and likewise the second terms of (A.14) and (A.15) sum to zero by the inductive hypothesis for $j = m$. Finally, from Lemma **??** the first terms on the right-side of each equation sum to zero.

Hence (A.10) vanishes, establishing the proposition.

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