DETERMINANTS OF ELLIPTIC BOUNDARY PROBLEMS IN QUANTUM FIELD THEORY

SIMON G. SCOTT AND KRZYSZTOF P. WOJCIECHOWSKI

Abstract. We review some recent advances in understanding the zeta-determinant of an elliptic boundary value problem for the Dirac operator. We discuss a recent adiabatic pasting formula for the determinant with respect to a partition of the underlying manifold, and outline some of the applications to geometric anomalies.

Introduction

The purpose of this exposition is to review recent results of the authors and collaborators in understanding the nature of the ζ-determinant and some applications of these ideas to *Quantum Field Theory*. This report covers some of the results in the works [21, 25, 31, 32, 34, 35, 41]. We refer to those papers for proofs and precise statements. Here we concentrate on ideas behind the results and we also discuss some examples.

The second author wants to express his deepest gratitutde to the organizers of the *Workshop on Noncommutative Differential Geometry and its Application to Physics* in Shonan-Kokusaimura for the extreme hospitality and for the creative atmosphere of the meeting, which lead to this paper.

Zeta function regularization and determinant line bundles have come to represent something of a Zeitgeist for the intense interchange of ideas in recent years between geometric analysis, topology and theoretical physics. On the physical side, the ζ-regularized determinant of an elliptic differential operator over a closed manifold M defines a putative value for path integral computations in string theory and QFT, while on the mathematical side, it is a delicate highly non-local analytic object encoding fundamental spectral and geometric information about the underlying manifold, bundle and elliptic operator. In either view point, ζ-determinants are somewhat mysterious and their computation is of great interest. Non-locality means that such computations are extremely hard on all but the simplest symmetric spaces, and a general elliptic operator will not even have a canonically defined ζ-determinant.

Quantum theory, on the other hand, tells us that anyway the meaningful quantity to think about is rather the democratic 'sum over all histories'. Mathematically one considers a submersion of Riemannian spin manifolds $\pi : Z \to B$ with fibre diffeomorphic to M, defining a family of twisted Dirac operators $\mathbb{D} = \{\mathcal{D}_b : b \in B\} : \mathcal{H}^+ \to \mathcal{H}^-$. Here

$\mathcal{H}^{\pm}$ are the infinite-dimensional Frechet bundles with fibre $\mathcal{H}^{\pm}_b = C^{\infty}(M_b; S^{\pm}_b \otimes E_b)$, with S the vertical spinor bundle and E a vertical Hermitian coefficient bundle with compatible connection. The family $\mathbb{D}$ defines two families of finite-dimensional vector spaces $Ker\ \mathbb{D} = \cup_b Ker\ \mathcal{D}_b$ and $Cok\ \mathbb{D} = \cup_b Cok\ \mathcal{D}_b$ associated to which one has an index bundle element $Ind\ \mathbb{D} = Ker\ \mathbb{D} - Cok\ \mathbb{D}$ and determinant line bundle $DET\ \mathbb{D}$ over B with fibre canonically isomorphic to the complex line $DetInd\mathcal{D}_b := \wedge^{max}(Ker\ \mathcal{D}_b)^* \otimes \wedge^{max} Cok\ \mathcal{D}_b$.

The prototypical physical problem is for Yang-Mills theory with gauge fields associated to some compact Lie group G coupled to Fermions over a compact even-dimensional Riemannian spin manifold M. A gauge field is represented by a connection on a principal G-bundle L over M, and a chiral Fermion as a section of a Hermitian bundle $S^{\pm} \otimes E$, where E is an associated G-bundle to L. The aim is to evaluate the (heuristic) path integral

$$Z(A,\psi) = \int_B e^{-S(A,\psi)}\ dA d\psi d\overline{\psi}, \tag{1}$$

with respect to the action $S(A,\psi) = \int_M \overline{\psi}\mathcal{D}_A\psi + \int_M \|F_A\|^2$, where $\mathcal{D}_A$ is the Dirac operator, and the parameter space B is the space of connections cross the space B of sections of $S^+ \otimes E$. *Anomalies* arise due to the first term in the action coupling Fermions to the gauge Bosons. Indeed, one is immediately faced with a number of difficulties. First of all, the measure in (1) is entirely fictitious. Second the action $S(A,\psi)$ is invariant under the action of the group $\mathcal{G}$ of gauge transformations of P, and so the integral should really be evaluated over the moduli space $\mathcal{M} = B/\mathcal{G}$, which with appropriate constraints may be a respectable finite-dimensional manifold.

The initial method of dealing with the first problem is to ignore it entirely, and pretend we are evaluating a finite-dimensional Gaussian integral. That is, we suppose the ψ are Fermionic variables in a finite-dimensional vector space and evaluate according to elementary formulae for finite-dimensional Fermionic integrals

$$Z(A,\psi) = \int_{\mathcal{A}} e^{\int_M \|F_A\|^2} \cdot det\mathcal{D}_A\ dA. \tag{2}$$

To proceed we must understand the integrand as a function and that means we must understand $det\mathcal{D}_A$ as a function. However, *a priori* $det\mathcal{D}_A$ arises as a section of the complex line bundle $DET\ \mathbb{D}$ and to realize it as a function over B therefore requires a trivialization of $DET\ \mathbb{D}$. Using, for example, ζ-function regularization one defines a metric and compatible connection on the bundle whose curvature, holonomy, and Chern class represent respectively local, global and topological obstructions to the existence of of a regularized determinant function on B. These quantities may be understood as anomalies for an appropriate QFT, while mathematically they are local, global and topological expressions for the degree 2 part of the family's index [3, 13, 28].

The ζ-function metric is defined using the determinant of the Dirac-Laplacian $\Delta_b = \mathcal{D}_b^*\mathcal{D}_b$. Since Δ_b is a self-adjoint elliptic operator it has a discrete spectral resolution $\{\lambda \in \mathbb{R}, \phi_\lambda\}$, and so formally its determinant is the infinite product $\prod\ \lambda$, which is undefined.

To define a regularized product one uses a spectral ζ-function. This is based upon a remarkable generalization of the method by which the classical Riemann ζ-function, defined initially only in a half-plane, is extended meromorphically to the whole complex plane, and the identity for a finite-rank operator $A \in \mathrm{End}(V)$

$$det\ A = e^{-\zeta_A^{'}(0)}, \tag{3}$$

where s a complex number $\zeta_A(s) := \mathrm{tr}\ A^{-s}$ and $\zeta_A^{'}(0) := \frac{d}{ds}\{\mathrm{Tr}\ A^{-s}\}|_{s=0}$.

For a general Laplacian, $det_\zeta(\Delta)$ is defined in a similar way. In Seeley's paper [36] it was explained that by standard operator norm estimates Δ^{-s} is trace-class for $Re(s) > dim(M)/2$, and so in this half-plane $\zeta_\Delta(s) := \mathrm{Tr}\,(\Delta^{-s})$ is a well-defined holomorphic function and, moreover, one has via the Mellin transform, the identity

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}\ e^{-t\Delta}\ dt, \tag{4}$$

where Γ is the gamma-function. Since Δ is positive, the heat operator $e^{-t\Delta}$ has a smooth kernel $P_t(x,y)$, and is of trace-class. As $t \to 0$ there is an asymptotic expansion

$$\mathrm{Tr}\,[P_t(x,x)] = \sum_{j=-n/2}^{k} a_j(x) t^j + o(t^k, x), \tag{5}$$

where $a_j(x)$ is a C^∞ function depending only on the local symbol of D. In particular, we can therefore use the right-side of (4) to define a meromorphic continuation of $\zeta_\Delta(s)$ to the whole complex plane with only simple poles. In particular $s = 0$ is not a pole. Hence $\zeta_\Delta'(0) = \frac{d}{ds}\{\zeta_\Delta(s)\}|_{s=0}$ is well-defined and we may define the ζ-determinant by $det_\zeta(\Delta) := e^{-\zeta_\Delta^{'}(0)}$. The metric on $Det\ \mathcal{D}$ over the region of B where the D_b are invertible is defined by

$$\|det\ \mathcal{D}_b\|_\zeta^2 = det_\zeta \Delta. \tag{6}$$

Let us end this Section with some explicit examples

Dimension 1: Let us consider the simplest case of interest. Take

$$\mathcal{D} = i\frac{d}{dx} + a : C^\infty(S^1;\mathbb{C}) \to C^\infty(S^1;\mathbb{C}), \quad a \in (0,1). \tag{7}$$

Then $\Delta = \mathcal{D}^2$ has spectrum $\{\lambda_n = (n+a)^2 : n = 0, 1, \ldots\}$ and so

$$\zeta_\Delta(s) := \mathrm{Tr}\,(\Delta^{-s}) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^{2s}} \tag{8}$$

which is precisely the classical Riemann-Hurwitz zeta-function $\zeta(2s, a)$. This is well-defined for $Re(s) > 1/2$, and has a meromorphic continuation to the entire complex plane with only simple poles. Around $s = 0$ the continuation is holomorphic with

$$\zeta_\Delta^{'}(0) = -\log(4\sin^2(\pi a)). \tag{9}$$

Hence we obtain

$$det_\zeta(\Delta) := e^{-\zeta_\Delta^{'}(0)} = 4\sin^2(\pi a). \tag{10}$$

More generally, in dimension one there is a canonical family of Dirac operators. Take $M = S^1$, and consider a rank n Hermitian bundle E over S^1. Let B be the Banach manifold of unitary connections on E, parameterizing the family $\mathbb{D} = \{\mathcal{D} = i\nabla_{d/dx} : \nabla \in B\}$. The determinant line bundle is canonically trivial in this case, and from formulas for the Riemann-Hurwitz ζ-function it is well-known how to compute that

$$det_\zeta \mathcal{D} = det(I + h_\nabla), \tag{11}$$

where h_∇ is the holonomy of ∇ around the circle. In particular, we see therefore that $det_\zeta \mathcal{D}$ is a smooth function invariant under the action of the group $\mathcal{G}$ of unitary gauge transformations acting on B, and hence passes to a function on the finite-dimensional moduli space $B/\mathcal{G} = U(n)/conjugation$. So no $\mathcal{G}$ anomaly arises, reflecting the fact that $\nabla^{(\zeta)}$ is $\mathcal{G}$-invariant and the curvature of the ζ-connection is identically zero. In fact,

$$det_\zeta(\mathcal{D}^*\mathcal{D}) = |det_\zeta D|^2, \tag{12}$$

means that $\|det\mathcal{D}\|^2_\zeta$ is the norm square of a holomorphic function, and so the metric is flat.

Dimension 2: Consider first the case of an elliptic curve $\Sigma_\tau = \mathbb{R}^2/(\mathbb{Z} \times \tau\mathbb{Z})$, where $\tau = \tau_1 + i\tau_2$ is in the upper-half plane $\tau_2 > 0$, parameterizing the Teichmuller space of complex structures on the torus. Then for $\bar{\partial}_\tau$ acting on the space $\Omega^{m,0}(\Sigma_\tau)$ of differentials of order m it is well-known that

$$\det_\zeta(\bar{\partial}_\tau) = q^{\frac{6m^2-6m+1}{12}} \prod_{n=1}^\infty (1-q^n)^2, \qquad \det_\zeta(\bar{\partial}^*_\tau\bar{\partial}_\tau) = \frac{1}{\tau_2}|det_\zeta(\bar{\partial}_\tau)|^2, \tag{13}$$

where $q = e^{2\pi i\tau}$. This was first computed in [29], as analytic Reidemeister Torsion, but is popular in the String Theory literature since its explicit form allows one to construct the Polyakov integral exactly. Notice that $\det_\zeta(\bar{\partial}_\tau)$ is a holomorphic function of τ, while, unlike dimension one, the Laplacian fails to be an absolute function squared due to zero mode.

For higher genus surfaces, no general closed formula for $det_\zeta(\bar{\partial}_\tau)$ is known, though one can say it will be a generalized theta-function. However, there is the relative 'conformal anomaly' formula:

$$\frac{\det_\zeta(\bar{\partial}^*\bar{\partial})_{g'}}{\det_\zeta(\bar{\partial}^*\bar{\partial})_g} = \exp[\frac{6m^2-6m+1}{6\pi i}L(g,f)], \tag{14}$$

where $L(g,f) = \int_\Sigma[\bar{\partial}f \wedge \bar{\partial}f + fR]$. Here $(\bar{\partial}^*\bar{\partial})_g$ is the $\bar{\partial}$-Laplacian defined by the complex structure associated to a choice of metric g on Σ, with curvature R, and $g^{'}$ is the conformally equivalent metric $e^{2f}g$.

Higher dimensions: There is a straightforward generalization of (13) to operators of the form $\mathcal{D} = d/du + A$ on higher dimensional cylinders $Y \times S^1$, where A is a self-adjoint elliptic operator of order 1, with discrete real spectrum $\{\lambda_k : k \in \mathbb{Z}\}$. One finds, up to a constant, $det_\zeta\mathcal{D} = \prod_k(1 - e^{-\sigma(\lambda_k)})$, where $\sigma(z) = sgn(Re\ z)$.

0.1. **Elliptic boundary Value Problems.** Topological (or functorial) QFT tells that in fact our theories must include not only histories defined by closed manifolds, but also manifolds with boundary, and that axiomatically this should be realizable as a functor from geometric fibrations to vector bundles (of Hilbert spaces) satisfying pasting laws suggested by heuristic path integral formulae. Whatever the merits of such a theory may be, it has presented the need for a mathematical theory of determinant lines bundles for families of elliptic operators over manifolds with boundary and pasting laws for the ζ-determinant with respect to a partition of the manifold. We return in Section 4 with a discussion of the functorial quantum field theory (FQFT) description of determinant anomalies, while in Section 3 we describe recent results on an adiabatic pasting law.

First, we need to review briefly some facts about elliptic boundary value problems (EBVPs) for a Dirac-type operator over a compact Riemannian spin manifold X with boundary $\partial X = Y$. Let $\mathcal{D}$ be the corresponding Dirac operator. We assume that that the Riemannian metric on X and the Hermitian structure on the Clifford bundle S over X are products in a collar neighborhood $N = [0,1] \times Y$ of the boundary. In N the operator $\mathcal{D}$ has the form

$$\mathcal{D} = G(\partial_u + B) \ , \tag{15}$$

where $G : S|Y \to S|Y$ is a unitary bundle isomorphism and $B : C^\infty(Y; S|Y) \to C^\infty(Y; S|Y)$ is the corresponding Dirac operator on Y.

In order to obtain an unbounded Fredholm operator with good elliptic regularity properties we have to impose a boundary condition on the operator $\mathcal{D}$. The space $L^2(Y; S|Y)$ has a preferred orthogonal decomposition $L^2(Y; S|Y) = H^+ \oplus H^-$, where subspace $H^\pm$ are the subspaces spanned by the eigenvectors corresponding to the $\pm$ eigenvalues of B. Let $\Pi_>$ denote the spectral projection onto the subspace H^+, and $\Pi_< = I - \Pi_>$ the projection onto H^-. It is well known that $\Pi_>$ is an elliptic boundary condition for the operator $\mathcal{D}$ (see [1], [7]). More, generally, any pseudodifferential projection (self-adjoint indempotent) operator P on $L^2(Y; S|Y)$ such that $P - \Pi_>$ is a pseudodifferential operator of order -1 defines a good boundary condition for $\mathcal{D}$. This means that the unbounded operator

$$\mathcal{D}_P = \mathcal{D} : dom(\mathcal{D}_P) \to L^2(M; S) \tag{16}$$

with domain

$$dom\ \mathcal{D}_P = \{s \in H^1(M; S)\ :\ P(s|Y) = 0\}\,,$$

with H^1 the first Sobolev space, is a Fredholm operator with kernel and cokernel consisting only of smooth sections. The parameter space of all such projections is an infinite-dimensional Grassmannian $Gr_{-1}(\mathcal{D})$, which has the homotopy type $\mathbb{Z} \times BU$ of the space of Fredholm operators; two boundary conditions P_1 and P_2 belong to the same connected component if and only if

$$index\ \mathcal{D}_{P_1} = index\ \mathcal{D}_{P_2}\,.$$

For analytical reasons associated with the existence of the ζ-determinant, we shall restrict our comments to the *Smooth Grassmannian*, defined by

$$Gr_\infty(\mathcal{D}) = \{P \in Gr(\mathcal{D}) \ : \ P - \Pi_\geq \ \ \textit{has a smooth kernel}\} \ . \tag{17}$$

The crucial fact is that, as a consequence of the *Unique Continuation Property* for the Dirac operators, there is a canonical identification between $H(\mathcal{D})$ and the *infinite-dimensional* solution space

$$\{s \in C^\infty(M : S) \ ; \ \mathcal{D}s = 0 \ \ in \ M \setminus Y\}.$$

The restriction of smooth sections to the boundary extends to a continuous map:

$$\gamma_0 : H^s(M; S) \to H^{s-\frac{1}{2}}(Y; S|Y) \ \ ,$$

which is well-defined for $s > 1/2$ (see [7]). For any real s the *Poisson operator* of $\mathcal{D}$ is a mapping

$$\mathcal{K} : C^\infty(Y; S|Y) \to C^\infty(M; S)$$

which extends to a continuous map $\mathcal{K} : H^{s-1/2}(Y; S|Y) \to H^s(M; S)$, with range equal to the space $ker(\mathcal{D}, s) = \{f \in H^s(M; S) \ : \ \mathcal{D}f = 0 \ in \ M \setminus Y\}$ and

$$\mathcal{K} : \mathcal{H}(\mathcal{D}, s) = Ran\{P(\mathcal{D}) : H^{s-1/2}(Y; S|Y) \to H^{s-1/2}(Y; S|Y)\} \to ker(\mathcal{D}, s) \tag{18}$$

is an isomorphism (see [7]). The Poisson operator defines the Calderon projection:

$$P(\mathcal{D}) = \gamma_0 \mathcal{K} \tag{19}$$

We refer to [7] for the details of the construction, which is originally due to Calderon and Seeley. The Poisson operator $\mathcal{K}$ implies that constructing solutions for the boundary operator

$$\mathcal{S}(P) = P \circ P(\mathcal{D}) : H(\mathcal{D}) \longrightarrow \text{range}\,(P),$$

is equivalent to constructing solutions to the elliptic boundary value problem $\mathcal{D}_P$ (and the same for the adjoints). The Fredholm alternative then gives us:

$$index \ \mathcal{D}_P = index \ \mathcal{S}(P), \tag{20}$$

and the relative index formula

$$index \ \mathcal{D}_{P_1} - index \ \mathcal{D}_{P_2} = index \ (P_1, P_2), \tag{21}$$

where $(P_1, P_2) = P_2 \circ P_1 : \text{range}\,(P_1) \longrightarrow \text{range}\,(P_2)$.

In particular $\mathcal{D}_P$ is an invertible operator if and only if the operator $\mathcal{S}(P) = PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to Ran \ P$ is invertible. The crucial fact proved in [35] is the following exact relative inverse formula giving a a far more delicate analytic realization of (21):

Theorem 0.1. [35] *Let $P_1, P_2 \in Gr_\infty(\mathcal{D})$ such that the EBVPs $\mathcal{D}_{P_i}$ are invertible. Then one has the relative inverse formula*

$$\mathcal{D}_{P_1}^{-1} = \mathcal{D}_{P_2}^{-1} - \mathcal{K}\mathcal{S}(P_1)^{-1} P_1 \gamma_0 \mathcal{D}_{P_2}^{-1} \ . \tag{22}$$

In particular, $\mathcal{D}_{P_1}^{-1} - \mathcal{D}_{P_2}^{-1}$ is a smoothing operator.

This tells us that if the ζ-determinants $det_\zeta \mathcal{D}_{P_i}$ exist, that we can expect the relative determinant $det_\zeta \mathcal{D}_{P_1} / det_\zeta \mathcal{D}_{P_2}$ to be a Fredholm determinant associated to the boundary operators $\mathcal{S}(P_1), \mathcal{S}(P_2)$.

Observe furthermore that in the examples outlined earlier for closed manifolds, in each case where one has an explicitly computable zeta-determinant the problems are naturally posed as elliptic boundary value problems on the appropriate cylinder (apart possibly from the relative conformal anomaly). Furthermore, the regularized determinants are expressed *purely in terms of boundary Cauchy data*, i.e. there is a canonical basis for the solution space of the Dirac operator on the cylinder constructed from the spectral resolution of the boundary operator. In summary:

- The ζ-determinant (and other spectral invariants) for preferred classes of elliptic boundary value problems is a more computable invariant than for the Dirac operator over a closed manifold. This is because of an isomorphism between the (infinite-dimensional) solution space of the Dirac operator and the boundary Cauchy data, defined by the *Poisson operator*.
- In general, one cannot hope to explicitly compute the ζ-determinant over topologically non-trivial manifolds. Nevertheless, it may be possible to compute relative determinants, i.e. relative to a basepoint. cf equation (14).

These points are made concrete in a recent formula proved by the authors, which explicitly computes the relative ζ-determinant for an elliptic boundary value problem for the Dirac operator over an odd-dimensional manifold with boundary in terms of a Fredholm determinant of a canonically defined operator over the *boundary.* The objective is to compute the ζ-determinant on a manifold by computing it on 'simpler' codimension 0 submanifolds and then using pasting formulae. This is a process suggested by Topological QFT. The precise formula is explained in Section 1. First, we make some brief remarks on some global properties of determinants associated to families of EBVPs.

0.2. **The Determinant Line Bundle.** Since the boundary condition is something one must choose from an appropriate parameter space of projections, the sum over histories must therefore include a sum over boundary data (fermionic variables) in addition to the sum over geometric (bosonic) data. The corresponding family parameterized by the manifold B becomes a smooth family of elliptic boundary value problems (EBVPs). Thus in addition to the family of Dirac operators $\mathbb{D}$ over a compact manifold with boundary X, we define a smoothly varying family $\mathbb{P}$ of pseudodifferential projections, called a *Grassmann section* (these include the 'spectral sections' of [19]), assigning to each operator

$\mathcal{D}_b$ an elliptic boundary condition P_b. In particular, $\mathbb{D}$ defines canonically the Calderon (Grassmann) section $P(\mathbb{D})$ equal to the Calderon projection $P(\mathcal{D}_b)$ at $b \in B$. We denote the corresponding smooth family of EBVPs by $(\mathbb{D}, \mathbb{P})$. Associated to such a family one has a well-defined determinant line bundle $DET(\mathbb{D}, \mathbb{P})$ and index bundle $Ind(\mathbb{D}, \mathbb{P})$. The topological realization of the relative inverse formula, and relative index property are the following global identifications for families of EBVPs:

Proposition 0.2. [31] *Let $\mathbb{P}, \mathbb{P}_1, \mathbb{P}_2$ be Grassmann sections for a family of Dirac operators $\mathbb{D}$. There are canonical isomorphisms of determinant line bundles*

$$DET(\mathbb{D}, \mathbb{P}) \cong DET(\mathbb{S}(\mathbb{P})), \tag{23}$$

and

$$DET(\mathbb{D}, \mathbb{P}_0) \cong DET(\mathbb{D}, \mathbb{P}_1) \otimes DET(\mathbb{P}_1, \mathbb{P}_0) \ . \tag{24}$$

If B is compact, as elements of K-theory $K(B)$

$$Ind(\mathbb{D}, \mathbb{P}_0) = Ind(\mathbb{D}, \mathbb{P}_1) + Ind(\mathbb{P}_1, \mathbb{P}_0) \ . \tag{25}$$

Here $(\mathbb{P}_1, \mathbb{P}_0)$ is the family of boundary Fredholm operators $P_{1,b} \circ P_{0,b} : \mathrm{range}\,(P_{0,b}) \to \mathrm{range}\,(P_{1,b})$ and $\mathbb{S}(\mathbb{P}) := (P(\mathbb{D}), \mathcal{P})$. We refer to [31] for details of these constructions.

Since the anomalies discussed earlier are statements about the local geometric and global topological structure of the determinant bundle, we can see that the presence and determination of anomalies depends explicitly on the choice of boundary conditions, that is, on the Fermionic degrees of freedom. In particular, notice that one can always choose $\mathbb{P}$ such that $DET(\mathbb{D}, \mathbb{P})$ is trivial, so in such a case no topological anomaly is present. In practice, physical considerations in general prohibit such choices.

1. Odd-dimensions

We assume now that X is an odd-dimensional manifold. In this case G and B in (15) satisfy the identities

$$G^2 = -Id \quad \text{and} \quad GB = -BG \ . \tag{26}$$

Since Y is even-dimensional the boundary spinor bundle $S|Y$ decomposes into its positive and negative chirality components $S|Y = S^+ \bigoplus S^-$ leading to the orthogonal decomposition $F = F^+ \oplus F^-$ of the boundary spinor fields. Equation (15) can then be rewritten in the form

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left(\partial_u + \begin{pmatrix} 0 & B_- \\ B_+ & 0 \end{pmatrix} \right) \tag{27}$$

where B_+ is the chiral Dirac operator over the boundary. To simplify the exposition we assume that B is invertible.

Consider first the case where X is closed. In this case, in odd-dimensions, since the Dirac operator D is self-adjoint, the index theorem has nothing to say. However, new more subtle secondary invariants arise. From the viewpoint of determinants and index theory, the most important of these is the η-invariant, which measures the difference between the positive and negative (real) spectrum of D, and may be regarded as the analogue of the index in odd-dimensions. It is a holomorphic function for $Re(s) > \dim(M)$ defined by

$$\eta_D(s) = \mathrm{Tr}\,[D(D^2)^{(s+1)/2}] = \sum_{\lambda} \frac{\mathrm{sign}(\lambda)}{\lambda^s}. \tag{28}$$

¿From the heat kernel representation

$$\eta_D(s) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}}\,\mathrm{Tr}\,(De^{-tD^2})dt \tag{29}$$

one again sees that $\eta_{\mathcal{D}}(s)$ has a meromorphic continuation to $\mathbb{C}$ with isolated simple poles along the real axis. The point $s = 0$ is not a pole and so the η-invariant $\eta_D(0)$ is defined. If D is a compatible (true) Dirac operator then $\eta_D(s)$ is actually a holomorphic function for $Re(s) > -2$ [4] and so then

$$\eta_D(0) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}}\mathrm{Tr}\,(De^{-tD^2})dt. \tag{30}$$

This leads to the following ζ-function regularization of the determinant of D:

$$\zeta_D(s) = \frac{1}{2}(1 + e^{\pm i\pi s})\zeta_{D^2}(\frac{s}{2}) + \frac{1}{2}(1 - e^{\pm i\pi s})\eta_D(s), \tag{31}$$

and hence that

$$det_\zeta D := e^{-\zeta_D'(0)} = e^{\pm\frac{i\pi}{2}(\zeta_{\mathcal{D}^2}(0) - \eta_{\mathcal{D}}(0))}.e^{-\frac{1}{2}\zeta'_{\mathcal{D}^2}(0)} = e^{\pm\frac{i\pi}{2}(\zeta_{\mathcal{D}^2}(0) - \eta_{\mathcal{D}}(0))}|det_\zeta D|. \tag{32}$$

(see [35, 38] for more details). In particular, one has

$$det_\zeta \Delta = |det_\zeta D|^2, \tag{33}$$

and so the ζ-metric (see [3, 28] is the norm square of a function, and therefore is the flat metric providing no curvature. This is precisely because D is self-adjoint. Though there is therefore no geometric anomaly present, there is another more subtle parity anomaly due to the ambiguity in the choice of the sign in equation (31) and hence in the definition of $det_\zeta D$. We choose the '+' sign.

Now return to the case where X has boundary. In the odd-dimensional case self-adjoint realizations of the operator $\mathcal{D}$ are of particular interest. The involution $G : S|Y \to S|Y$ equips $L^2(Y; S|Y)$ with a symplectic structure, and by Green's formula (see [7]) we have that that the boundary condition $P \in Gr(\mathcal{D})$ provides a self-adjoint realization $\mathcal{D}_P$ of the operator $\mathcal{D}$ if and only if

$$-GPG = Id - P \quad . \tag{34}$$

(See [5], [7], [12].) We therefore introduce the submanifold $Gr^*_\infty(\mathcal{D})$ of $Gr_\infty(\mathcal{D})$ parameterizing such projections.

For any $P \in Gr^*_\infty(\mathcal{D})$ the operator $\mathcal{D}_P$ has a discrete spectrum nicely distributed along the real line (see [5], [12]) and the second author proved:

Theorem 1.1. [41] *For any projections $P, P_1, P_2 \in Gr^*_\infty(\mathcal{D})$:*

1. *$\eta_{\mathcal{D}_P}(s)$ and $\zeta_{\mathcal{D}^2_P}(s)$ are holomorphic functions of s in the neighborhood of $s = 0$.*
2. *The value of ζ-function at $s = 0$ is constant on $Gr^*_\infty(\mathcal{D})$ i.e.*
$$\zeta_{\mathcal{D}^2_{P_1}}(0) = \zeta_{\mathcal{D}^2_{P_2}}(0) \quad .$$
3. *The function $\eta_{\mathcal{D}_P}(s)$ is a holomorphic function of s in the half-plane $Re(s) > -1$.*
4. *The function $\Gamma(s)\zeta_{\mathcal{D}_P}(s) = \int_0^\infty t^{s-1}\mathrm{Tr}\ e^{-t\mathcal{D}^2_P}dt$ has a simple pole at $s = 0$. Hence $\zeta_{\mathcal{D}^2_P}(0)$ and, according to formula (32), $ln\ det_\zeta(\mathcal{D}_P)^2) = -d/ds(\zeta_{\mathcal{D}^2_P})|_{s=0}$ are well-defined.*

Therefore $det_\zeta \mathcal{D}_P$ is a well-defined, smooth function on $Gr^*_\infty(\mathcal{D})$. The question now is how to compute it. To see this, first observe that the grading $F = F^+ \oplus F^-$ is defined by the self-adjoint involution

$$B^{-1}|B| = \begin{pmatrix} 0 & B_+^{-1}(B_+B_-)^{1/2} \\ (B_+B_-)^{-1/2}B_+ & 0 \end{pmatrix}, \tag{35}$$

where $|B| = +\sqrt{B^2} = B\Pi_> - B\Pi_<$, then

$$\Pi_\geq = \frac{1}{2}(I + B^{-1}|B|) = \frac{1}{2}\begin{pmatrix} I & w_+^{-1} \\ w_+ & I \end{pmatrix}, \tag{36}$$

where

$$w_+ = (B_+B_-)^{-1/2}B_+. \tag{37}$$

Equivalently,

$$\mathrm{range}\,(\Pi_>) = graph(w_+ : F^+ \to F^-). \tag{38}$$

In particular, we see that $\Pi_>$ is a pseudodifferential operator of order 0, and inverting the boundary value problem (16) is the same thing as inverting the boundary value problem

$$(\partial_u - |B|)f = k, \quad f \in graph(-w_+ : F^+ \to F^-). \tag{39}$$

Anyway, we have that H^+ is the graph of $w_+ : F^+ \to F^-$. Moreover, as we have assumed that $ker\ B = \{0\}$, then $\Pi_>$ is an element of $Gr^*_\infty(\mathcal{D})$. A generalization [10, 30] of this, extends the graph$\longleftrightarrow$ self-adjoint boundary condition correspondence for H^+ to the whole Grassmannian $Gr^*_\infty(\mathcal{D})$: There is a $1-to-1$ correspondence of $Gr^*_\infty(\mathcal{D})$ with unitary maps $g : F^+ \to F^-$, such that the difference $g - g_+$ is an operator with a smooth kernel. The corresponding orthogonal projection P is given by the formula

$$P = \frac{1}{2}\begin{pmatrix} Id_{F^+} & g^{-1} \\ g & I_{F^-} \end{pmatrix}, \quad g = w_+ + S,$$

where $S : F^+ \to F^-$ is a smoothing operator. By choosing a basepoint, the correspondence defined above allows us to establish an isomorphism between $Gr^*_\infty(\mathcal{D})$ and the group $U^\infty(F^-)$ of unitaries acting on $F^- = L^2(Y; S^-)$ which differ from Id_{S^-} by an operator with a smooth kernel. The Calderon projection defines a preferred basepoint, and hence, letting $K : C^\infty(Y; S^+) \to C^\infty(Y; S^-)$ denote the unitary such that $\mathcal{H}(\mathcal{D})$ is equal to the $graph(K)$, we have a natural isomorphism $Gr^*_\infty(\mathcal{D}) \cong U^\infty(F^-)$ defined by the map $P \to TK^{-1}$. This means that the determinant line bundle associated to the family $\{\mathcal{D}_P : P \in Gr^*_\infty(\mathcal{D})\}$ is canonically trivial. The following theorem asserts that up to a natural constant this is precisely the trivialization defining the ζ-determinant:

Theorem 1.2. [34, 35] *Let* $P_1, P_2 \in Gr^*_\infty(\mathcal{D})$ *such that* $\mathcal{D}_{P_1}$,$\mathcal{D}_{P_2}$ *are invertible. If* range$(P_i) = graph(T_i)$, *then*

$$\frac{det_\zeta \mathcal{D}_{P_1}}{det_\zeta \mathcal{D}_{P_1}} = \frac{det_{Fr}\ \frac{1}{2}(Id + KT_1^{-1})}{det_{Fr}\ \frac{1}{2}(Id + KT_2^{-1})}. \tag{40}$$

Here det_{Fr} denotes the Fredholm determinant, defined for any operator of the form Id plus trace-class. The determinants on the right-side are the Fredholm determinants of the boundary operators $\mathcal{S}(P_i)$ computed in the above trivialization.

In particular, it is easy to see that Fredholm determinants are invariant under the action of the gauge group $\mathcal{G}$, and hence this regularization reduces to the moduli space $\mathcal{M}$, as required, corresponding to the fact that for families of self-adjoint EBVPs in odd-dimensions there is no gauge anomaly.

2. Relative Determinant on the Infinite Cylinder

In this Section we consider the simplest non-compact case in which we can define *relative* ζ-determinant. We discuss an explicit computation of formula (40) in the case of the non-compact half-infinite cylinder $[0, \infty) \times Y$. This is possible because we have in this case explicit knowledge of the Schwartz kernel and heat kernel for $\mathcal{D}_P$. Details will appear elsewhere.

Again, let Y be a compact spin manifold of any dimension. We consider the operator

$$D = G\left(\frac{\partial}{\partial u} + B\right), \tag{41}$$

on the half-infinite cylinder $Y \times [0, \infty)$, where the Dirac operator B over Y we take to be invertible, and with the usual identities (26). We assume that B is an invertible operator to simplify the exposition. Associated to B we have the Grassmannian parameterising 'self-adjoint' elliptic boundary conditions for D

$$Gr^*_\infty(D) = \{P : P^2 = P, P^* = P, P - \Pi_> = smoothing, -\Gamma P \Gamma = I - P\}.$$

Let P_r be a path in $Gr^*_\infty(D)$ connecting $\Pi_>$ and P. Then we can find a corresponding path of unitaries $U_r \in \mathcal{U}_\infty(F) = \{U \in \mathcal{U}(F) : g = I + smoothing\}$ with

$$P_r = U_r \Pi_r U_r^{-1}. \tag{42}$$

We may assume that g_r commutes with G, so that

$$-U_r = G U_r G. \tag{43}$$

Because we are working on a non-compact manifold the spectrum of $\mathcal{D}_P$ will in general be continuous. However, we can still define a relative ζ–determinant using the relative ζ-function, which is determined by the *couple* of boundary conditions

$$\zeta(s; \mathcal{D}; P, \Pi_>) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}\,(e^{-tD_P} - e^{-tD_{\Pi_>}})dt, \tag{44}$$

The function $\zeta(s; \mathcal{D}; P, \Pi_>)$ exists and shares the standard properties of the ζ-function on a closed manifold, since $\mathcal{D}_P^{-1}$ and $\mathcal{D}_{\Pi_>}^{-1}$ differ by only a smoothing operator. In the case of the compact manifold M of Section 2 , $\zeta(s; \mathcal{D}; P, \Pi_>)$ reduces to the difference of the individual ζ-functions. For spectral invariants on manifolds with cylindrical ends we refer to [15] and [24].

The operators D_{P_r} have domains varying with r and so we again use a 'unitary twist' $U_{rf(u)}$ as in Section 2. Fix $\delta_1 > \delta_0 > 0$ and define a smooth non-decreasing function $f(u)$ such that

$$f(u) = 1 \ for\ u < \delta_0 \ \ and \ \ f(u) = 0 \ for\ u > \delta_1 \ \ .$$

For each r introduce the 2-parameter family

$$U_{r,u} = U_{rf(u)}, \quad u \in [0, \infty) \quad -\epsilon \le r \le \epsilon.$$

The operators D_{P_r} and $(U^{-1}_{rf(u)} D U_{rf(u)})_{\Pi_>}$ are unitary equivalent operators and hence their ζ-determinants have the same variation. We write

$$D^r = U^{-1}_{rf(u)} D U_{rf(u)}.$$

Again we have that $U_{rf(u)}$ commutes with G:

$$U_{rf(u)} = G U_{rf(u)} G. \tag{45}$$

Then we may define the ζ-determinant of D_{P_r} as the relative ζ-determinant. We can define both phase and the modulus of such a determinant. The phase is determined by the relative ζ-function defined above and by the value at $s=0$ of the relative η-function, which is defined similarly to the equation (44), with ζ-function replaced by the η-function

$$\eta(s;\mathcal{D};P,\Pi_>) = \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{\frac{s-1}{2}}\operatorname{Tr}(\mathcal{D}e^{-t\mathcal{D}_P} - \mathcal{D}e^{-t\mathcal{D}_{\Pi_>}})dt, \tag{46}$$

It is not difficult to show that $\eta(s;\mathcal{D};P,\Pi_>)$ is well-defined (see [33] for related computations on the cylinder) and in particular that it is regular at $s=0$.

As usual the modulus of the determinant is more difficult to handle. Its value is given by

$$det_\zeta(D^2_{P_r};D^2_{\Pi_>}) := e^{-d/ds(\zeta(s;D^2_{P_r};D^2_{\Pi_>}))|_{s=0}} \ . \tag{47}$$

When $Y\times[0,\infty)$ is odd-dimensional the tangential operator splits into its positive and negative parts $B_\pm$ according to the chiral splitting $H_Y = F^+\oplus F^-$ and D has the form (27) and $P\in Gr^*_\infty$ has the form

$$P := P_T = \frac{1}{2}\begin{pmatrix} I & T^{-1} \\ T & I \end{pmatrix}, \tag{48}$$

where $T = w_+ + S : F^+ \longrightarrow F^-$ is unitary in the L^2-metric, S is a smoothing operator. The unitary twist has the form

$$U_{rf(u)} = \begin{pmatrix} I & 0 \\ 0 & g_{rf(u)} \end{pmatrix} \tag{49}$$

with $g_{rf(u)}\in U_\infty(F^-)$, so

$$U_{rf(u)}P_TU^{-1}_{rf(u)} = P_{g_rT}. \tag{50}$$

Now a path $\{P_r\}$ between $\Pi_>$ and P_T can be given by choosing a path $\{T_r\}$ between w_+ and T and we can set

$$g_r = T_rw_+^{-1} \tag{51}$$

More general, for any pair $\{P_1,P_2\}$ of elements of the self-adjoint smoothing Grassmannian, where $Ran\ P_i = graph\ T_i$ we can find a path between those two projections of the form $\{U_rP_1U_r^{-1}\}$ with

$$U_r = \begin{pmatrix} I & 0 \\ 0 & h_r \end{pmatrix} \tag{52}$$

where $h_r\in U_\infty(F^-)$. Now we can follow the method of [35] (see also [34]) and prove the following result, which is an analogue of Theorem 1.2.

Theorem 2.1. *Let $Y \times [0, \infty)$ be odd-dimensional and let $P_i = P_{T_i} \in Gr^*_\infty(D)$. Then*

$$det_\zeta(\mathcal{D}; P_1, P_2) = det_{Fr}\left[\frac{1}{2}(Id + T_2 T_1^{-1})\right] . \tag{53}$$

One can also prove this Theorem by direct computation following the method presented in [6] using the fact that as in the one-dimensional case we know an explicit formula for the kernel of the inverse operators $\mathcal{D}_{P_i}^{-1}$ and the heat kernel of the Atiyah-Patodi-Singer problem.

3. An Adiabatic Pasting Formula

Given that we have a precise formula for a preferred class of self-adjoint elliptic boundary value problems, it is natural to look for a pasting formula for the determinant on a closed manifold endowed with a partition. In this Section we give a brief review of recent formulas obtained by K.Wojciechowski and J.Park.

We consider the case of a closed odd-dimensional manifold M. Let $\mathcal{D} : C^\infty(M; V) \to C^\infty(M; V)$ be a compatible Dirac operator acting on sections of a bundle of Clifford modules V over M. Assume that we have a partition of M as $M_0 \cup_Y M_1$, where M_0 and M_1 are compact manifolds with boundary such that

$$M_0 \cap M_1 = Y = \partial M_0 = \partial M_1 \ . \tag{54}$$

We also assume that the Riemannian metric on X and the Hermitian product on V are products in the bicollar neighborhood $\tilde{N} = [-1, 1] \times Y$ of Y, where $M_0 \cap \tilde{N} = [-1, 0] \times Y$. The operator $\mathcal{D}$ is given by the formula (15) in $\tilde{N}$. Let $\mathcal{D}^i = \mathcal{D}|M^i$ $(i = 0, 1)$ and $P_0, P_1 \in Gr^*_\infty(\mathcal{D})$ and let $\eta(P_0, P_1)$ denote the η-invariant of the operator $G(\partial_u + B)$ on the manifold $[-1, 1] \times Y$ subject to the boundary condition P_0 at $u = -1$ and the boundary condition $Id - P_1$ at $u = 1$. The second author proved the following additivity formula for the η-invariant (see also [12], [15], [17], [39] for partial results and related topics).

Theorem 3.1. [40][41] *For any $P_0, P_1 \in Gr^*_\infty(\mathcal{D})$ one has the following formula*

$$\eta_{\mathcal{D}} = \eta_{\mathcal{D}^0_{Id-P_0}} + \eta_{\mathcal{D}^2_{P_1}} + \eta(P_0, P_1) \quad mod\ \mathbf{Z} \ . \tag{55}$$

Results analogous to Theorem 3.1 of [40] were obtained and discussed by other authors. We refer especially to the papers [9], [11], [14], [18], [22], [23]. Theorem 3.1 extends the formula on the variation of the η-invariant under a change of boundary condition from the work [17].

The crucial result from which the Theorem 3.1 follows is:

Theorem 3.2. [41] *Let* $P_0, P_1 \in Gr^*_\infty \mathcal{D}$ *, then*

$$\eta_{\mathcal{D}_{1_{P_0}}} - \eta_{\mathcal{D}_{1_{P_1}}} = -\frac{1}{\pi}\int_0^1 dr \int_0^1 du \ \mathrm{Tr} \ G(g^{-1}\dot{\frac{\partial g}{\partial u}}|_r) \ \ mod \ \mathbf{Z} \ , \tag{56}$$

$$\eta(P_0, P_1) = -\frac{1}{\pi}\int_0^1 dr \int_0^1 du \ \mathrm{Tr} \ G(g^{-1}\dot{\frac{\partial g}{\partial u}}|_r) \ \ mod \ \mathbf{Z} \ , \tag{57}$$

where $\{g_{r,u}\}$ *is any family connecting* P_0 *with* P_1 *in the way described above.*

Thus, in this case we get an exact decomposition of the η-invariant $\eta_{\mathcal{D}} = \eta_{\mathcal{D}}(0)$ of the operator $\mathcal{D}$ into contributions coming from the different parts of the manifold. An interesting consequence of Theorem 3.1 and Theorem 1.1(2) is that they imply:

Corollary 3.3. *The phase of the* ζ*-determinant is additive of under the pasting of two manifolds with the same boundary.*

That might lead one to expect that the ζ-determinant might be multiplicative under pasting of manifolds, but that is not true. The ζ-determinant on a closed manifold is independent of the partition, and indeed any choice of boundary conditions. Hence we should expect a formula which 'averages away' the choice of boundary condition. This is suggested formally by path integral formulae (see Section 4), and in fact in the correct analytic formula the boundary condition is scaled away adiabatically. The problem here is with the modulus of the determinant

$$|det_\zeta \mathcal{D}| = e^{-\frac{1}{2}\zeta'_{\mathcal{D}^2}(0)} \ .$$

This is not a local quantity and its variation is not local either. Assuming that *dim M* is odd it is equal to (see [38, 41]

$$ln \ det_\zeta(\mathcal{D}^2) = -d/ds(\zeta_{\mathcal{D}^2})|_{s=0} = -\int_0^\infty \frac{1}{t} Tr \ e^{-t\mathcal{D}^2} \ . \tag{58}$$

However, we can discuss the adiabatic splitting formula in this case. We replace the bicollar $\tilde{N}$ by $\tilde{N}_R = [-R, R] \times Y$ where $R \to \infty$. In other words we stretch the manifold M to M_R by replacing $\tilde{N}$ by a cylinder of length $2R$. Due to the construction of the heat kernel the right side of (58) now splits into the contribution from the interior, the cylinder contribution, and an error term. If we assume that tangential operator B is invertible, then the error term disappears as $R \to \infty$. Therefore we can study the decomposition of the contribution coming from the different part of the manifolds. The first study of this type was made in [16]. The focus of the Klimek-Wojciechowski paper was on the η-invariant of chiral boundary problems and on the analytic torsion, hence there was no direct reference to the ζ-determinant. However, the results of [16] allow us to study the adiabatic decomposition of the modulus of the ζ-determinant of chiral boundary problems.

This is explained in recent work of Park and Wojciechowski (see [25]). Park and Wojciechowski study also a pasting law, which involves the Atiyah-Patodi-Singer conditions. We describe the result and refer to [25] for more details and proofs. Let $\mathcal{D}_R$ denote a Dirac operator $\mathcal{D}$ on the manifold M_R , which is the manifold M with the bicollar $\tilde{N}$ replaced by $\tilde{n}_R = [-R, R] \times Y$. Let $\mathcal{D}_{i,R}$ denote the Dirac operator on the manifold $M_{i,R}$ (equal to M_i with a cylinder of length R attached). Now the ζ-determinant of the Dirac Laplacian $\mathcal{D}_R^2$ blows up as $R \to \infty$. The same happens with the determinant of the operator $(\mathcal{D}_{i,R})^2_{\Pi_>}$. The corresponding quotient however is well-defined and we have the following formula

Theorem 3.4 (Park-Wojciechowski)**.**

$$\lim_{R\to\infty} \frac{det_\zeta \mathcal{D}_R^2}{det_\zeta((\mathcal{D}_{0,R})^2_{\Pi_>}) \cdot det_\zeta((\mathcal{D}_{0,R})^2_{\Pi_>})} = 2^{-\zeta_{B^2}(0)} \quad . \tag{59}$$

Remark 3.5. *(1) Of course formula (59) works only if the operators $\mathcal{D}_R$ and $(\mathcal{D}_{i,R})_{\Pi_>}$ are invertible (say for large R). We have already assumed that the tangential operator B is invertible, hence the sufficient condition here is that operators $\mathcal{D}_{i,\infty}$ (= the operator $\mathcal{D}$ on the manifold M_i with an infinite cylinder attached) do not have L^2 solutions (see for instance the discussion in* [39]*).*

(2) Besides the techniques introduced and used in [12] *(see also* [16]*) the proof of the Theorem 3.4 employs technical results of the beautiful work* [8].

We end this Section with the discussion of the simplest possible pasting situation. Let us recompute the determinant of the $\bar{\partial}$-operator over a mapping cylinder, using the canonical regularization above.

Let $M = M^0 \cup_Y M^1$ be a partitioned manifold. And suppose that that $H(D^0) = graph(K^0 : H_Y^+ \to H_Y^-)$ and $H(D^1) = graph(K^1 : H_Y^- \to H_Y^+)$, where the K^i are at least Hilbert-Schmidt operators. From the above identifications we have a canonical isomorphism

$$Det(D) \cong Det(P(D^0), I - P(D^1)).$$

Hence, as before, we obtain a canonical regularization of the determinant of D as the regularized determinant of the operator $\mathcal{S}(D) = (I - P(D^1))P(D^0) : H(D^0) \to H(D^1)^\perp$ which we denote $det_{\mathcal{C}}(D)$. This yields [30]

$$det_{\mathcal{C}}(D) = det_{Fr}(I - K^1K^0). \tag{60}$$

We consider the cylinder $M = S^1 \times Y$ with the operator $D = \partial/\partial u + B$, where B is an invertible first-order elliptic self-adjoint operator over Y acting on sections of a bundle E. Now decompose M as the sum of two cylinders $M_R = [0, R] \times Y$ and $M'_R = [R, 2\pi] \times Y$ with restricted operators D^R and $D^{R'}$. Let $\{\lambda_n, \phi_n\}$ be a spectral resolution of B. Then

$$Ker\ D^R = span\{e^{-u\lambda_n}\phi_{\lambda_n}(y) : n \in \mathbb{Z}\},$$

so that

$$H(D) = span\{(\phi_{\lambda_n}, e^{-R\lambda_n}\phi_{\lambda_n}) : n \in \mathbb{Z}\}. \tag{61}$$

The cylinder has boundary $\partial M_R = Y \sqcup \overline{Y}$, where the second component is Y with orientation reversed. Relative to this $H_Y = H_0 \oplus H_R$ and the tangential component of D^R is $B^R = B_0 \oplus B_R$, where $B_0 = B$ and $B_R = -B$. Hence the energy grading of H_Y defined by B^R is

$$H^+ = H_0^+ \oplus H_R^- \quad H^- = H_0^- \oplus H_R^+, \tag{62}$$

where $H_i^{\pm}$ are the positive and negative gradings associated to the B_i. From (61) and (62) we have that $H(D^R)$ is the graph of

$$K : H_0^+ \oplus H_R^- \longrightarrow H_0^- \oplus H_R^+, \quad K(f_0^+, f_R^-) = (e^{-RB}f_0^+, e^{RB}f_R^-).$$

That is,

$$H(D^R) = graph(K = e^{-R|B|} : H^+ \longrightarrow H^-). \tag{63}$$

Applying the same argument to $D^{R'}$, we obtain from (60) and (63)

$$det_{\mathcal{C}}(D) = det_{Fr}(I - e^{-2\pi|B|}) = \prod_{n\in\mathbb{Z}}(1 - e^{-2\pi\lambda_n}). \tag{64}$$

If we apply this to the case of the $\bar{\partial}$-operator $\bar{\partial}_\tau = \frac{d}{du} + \frac{1}{\tau}\frac{d}{dy}$ over an elliptic curve, defined by τ in the upper-half plane, we obtain

$$det_{\mathcal{C}}(\bar{\partial}_\tau) = c{\cdot}det_{Fr}(I - e^{-2\pi|B|}) = c{\cdot}\prod_{n=1}^{\infty}(1 - q^n)^2, \tag{65}$$

where $q = e^{2\pi i\tau}$. The operator $\bar{\partial}_\tau$ is not invertible, so we choose an isomorphism between the kernel and cokernel of $\mathcal{S}(\bar{\partial}_\tau)$, the constant c is the determinant of this isomorphism.

Notice that the choice of a decomposition of $M \cong S^1 \times S^1$ defines a homology 1-cycle on M and hence an element γ of $H_1(M, \mathbb{Z})$. Because $Det(\bar{\partial}_\tau)$ is the dual line to the holomorphic differentials on M, we obtain a canonical element $\xi_\gamma \in Det(\bar{\partial}_\tau)$. In [37], Segal showed that

$$det\bar{\partial}_\tau = \prod_{n=1}^{\infty}(1-q^n)^2 \cdot \xi_\gamma \ .$$

Thus we see the canonical trivialization is essentially the trivialization ξ_γ.

From (13)

$$det_\zeta(\bar{\partial}_\tau) = \frac{q^{\frac{1}{12}}}{c} det_C(\bar{\partial}_\tau).$$

Similarly one can compute the canonical determinant of $\bar{\partial}_\tau$ acting on holomorphic differentials of degree m, which introduces an additional factor of $q^{(m^2-m)/2}$, and the case where $\bar{\partial}_\tau$ is coupled to a flat connection on a coefficient bundle. Details of these computations will appear elsewhere.

4. Functorial QFT and Gauge Anomalies

Functorial quantum field theory (FQFT) is an attempt to abstract the algebraic framework that a path integral would create if it existed as a rigorous mathematical object. Thus one begins with a myth, and aims to work towards some kind of technology. Roughly speaking, a FQFT is kind of generalized cohomology theory defined by a functor from manifolds with boundary to vector (Hilbert) spaces satisfying axioms formally satisfied by the path integral.

More precisely, a $d+1$-dimensional FQFT assigns to a closed manifold Y of dimension d a vector space $Z(Y)$, while to a manifold of dimension $d+1$ it assigns a vector $Z_X \in Z(Y)$, where $Y = \partial X$ is the boundary of X. By *fiat* $Z(\emptyset) = \mathbb{C}$, so that if X is closed then Z_X is a complex number. In practice, the class of manifolds being considered needs to be specified and usually additional data, such as geometric data and boundary conditions.

For $(d+1)$-dimensional manifolds X, X_0, X_1 and d-dimensional manifolds Y, Y_0, Y_1, the 'functor' Z is required to satisfy certain axioms; notably, that

$$Z_{X_0 \sqcup X_1} = Z_{X_0} \otimes Z_{X_1}, \quad Z(Y_0 \sqcup Y_1) = Z(Y_0) \otimes Z(Y_1),$$

while if $\overline{Y}$ denotes Y with reversed orientation then

$$Z(\overline{Y}) = Z(Y)^*.$$

This means that a cobordism X induces a linear transformation $Z_X : Z(Y_0) \to Z(Y_1)$ through the identifications

$$Z_X \in Z(\overline{Y_0} \sqcup Y_1) = Z(\overline{Y_0}) \otimes Z(Y_1) = Z(Y_0)^* \otimes Z(Y_1) = \mathrm{Hom}(Z(Y_0), Z(Y_1)).$$

One further requires that if $M = X_0 \cup_Y X_1$ with $\partial X_0 = \overline{Y_0} \sqcup Y$ and $\partial X_1 = \overline{Y} \sqcup Y_1$, then

$$Z_M = Z_{X_1} \circ Z_{X_0}.$$

This in turn induces a canonical pairing $Z(\overline{Y_0})\otimes Z(Y)\otimes Z(\overline{Y})\otimes Z(Y_1) \longrightarrow Z(\overline{Y_0})\otimes Z(Y_1)$, and in the case when $Y_0 = Y_1 = \emptyset$, so M is a closed manifold , this becomes a pairing

$$(\, , \,) : Z(Y) \otimes Z(\overline{Y}) \longrightarrow \mathbb{C}, \tag{66}$$

with the *sewing property*

$$Z_M = (Z_{X_0}, Z_{X_1}). \tag{67}$$

Thus the number Z_M can be computed by evaluating over the submanifolds and then sewing together the results via the bilinear pairing. These axioms are 'idealized', and in for each case modifications are needed.

In the case where X_0 has connected boundary equal to Y_0, so $Y = \emptyset$, then $Z_X \in \mathrm{Hom}(Z(Y_0), \mathbf{C})$. Formally setting $\mathcal{E}_f(X)$ to be a space of fields on X with boundary value $f \in \mathcal{E}(Y_0) := Z(Y_0)$, the vector Z_X corresponds to a partition function given by a path integral

$$Z_X : \mathcal{E}(Y) \longrightarrow \mathbb{C}, \quad Z_X(f) = \int_{\mathcal{E}_f(X)} e^{-S(\psi)} \, d\psi, \tag{68}$$

where $d\psi$ is a formal measure and $S : \mathcal{E}_f(X) \to \mathbb{C}$ an action functional. The path integral version of the algebraic sewing formula (67) then takes the form

$$\begin{aligned} \int_{\mathcal{E}(M)} e^{-S(\psi)} \, d\psi &= \int_{\mathcal{E}(Y)} df \int_{\mathcal{E}_f(X_0)} e^{-S(\psi_0)} \, d\psi_0 \int_{\mathcal{E}_f(X_1)} e^{-S(\psi_1)} \, d\psi_1 \\ &= \int_{\mathcal{E}(Y)} Z_{X_0}(f) Z_{X_1}(f) \, df. \end{aligned} \tag{69}$$

To see how to describe the chiral anomaly in this framework, consider first the closed manifold M which we assume to be even-dimensional and spin. The chiral Dirac operator acts between positive and negative chirality fields, $\mathcal{D} = \mathcal{D}^+_A : C^\infty(M; E \otimes S^+) \longrightarrow C^\infty(M; E \otimes S^-)$ coupled to a coefficient bundle E with connection A. Evaluating the Euclidean path integral (1), we seek a regularized determinant of $\mathcal{D}_A$ considered as a function on the affine space B of gauge potentials on E. To do that we must choose an operator $T_A : C^\infty(M; E \otimes S^-) \longrightarrow C^\infty(M; E \otimes S^+)$, and then try to regularize $det T_A \mathcal{D}$. However, there is no reason to expect $det_r \mathcal{D} := det_r T_A \mathcal{D}$ to transform equivariantly under gauge transformations, and indeed in general it does not. One finds

$$det_r \mathcal{D}^+_{A^g} = det_r \mathcal{D}^+_A \; \omega(g, A),$$

where $\log \omega$ is an integral over M of a local differential polynomial in g, A and the metric on M. ω defines a one-cocycle on the gauge transformation group $\mathcal{G}$, the transgression of ω to a 2-form on the moduli space $\mathcal{M}$ is the first Chern class of the determinant line bundle pushed-down to $\mathcal{M}$. The non-triviality of this class is the topological obstruction to the existence of $\mathcal{G}$-invariant determinant regularization [2]. It is called the chiral anomaly. From the gauge transformation group view point, the 1-form ω defines a representation of an abelian extension $\widehat{\mathcal{G}}$ of $\mathcal{G}$ for which ω is the extension cocycle [20].

This description of the chiral anomaly corresponds to the left-side of equation (69). Let us turn then to the right-side of that formula. (For a detailed presentation of the following constructions we refer to [21].) Let $\mathcal{D}^i,\ i = 0, 1$, denote the restrictions of $\mathcal{D}$ to the two halves X_i of M. We assume the geometric set-up of previous sections. For analytic reasons, we must now replace the local boundary condition f for $\mathcal{D}^0$ by a global condition $W = \mathrm{range}\,(P)$ with P is a restricted Grassmannian of the boundary fields, and we shall choose $Gr_\infty(\mathcal{D}^0)$. The path integral (1) now becomes, ignoring the pure gauge component,

$$Z_X(P) := det(\mathcal{D}^0_P) = \int_{\mathcal{E}_P} e^{\int_X \psi^* \mathcal{D}^0 \psi \, dm} \; d\psi d\psi^*, \tag{70}$$

where $\mathcal{E}_P = dom(\mathcal{D}_P)$, while the sewing formula becomes

$$det(\mathcal{D}) = \int_{Gr_Y} det\mathcal{D}^0_P \cdot det\mathcal{D}^1_{I-P} dP \ . \tag{71}$$

This is entirely formal, but it asserts that $det(\mathcal{D})$ is obtained by integrating away the boundary data. Indeed, the determinant over the closed manifold clearly does not depend on boundary conditions, and so this is formally what we would expect. In practice, to tell the ζ-function over the closed manifold where the partition lies we have to send it an 'impulse function', the rigorous formulation of this idea is the adiabatic pasting formula of the previous section.

However, (71) can be given a precise formulation in terms of a FQFT. Notice first that there is no regularization involved in the formula, it is a relation between sections of appropriate determinant line bundles. Hence the corresponding FQFT pairing (67) should be a pairing on spaces of sections of those determinant line bundles.

To see what those spaces are, return for a moment to the action of the gauge group. For a manifold with boundary the essential data needed to detect the chiral anomaly can be seen from the action of the boundary gauge transformation group. To begin with, consider the case of a Riemann surface with connected boundary diffeomorphic to S^1. The boundary gauge group in this case is the loop group LG where G is the (compact) structure group of the (necessarily trivial) complex bundle over the circle. It was explained in the text [27] that a LG has a fundamental projective representation in the Fock space of holomorphic sections of the dual of the determinant line bundle DET_{H^+} over the Grassmannian. Here DET_{H^+} is the bundle based at H^+ with fibre at P equal to the complex line $Det(H^+, W) := Det(P\Pi_> : H^+ \to W),\ \ W = \mathrm{range}\,(P)$. This suggests the general construction for the Dirac operator $\mathcal{D}^0$ over the manifold X_0 with boundary. First, for each $P \in Gr_\infty(\mathcal{D}^0)$ one has a determinant line bundle DET_W

based at $W = \mathrm{range}\,(P)$ with fibre at $W' = \mathrm{range}\,(P^{'})$ equal to $Det(W, W^{'})$. The Fock space based at W is defined by

$$\mathcal{F}_W = \Gamma_{hol}(Gr_\infty(\mathcal{D}^0); DET^*_W),$$

where Γ_{hol} denotes holomorphic sections. (Here we can replace $Gr_\infty(\mathcal{D}^0)$ by the Hilbert-Schmidt Grassmannian, but we ignore this point.) Under a change of base point one has from (23) a canonical isomorphism

$$\mathcal{F}_{W_1} \cong \mathcal{F}_{W_2} \otimes DET(W_1, W_2), \tag{72}$$

where the second factor on the right-side is the trivial bundle with fibre $Det(W_1, W_2)$. Moreover, under the Plucker embedding

$$DET_W \longrightarrow \mathcal{F}_W$$

the determinant section of DET_W is identified with the 'vacuum vector' $\nu_W \in \mathcal{F}_W$. The basic fact is the following Theorem:

Theorem 4.1. [21] *There are functorial bilinear pairings*

$$(\ ,\ ) : \mathcal{F}_{K(D^0)}(H_Y) \times \mathcal{F}_{W^\perp}(\overline{H}_Y) \longrightarrow Det(D^0_P), \tag{73}$$

($W = \mathrm{range}\,(P)$) with

$$(\nu_{K(D^0)}, \nu_{W^\perp}) = det(D^0_P), \tag{74}$$

and

$$(\ ,\ ) : \mathcal{F}_{K(D^0)}(H_Y) \times \mathcal{F}_{K(D^1)}(\overline{H}_Y) \longrightarrow Det(D), \tag{75}$$

with

$$(\nu_{K(D^0)}, \nu_{K(D^1)}) = det(D). \tag{76}$$

These pairings are of course naturally the bilinear pairing needed to define the FQFT we are seeking. This requires that to each closed d-dimensional manifold Y we associate an admissible (W an element of the Grassmannian) polarization $H_Y = W^+ \oplus W^-$ and then to the pair (Y, W) we assign the Fock space $Z(Y, W) := \mathcal{F}_W$. While to a $d+1$-dimensional cobordism X with boundary $Y_0 \sqcup Y_1$ with assigned polarizations W_0, W_1 one obtains a canonical vector space homomorphism $Z_X \in \mathrm{Hom}(\mathcal{F}_{W_0}, \mathcal{F}_{W_1})$ via the Plucker embedding. In the case that X is a closed manifold one defines $Z_X = det\mathcal{D}_X \in Det(\mathcal{D}_X)$. Comparing (73) with (67) we see this defines the required FQFT. In particular, in the graph trivialization the pairing (76) coincides with the canonical determinant regularization (60).

All of these constructions have obvious generalizations to families of Dirac operators. For a smooth family of EBVPs $(\mathbb{D}, \mathbb{P})$ parameterized by a manifold B one has associated *Fock bundles* $\mathcal{F}_{K(\mathbb{D})}$ and $\mathcal{F}_{\mathbb{P}}$ and the pairing (73), for example, becomes a pairing from Fock bundles to the determinant line bundle $DET(\mathbb{D}, \mathbb{P})$. Now return to the action of the gauge group where B is as before the parameter space of gauge potentials. We have the gauge transformation group $\mathcal{G}$ acting on the base B and using ideas of [20] it is not hard to see how to lift the action to an induced *projective* action of $\mathcal{G}$ in the Fock bundle intertwining with the family of of quantized Dirac Hamiltonians in the fibres. The essential difference here to [20] is that the action is not a true projective representation of $\mathcal{G}$, but rather a linear isometric action between the different fibres of the Fock bundle. This leads to a new description of the chiral anomaly for a family of EBVPs and, when applied to (75), for a family of Dirac operators over a closed manifold (see [21]). However, rather than pursue that here we shall finish by mentioning a differential geometric description of gauge anomalies via the ζ-function geometry of the determinant bundle.

The essential point is that for a family $(\mathbb{D}, \mathbb{P})$ of EBVPs there are two natural ways to put a metric and connection on the determinant line bundle $DET(\mathbb{D}, \mathbb{P})$. First, by pull-back through the isomorphism (23) the determinant bundle $DET(\mathbb{S}(\mathbb{P}))$ has a canonical metric defined over the open subset U where the operators are invertible by

$$\|det(\mathcal{D}_P)\|^2_{\mathcal{C}} = det_{\mathcal{C}}(\Delta_P) := det_{Fr}(\mathcal{S}(P)^*\mathcal{S}(P)), \tag{77}$$

where $\Delta_P = (\mathcal{D}_P)^*\mathcal{D}_P$ and, recall, $\mathcal{S}(P) := PP(\mathcal{D}) : K(\mathcal{D}) \to \mathrm{range}\,(P)$ with $P(\mathcal{D})$ the Calderon projection of the operator $\mathcal{D}$. On the other hand, $DET(\mathbb{D}, \mathbb{P})$ has a Quillen metric, defined over U by

$$\|det(\mathcal{D}_P)\|^2_{\zeta} = det_{\zeta}(\Delta_P) := e^{-\zeta'_{\Delta_P}(0)} \tag{78}$$

where $\zeta_{\Delta_P}(s) = \mathrm{Tr}\,(\Delta_P^{-s})$ is defined around 0 by analytic continuation.

Furthermore, in each case there are natural connections $\nabla^{\mathcal{C}}, \nabla^{\zeta}$ defined on $DET(\mathbb{D}, \mathbb{P})$ compatible with these metrics, whose curvatures may represent geometric anomalies, as described in the Introduction. Naturally, we would like to know the relation between the quite simple construction of the metric and connection $\|\,.\,\|_{\mathcal{C}}, \nabla^{\mathcal{C}}$ and their delicate and complicated ζ-function partners.

Theorem 4.2. [32] *Let $\mathbb{P}_1, \mathbb{P}_2$ be Grassmann sections for the family of Dirac operators $\mathbb{D}$. Then over the subset of B where the operators $\mathcal{D}_{P_i}$ are invertible one has*

$$\frac{\|det(\mathcal{D}_{P_1})\|_{\zeta}}{\|det(\mathcal{D}_{P_2})\|_{\zeta}} = \frac{\|det(\mathcal{D}_{P_1})\|_{\mathcal{C}}}{\|det(\mathcal{D}_{P_2})\|_{\mathcal{C}}}\,. \tag{79}$$

Let $\mathbf{R}^i_\zeta, \mathbf{R}^i_{\mathcal{C}} \in \Omega^2(B)$ *denote the respective curvature 2-forms of the* ζ *and* $\mathcal{C}$*-connections on* $DET(\mathbb{D}, \mathbb{P}_i)$*. Then:*

$$\mathbf{R}^1_\zeta - \mathbf{R}^2_\zeta = \mathbf{R}^1_{\mathcal{C}} - \mathbf{R}^2_{\mathcal{C}}. \tag{80}$$

Thus (80) is a *relative geometric anomaly* formula. This immediately, for example, leads to an identification between the $\mathcal{G}$ cocycle and ζ-function curvature description of gauge anomalies.

Finally, we point out that a corresponding adiabatic pasting formula will hold for the curvatures with respect to a partition of a closed manifold along the lines of the Park-Wojciechowski formula of Theorem 3.4. That will be explained in a future publication, for the b-calculus analogue of this see [26].

References

[1] Atiyah, M.F., Patodi, V.K., and Singer, I.M.: 1975, 'Spectral asymmetry and Riemannian geometry. I', *Math. Proc. Cambridge Phil. Soc.* **77**, 43–69.

[2] Atiyah, M.F. and Singer, I.M.: Dirac operators coupled to vector potentials. Proc. Nat. Acad. Sci. USA **81**, 2597 (1984).

[3] Bismut, J-M and Freed, D.S.: 1986, D., The analysis of elliptic families I, *Comm. Math. Phys.* **106**, 159–176.

[4] Bismut, J-M and Freed, D.S.: 1986, 'The analysis of elliptic families.II. Dirac operators, eta invariants, and the holonomy theorem', *Comm. Math. Phys.* **107**, 103–163.

[5] Booß–Bavnbek, B., and Wojciechowski, K.P.: 1989, 'Pseudo-differential projections and the topology of certain spaces of elliptic boundary value problems', *Comm. Math. Phys.* **121**, 1–9.

[6] Booß–Bavnbek, B., Scott, S. G., and Wojciechowski, K. P.: 1998, 'The ζ-determinant and $\mathcal{C}$-determinant on the Grassmannian in dimension one', *Letters in Math. Phys.* **45**, 353–362.

[7] Booß–Bavnbek, B., and Wojciechowski, K.P.: 1993, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, Boston.

[8] Brünning, J. and Lesch, M.: 1999, On the η-invariant of certain nonlocal boundary value problems, *Duke Math. J.* **96**, 425–468.

[9] Bunke, U.: 1995, 'On the glueing problem for the η-invariant', *J. Diff. Geom.* **41/2**, 397–448.

[10] Daniel, M. and Kirk, P., with an appendix by Wojciechowski, K. P.: 1999, 'A general splitting formula for the spectral flow', *Michigan Math. J.* **46**, 589–617.

[11] Dai, X., and Freed, D.: 1994, 'η-invariants and determinant lines', *J. Math. Phys.* **35**, 5155–5195.

[12] Douglas, R.G., and Wojciechowski, K.P.: 1991, 'Adiabatic limits of the η–invariants. The odd–dimensional Atiyah–Patodi–Singer problem', *Comm. Math. Phys.* **142**, 139–168.

[13] Freed, D.: 1986, 'Determiants, torsion and strings', *Commun. Math. Phys.* **107**, 483–513.

[14] Hassell, A., Mazzeo, M. and Melrose, R.B.: 1996, 'A signature formula for manifolds with corners of codimension 2', Preprint.

[15] Klimek, S., and Wojciechowski, K.P.: 1993, 'η–invariants on manifolds with cylindrical ends', *Diff. Geom. and Appl.* **3**, 191–201.

[16] Klimek, S., and Wojciechowski, K.P.: 1996, 'Adiabatic cobordism theorems for analytic torsion and η–invariant', *J. Funct. Anal.* **136**, 269–293.

[17] Lesch, M., and Wojciechowski, K.P.: 1996, 'On the η–invariant of generalized Atiyah–Patodi–Singer problems', *Illinois J. Math.* **40**, 30–46.

[18] Mazzeo, R.R. and Melrose, R.B.: 1995, 'Analytic surgery and the η-invariant', *GAFA* **5**, 14–75.

[19] Melrose, R.B., Piazza, P.: 1997, 'Families of Dirac operators, boundaries and the b-calculus', *J. Diff. Geom.* **46**, 99–167.

[20] Mickelsson, J.: *Current algebras and groups.* London and New york: Plenum Press, 1989.

[21] Michelsson, J., Scott, S.G.: 1999, 'Functorial QFT, gauge anomalies and the Dirac determinant bundle', preprint; hep-th/9908207.

[22] Müller, W.: 'Eta invariants and the manifolds with boundary', *J. Diff. Geom.* **40**, 311–377.

[23] Müller, W.: 'On the index of Dirac operator on manifolds with corners of codimension two.I.', *J. Diff. Geom.* **44**, 97–177.

[24] Müller W.: 1997, 'On the relative determinant', *Comm. Math. Phys.* **192**, 309–347.

[25] Park, J. and Wojciechowski, K. P.: 2000, *Relative ζ-determinant and Adiabatic Pasting Formula for the ζ-determinat of the Dirac Laplacian*, Preprint.

[26] Piazza, P.: 1996, 'Determinant bundles, manifolds with boundary and surgery', I *Comm. Math. Phys.* **178**, 597–626; 1998, II, *Comm. Math. Phys.* **193**, 105-124.

[27] Pressley, A. and Segal, G.B.: *Loop Groups.* Oxford: Clarendon Press, 1986.

[28] Quillen, D. G.: 1985, 'Determinants of Cauchy-Riemann operators over a Riemann surface', *Funkcionalnyi Analiz i ego Prilozhenya* **19**, 37–41.

[29] Ray, D., and Singer, I.M.: 1971, 'R–torsion and the Laplacian on Riemannian manifolds', *Adv. Math.* **7**, 145–210.

[30] Scott, S.G.: 1995, 'Determinants of Dirac boundary value problems over odd–dimensional manifolds', *Comm. Math. Phys.* **173**, 43–76.

[31] Scott, S.G.: 'Splitting the curvature of the determinant line bundle', *Proc. Am. Math. Soc.*, to appear.

[32] Scott, S.G.: 'Relative Zeta determinants and the Quillen metric', preprint; math.AP/9910148.

[33] Scott, S.G., and Wojciechowski, K.P.: 1997, 'Abstract Determinant and ζ–Determinant on the Grassmannian', *Letters in Math. Phys.* **40**, 135–145.

[34] Scott, S.G., and Wojciechowski, K.P.: 1999, 'ζ-determinant and the Quillen determinant on the Grassmannian of elliptic self-adjoint boundary conditions', *C. R. Acad. Sci., Serie I,* **328**, 139–144.

[35] Scott, S.G., and Wojciechowski, K.P.: 'The ζ-determinant and Quillen determinant for a Dirac operator on a manifold with boundary', *GAFA*, to appear.

[36] Seeley, R. T.: 1967 'Complex powers of an elliptic operator'. AMS *Proc. Symp. Pure Math. X.* AMS Providence, 288–307.

[37] Segal, G.B.: 'The definition of conformal field theory', Oxford preprint, 1990.

[38] Singer, I.M.: 1985, 'Families of Dirac operators with applications to physics', *Asterisque, hors série*', 323–340.

[39] Wojciechowski, K.P.: 1994, 'The additivity of the η-invariant: The case of an invertible tangential operator', *Houston J. Math.* **20**, 603–621.

[40] Wojciechowski, K.P.: 1995, 'The additivity of the η-invariant. The case of a singular tangential operator', *Comm. Math. Phys.* **169**, 315–327.

[41] Wojciechowski, K.P.: 1999, 'The ζ-determinant and the additivity of the η-invariant on the smooth, self-adjoint Grassmannian', *Comm. Math. Phys.* **201**, 423–444.

Department of Mathematics, King's College, London WC2R 2LS, U.K.
E-mail address: `sscott@mth.kcl.ac.uk`

Department of Mathematics, IUPUI (Indiana/Purdue), Indianapolis IN 46202–3216, U.S.A.
E-mail address: `kwojciechowski@math.iupui.edu`